

Maximizing Interactions in Class and at Social Mixers

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Introduction

Have you ever noticed that at parties, wedding receptions, and class reunions, people tend to hang out with the people whom they already know rather than to mingle with everyone else? Not surprisingly, faculty have this same problem in their classrooms. Students tend to sit with their friends and are reluctant to meet and work with the other students in the class.

While this may not be a problem in a generic lecture course, it can be a significant problem in courses where students routinely work in small groups. Unless they are regularly reassigned to groups, students can get stuck in a group of strong students or a group of weak students, or with someone who doesn't pull their own weight for the entire semester. All of these situations hinder the professor's ability to assess a student's individual work accurately.

If students are asked to form new groups on their own, they frequently try to re-form to be with acquaintances again, to be with students at the same skill level, or to avoid "problem" students. As an alternative, the professor could spend a lot of time carefully constructing new groups; but then students who are in groups that they don't want to be in feel "stuck" there by the professor, and this generates negative feelings.

We have found an alternative solution in the Quantitative Problem Solving course at Valparaiso University. In this course, students usually work in seven groups of four, with each group at a small table. To address the problems described above, each time new groups are formed, we ask the students to rearrange themselves according to the following two rules:

1. No person may sit at the same table more than once.
2. No two people may sit together more than once.

These two rules ensure that students do not become dependent on a particular group member and that no student can become physically isolated in the classroom. Students get invested in the answer, which involves problem-solving on their part.

Since we change groups four or five times in a semester, the students find that solving this problem becomes progressively harder. The problem leads us to the following natural generalization and to the specific question of interest to us:

Suppose that you have k people sitting at each of n tables. They get up and rearrange themselves according to the rules stated above. How many times can a new seating arrangement be generated?

We say that a set of seating arrangements is *valid* if, given any two seating arrangements from the set, one can be obtained from the other by a move satisfying both conditions. We denote the total number of arrangements (the original plus all rearrangements) by $v(n, k)$.

We describe the connections of this problem to similar problems, provide a partial solution, and show the reader where to begin the search for the complete solution.

Two Basic Examples

Let capital letters denote the people. Suppose that we have $n = 4$ tables and let $k = 3$ people sit at each of them. Table 1 shows the valid seating arrangements for the 12 people, A, . . . , L.

Table 1.
Arrangements with 3 people at each of 4 tables.

Table 1	Table 2	Table 3	Table 4
ABC	DEF	GHI	JKL
FHJ	AIK	BDL	CEG
DGK	CHL	AEJ	BFI
EIL	BGJ	CFK	ADH

Since A has sat at all four tables, no further rearrangements are possible. Thus, $v(4, 3) = 4$.

However, if we take the same four tables and allow $k = 4$ people to sit at each of them, we run into a surprising difficulty, shown in Table 2. After we place each of A, B and C at new tables, where do we place D?

Since there is no way to resolve the problem of where to put D, we conclude that $v(4, 4) = 1$.

Table 2.

Attempt at a rearrangement with 4 people at each of 4 tables.

Table 1	Table 2	Table 3	Table 4	
ABCD	EFGH	IJKL	MNOP	
	A	B	C	D???

A naïve approach to solving the problem is simply to attempt to try every possible seating arrangement. Since this involves considering each of the $(nk)!$ permutations of the people, it is impractical.

Kirkman's Schoolgirl Problem and the Golfer's Dilemma

Our problem sounds strikingly similar to Kirkman's schoolgirl problem [Kirkman 1850a, Kirkman 1850b]:

Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily, so that no two walk twice abreast.

It also sounds very similar to the Golfer's Dilemma [Mellinger 2004]:

John regularly plays golf with a group of 16 people. Three days a week for the entire summer, they go out in 4 groups of 4 players each to hit the course. Is there some way they can arrange the players in the groups each day so that everybody plays with everybody else in some sort of regular way?

Unfortunately—or fortunately, depending on your perspective—the problem that we pose in this paper is distinct from both of these other problems because of our first rule, not sitting at the same table twice. It would require the schoolgirls not only to walk with different classmates but also to walk in a different row each day; it would require the golfers not only change their partners but also change their tee-off sequence.

These two problems do give us a place to begin our search for a solution. Since both are solved by creating a block design, considering them eventually leads us to consider mutually orthogonal Latin squares (MOLS).

A Latin square is an $n \times n$ array on a set of n elements such that each of the elements occurs once in each row and once in each column, as shown in Table 3.

Two Latin squares are *orthogonal* if and only if in the juxtaposed array each ordered pair occurs exactly once. A set of Latin squares are *mutually orthogonal* if they are pairwise orthogonal. Table 4 is a collection of four pairwise orthogonal Latin squares on $n = 5$ elements. (Each of the four 5×5 blocks is a Latin square.)

Table 3.
A 5×5 Latin square.

E	A	B	C	D
D	E	A	B	C
C	D	E	A	B
B	C	D	E	A
A	B	C	D	E

Table 4.
Four orthogonal 5×5 Latin squares.

