A COUNTER-INTUITIVE BAD EFFECT OF QUARANTINE

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Abstract. We analyze an epidemic model that resembles an infection transmitted through both direct contact and an intermediate host. We show that, if the intermediate host cannot be controlled, restricting the infected individuals to quarantine may worsen the spread of the disease or even cause an endemic situation. Specifically, the epidemic reproductive number is computed in the absence and presence of the quarantined population. We show that, if the infection rate among quarantined individuals and susceptible intermediate hosts is high enough, a stable endemic equilibrium will occur while ,at the same time, the absence of quarantine measures may still lead to a disease free situation. This theoretical model is inspired from the Bubonic Plague epidemic of the 14^{th} century.

1. Introduction

When an epidemic of an infectious disease occurs in a society, a common way of controlling the disease is to separate infected individuals from healthy individuals. This generally proves to be effective or at least helpful for controlling the disease, since the spread can only occur when healthy individuals come into contact with infected individuals. However, quarantine is not always the best solution or even beneficial at all. For example, when the epidemic known as the Black Death occurred in Europe in the 14th century, infected people were forced to stay in their homes, even against their will if necessary. However, quarantine had limited effectiveness as the Black Death continued to spread. In fact, it is possible that attempts to quarantine the infected population made the epidemic even worse. This is possible because the deadly disease is believed to be the bubonic plague, which is spread by fleas transferring blood between humans and rats. Since the rats are not quarantined, they can still spread the disease between isolated and non-isolated people. Furthermore, quarantining people resulted in overcrowded houses, which were ideal for rats to pick up the disease from an infected person and transfer it elsewhere.

The aim of this paper is to show that it is mathematically possible to obtain a reversed effect of quarantine measures whereby isolating infected individuals, not only makes the epidemic worse, but may cause the very epidemic we try to avoid. The next step in this analysis is to test these

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models against real data pertaining to bubonic plague or similar infectious diseases spread by carrier.

2. The model

$$
\begin{cases}
S' &= H - \lambda_1 SI - \lambda_2 SR_i + \delta I + \delta J - \epsilon S, \\
I' &= \lambda_1 SI + \lambda_2 SR_i - \delta I - \mu_1 I - qI - \epsilon I, \\
J' &= qI - \mu_1 J - \delta J - \epsilon J, \\
R_s' &= R - \lambda_3 R_s J - \lambda_4 R_s R_i - \lambda_5 R_s I - \eta R_s, \\
R_i' &= \lambda_3 R_s J + \lambda_4 R_s R_i + \lambda_5 R_s I - \mu_2 R_i - \eta R_i.
\end{cases} \tag{1}
$$

The meaning of the parameters is as follows:

- $S =$ susceptible human population,
- $I =$ infected human population,
- $J =$ quarantined infected humans,
- $R_s =$ susceptible rat population,
- $R_i =$ infected rat population,
- H, R are the recruitment constant rates of susceptible humans and rats,
- ϵ and η are the natural mortality rates for human and rats,
- μ_1 and μ_2 are the diseased induced mortality rates for infected humans and rats,
- δ is the recovery rate,
- λ_1 through λ_5 are infection rates assumed equal (for simplicity) except λ_3,
- q is the transition rate into the quarantined class.

3. The epidemic reproductive number in the presence and absence of quarantine

For simplicity, we assume $\lambda_2 = \lambda_4 = \lambda_5 := \lambda_1$. We want to show eventually that the infection between quarantined infected people and healthy rats may cause an endemic situation. Using the method from [6] we separate the equations corresponding to the infected classes, I, J and R_i into two parts: F corresponding to the new infections and V corresponding to the remaining terms after eliminating the new infections.
In the case $q = 0$ the jacobians of the above matrices at the DFE are:

$$
F = \begin{pmatrix} \lambda_1 S & 0 & \lambda_1 S \\ 0 & 0 & 0 \\ \lambda_1 R_s & \lambda_3 R_s & \lambda_1 R_s \end{pmatrix}
$$

$$V = \begin{pmatrix} \mu_1 + \delta + \epsilon & 0 & 0 \\ 0 & \mu_1 + \delta + \epsilon & \\ 0 & 0 & \mu_2 + \eta \end{pmatrix}$$

The epidemic reproductive number is give by the spectral radius of FV^{-1}. This is

$$\mathcal{R}_0 = \lambda_1 \frac{R_s^*}{\mu_2 + \eta} + \lambda_1 \frac{S^*}{\mu_1 + \delta + \epsilon}$$

where

$$R_s^* = \frac{R}{\eta}, \quad \text{and} \quad S^* = \frac{H}{\epsilon}.$$

Biological interpretation of $\mathcal{R}_0$: $\frac{1}{\mu_2+\eta}$ represents the expected life time of an infected rat. Therefore the term $\lambda_1 \frac{R_s^*}{\mu_2+\eta}$ represents the secondary cases of infections produced by a single infected rat introduced in the population. Similarly, $\lambda_1 \frac{S^*}{\mu_1+\delta+\epsilon}$ represents the scondary cases of infections among humans caused by a single infected individual.