EABCD	DEABC	CDEAB	BCDEA
DEABC	BCDEA	EABCD	CDEAB
CDEAB	EABCD	BCDEA	DEABC
BCDEA	CDEAB	DEABC	EABCD
ABCDE	ABCDE	ABCDE	ABCDE

Results from Orthogonal Latin Squares

Let $\{A_1, A_2, \dots, A_k\}$ be a family of mutually orthogonal family of Latin squares whose entries are defined on n elements. Construct an $n \times n$ seating matrix SM with (i, j) th entry

$$SM_{i,j} = (a_1(i,j), \dots, a_e(i,j), \dots, a_k(i,j)),$$

a k -tuple whose e th element is the (i, j) th entry of the e th Latin square. Now let each of our kn people be represented by a specific element in a specific Latin square. Thus, each person can be denoted by (v, e) , with v the person's unique element in the Latin square e the in which the person occurs. The k -tuple $SM_{i,j}$ can be viewed as a table, the j th table, with k people sitting at it. Person (v, e) is sitting at this table if and only if there is a v in the e th position of the k -tuple. Further, each row of SM represents a possible seating arrangement, since every v appears in each row of a given Latin square only once.

Example: Consider the case $n = 4$ and $k = 2$, so that we have 4 tables with 2 people each, with the two MOLS and the resulting seating matrix in Table 5. Think of each person as having a name (v, e) , where v is the first name (from a list A, B, ...) and e is the last name (1 or 2, which tells which Latin square, A_1 or A_2 , the person is from). Thus, each MOLS contains arrangements of 4 of the people.

Then, for example, in the seating matrix SM, the $(3, 3)$ entry (in its third row and third column) is the 2-tuple DA. The D comes from the D in position $(3, 3)$ in A_1 , which represents person $(v, e) = (D, 1)$; the A comes from the 1 in position $(3, 3)$ in A_2 , which represents person

$(v, e) = (A, 2)$. The entry tells us that persons $(D, 1)$ and $(A, 2)$ are sitting together at the same table in this seating plan.

Similarly, the 2-tuple CC in position $(4, 3)$ of SM represents persons $(C, 1)$ and $(C, 2)$ sitting together.

Table 5.

Two MOLS and the corresponding seating matrix SM.

A_1	A_2	SM
DCBA	BADC	DB CA BD AC
CDAB	DCBA	CD DC AB BA
BADC	CDAB	BC AD DA CB
ABCD	ABCD	AA BB CC DD

From this setup, we get the following theorem.

Latin Square Theorem. *If there exist k MOLS of order n , then the number of arrangements is $v(n, k) = n$.*

Proof. Let $\{A_1, A_2, \dots, A_k\}$ be a family of mutually orthogonal Latin squares whose entries are defined on n elements, and construct the seating matrix SM.

First, we show that no person sits at the same table more than once. Since A_e is a Latin square, element v is positioned in each column j exactly one time. Because of this, in column j of SM, v will be the e th element of a k -tuple exactly once. Since each column of SM represents a different table in a seating arrangement, no person sits at a particular table more than once. In addition, each person sits at each table at least once.

Next, we show that no one sits with anyone else more than once. Consider people (v, e) and (w, f) . Since A_e and A_f are orthogonal, elements v and w appear together in a k -tuple in the e th and f th positions exactly once in SM. Thus, we have persons (v, e) and (w, f) sitting together exactly once. Therefore, the set of rows of SM form a valid set of n seating arrangements and $v(n, k) = n$. \square

The *Handbook of Combinatorial Design* [Colbourn and Dinitz 1996] gives theorems on the existence of MOLS and lists the number of MOLS for $n < 10,000$. We reproduce the first 11 entries in Table 6. If k is less than or equal to the number given in the right column of the table, we have $v(n, k) = n$.

If n is a prime or a prime power, the number of MOLS is $n - 1$. If n has many prime factors, the number of MOLS may drop significantly. For $n = 6$, there is just one MOLS, as proven by Tarry [1900; 1901] in his investigation of Euler's Problem of the 36 Officers.

Table 6.

Numbers of orthogonal Latin squares.

n	Max # of MOLS
2	1
3	2
4	3
5	4
6	1
7	6
8	7
9	8
10	2
11	10
12	5

Table 7 demonstrates that $v(6, 3) = 6$. Hence, the converse of the Latin Square Theorem is false: It is not true that $v(n, k) = n$ implies that there are k MOLS of order n . Although $v(6, 3) = 6$, there are not 3 MOLS of order 6, just 1.

Table 7.

Six arrangements with 3 people at each of 6 tables.

Table 1	Table 2	Table 3	Table 4	Table 5	Table 6
ABC	DEF	GHI	JKL	MNO	PQR
JMP	KNQ	ADL	GOR	BEH	CFI
DNR	BIJ	FOQ	CHR	AGK	ELM
EGQ	HLR	CGN	AFM	DIP	BKO
FHK	AOP	BMR	EIN	CLQ	DGJ
ILO	CGJ	EKP	BDQ	FJR	AHN

Results from SOMAs

Table 7 suggests need for a more general structure to classify solutions.

Phillips and Wallis [1996] studied *simple orthogonal multi-arrays (SOMA)* and introduced the name, but they had been studied prior to that as special class of semi-Latin squares [Bailey 1992]. A $SOMA(n, k)$ is an $n \times n$ array A whose entries are k -subsets of a kn -set O , such that each element of O occurs exactly once in each row and exactly once in each column of A , and no 2-subset of O is contained in more than one entry of A . This is precisely the structure that we have been looking for!

Table 7 displays a $SOMA(6, 3)$. SOMAs are related to orthogonal Latin squares by the following result, whose converse is not true. (The existence of $SOMA(6, 3)$ is a counterexample to the converse.)

MOLS-SOMA Theorem. *If there are k MOLS of order n , then there is a SOMA(n, k).*

Since a SOMA(n, k) has precisely the structure that we seek, the following theorem is obvious.

SOMA Theorem. *If there is a SOMA(n, k), then $v(n, k) = n$.*

Soicher [1999] investigated when a SOMA(n, k) exists and identified many of the properties of SOMAs. Unfortunately, SOMAs do not exist for all values of n and k . In particular, Phillip and Wallis [1996] conducted an exhaustive computer search for a SOMA(6, 4) and determined that one does not exist. Table 8 represents a typical example of how the first three rows and a partial fourth row might look in this search.

Table 8.
Almost 4 arrangements with 4 people at each of 6 tables.

Table 1	Table 2	Table 3	Table 4	Table 5	Table 6
ABCD	EFGH	IJKL	MNOP	QRST	UVWX
FLPU	AKQV	BGMR	CHSW	DINX	EJOT
VRIE	SNJU	AWOF	LBXT	GCKP	QMHD
SOXG	RWLD	VJPH	IUQC	EAMT	NFBK

In Table 8, the italic T at the bottom of the second-last column cannot be placed in the position indicated because it has already appeared in that column. From this work, we see that $v(6, 4) = 3$ or $v(6, 4) = 4$, depending on whether it is possible to complete the fourth row. If a SOMA(n, k) doesn't exist, we can still compute a value for $v(n, k)$ directly.

Summary

We have seen that whenever a SOMA(n, k) exists, the number of arrangements of nk people sitting k people per table is $v(n, k) = n$. We've also seen, in our first example, that $v(n, k)$ may equal 1 and, in our last example, that $v(n, k)$ may take values between 1 and n . (We don't know if $v(n, k)$ can take all of the values between 1 and n as k varies.)

When we ask students to solve this rearrangement problem in practice in our classroom, we usually do not explain any of the mathematics that we have used in this paper. While we mention (after the fact) that it is a somewhat difficult problem, we leave out the technical details, since the course is about the fundamental mathematical and problem-solving skills necessary to be quantitatively literate.

However, students' attempts to implement the rearrangements yield interesting observations. Recall that we usually have four students sitting

at each of seven tables. (If there are fewer than 28 students, empty spaces are populated by pseudo-students who also need to be moved.) By the Latin Square Theorem, the students should be able to rearrange themselves six times over the course of the semester. In practice, this does not happen! The first rearrangement is easy, and students are mildly amused by the problem and process of moving around in class. The second rearrangement is more difficult and is usually successful only if one or two students take leadership roles and direct the others.

The third rearrangement usually fails, for three reasons:

- The second rearrangement was done ad-hoc, but the sorting process is more difficult and far more time-consuming the third time.
- Surprisingly, some students forget with whom they sat initially.
- At this point in the semester, students are no longer intrigued by this problem; it is simply one more thing in their busy lives.

However, this failure does raise an other interesting mathematical question:

When $(n - 1)$ rearrangements are possible, what is the expected number that one would hope to see using a random assignment process?

Final Note

This rearrangement problem has other social applications as well. "Three-minute dating" is used by matchmakers to introduce a group of strangers to one another. At these events, each participant moves to a new table and has a conversation with a new partner every three minutes. It works because it is simply our problem with $k = 2$ and a large value of n ; for most large n , there are at least two MOLS. (Matchmakers give an approximate solution when there are not two MOLS.)

Similarly, at a progressive dinner, the diners move to new tables, with new dinner partners, for each course. This stimulates the dinner conversation and provides entertainment throughout the evening. If you are hosting a progressive dinner, you should have a prime power p^α number of tables (with fewer guests per table than tables) to maximize the number of courses in the dinner: There are $p^\alpha - 1$ MOLS and hence p^α arrangements [Colbourn and Dinitz 1996]. Thus, you can have p^α courses in your meal. Four- and five-course dinners are reasonable for up to 12 and 20 diners, respectively. . . ; or you could be an extravagant host and serve an eight-course progressive dinner for up to 56 people!

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