Now the similar results for $q \neq 0$.

$$F^q = \begin{pmatrix} \lambda_1 S & 0 & \lambda_1 S \\ q & 0 & 0 \\ \lambda_1 R_s & \lambda_3 R_s & \lambda_1 R_s \end{pmatrix}$$

$$V^q = \begin{pmatrix} \mu_1 + \delta + \epsilon + q & 0 & 0 \\ -q & \mu_1 + \delta + \epsilon & \\ 0 & 0 & \mu_2 + \eta \end{pmatrix}$$

The epidemic reproductive number in the presence of quarantine is the spectral radius of $F^q(V^q)^{-1}$ which is given by the largest absolute value of the roots of the following characteristic polynomial

$$f(x) := x^3 - \left[\lambda_1 \frac{R_s^*}{\mu_2 + \eta} + \lambda_1 \frac{S^*}{\delta + \mu_1 + \epsilon + q}\right] x^2 - \frac{\lambda_1 \lambda_3 S^* R_s^* q}{(\mu_1 + \delta + \epsilon)(\mu_1 + \delta + \epsilon + q)(\mu_2 + \eta)} x$$

First notice that one eigenvalue is zero and the other two are the roots of the quadratic polynomial

$$g(x) := x^2 - \left[\lambda_1 \frac{R_s^*}{\mu_2 + \eta} + \lambda_1 \frac{S^*}{\delta + \mu_1 + \epsilon + q}\right] x - \frac{\lambda_1 \lambda_3 S^* R_s^* q}{(\mu_1 + \delta + \epsilon)(\mu_1 + \delta + \epsilon + q)(\mu_2 + \eta)}$$

Notice that the discriminant is strictly positive so there are two real distinct roots of which one is negative and one is positive. Denote the positive root by $\mathcal{R}_0^q$. We have the following characterization:

$$\mathcal{R}_0^q < 1 \quad \text{if} \quad g(1) > 0$$
$$\mathcal{R}_0^q > 1 \quad \text{if} \quad g(1) < 0$$

Notice that

$$g(1) = 1 - \left[\lambda_1 \frac{R_s^*}{\mu_2+\eta} + \lambda_1 \frac{S^*}{\delta+\mu_1+\epsilon+q}\right] - \frac{\lambda_1\lambda_3 S^* R_s^* q}{(\mu_1+\delta+\epsilon)(\mu_1+\delta+\epsilon+q)(\mu_2+\eta)}$$

Therefore

$$\mathcal{R}_0^q > 1 \Leftrightarrow \left[\lambda_1 \frac{R_s^*}{\mu_2+\eta} + \lambda_1 \frac{S^*}{\delta+\mu_1+\epsilon+q}\right] + \frac{\lambda_1\lambda_3 S^* R_s^* q}{(\mu_1+\delta+\epsilon)(\mu_1+\delta+\epsilon+q)(\mu_2+\eta)} > 1$$

and

$$\mathcal{R}_0^q < 1 \Leftrightarrow \left[\lambda_1 \frac{R_s^*}{\mu_2+\eta} + \lambda_1 \frac{S^*}{\delta+\mu_1+\epsilon+q}\right] + \frac{\lambda_1\lambda_3 S^* R_s^* q}{(\mu_1+\delta+\epsilon)(\mu_1+\delta+\epsilon+q)(\mu_2+\eta)} < 1$$

Which means, for all pratical purposes, we can denote the epidemic reproductiv number in the case ($q \neq 0$) as

$$\mathcal{R}_0^q = \left[\lambda_1 \frac{R_s^*}{\mu_2+\eta} + \lambda_1 \frac{S^*}{\delta+\mu_1+\epsilon+q}\right] + \frac{\lambda_1\lambda_3 S^* R_s^* q}{(\mu_1+\delta+\epsilon)(\mu_1+\delta+\epsilon+q)(\mu_2+\eta)}$$

We are interested in establishing a condition that proves the quarantine CAUSES and endemic situation rather than eliminating the disease. To this end, we assume that in the absence of quarantine ($q = 0$) the Disease Free Equilibrium (DFE) is stable, i.e.

$$\mathcal{R}_0 < 1,$$

and we want to investigate whether, at the same time, it is possible that

$$\mathcal{R}_0^q > 1,$$

which means the disease is endemic in the presence of quarantine.

That means we want to establish

$$\mathcal{R}_0 < 1 < \mathcal{R}_0^q$$

First, we need to see whether $\mathcal{R}_0 < \mathcal{R}_0^q$ is possible at all. This is equivalent to

$$\frac{\lambda_3 R_s^*}{\mu_2+\eta} > 1$$

This condition can be interpreted as follows: The secondary cases of infected rats produced by a typical infected rat who, furthermore infects a human that become quarantined who, in turns, infects $\lambda_3 R_s$ healthy rats.

Notice also that, from $\mathcal{R}_0 < 1$ we have $\frac{\lambda_1 R_s^*}{\mu_2+\eta} < 1$. Together with $\frac{\lambda_3 R_s^*}{\mu_2+\eta} > 1$, it follows that

$$\lambda_3 > \lambda_1$$

which is expected since the endemic situation with quarantine results primarily from a higher infection rate between quarantined infected people and healthy rats. It is clear that the condition above is satisfied if λ_1 is small enough and λ_3 is large enough. We establish now a condition on q such that $\mathcal{R}_0^q > 1$. Solving this inequality for q we obtain the solution

$$q > \frac{(\mu_1+\delta+\epsilon)(1-\mathcal{R}_0)}{\frac{\lambda_1 R_s^*}{\mu_2+\eta}\left[1 + \frac{\lambda_3 S^*}{\mu_1+\delta+\epsilon}\right] - 1}$$

together with the condition that the denominator is positive, i.e.

$$\frac{\lambda_1 R_s^*}{\mu_2+\eta}\left[1+\frac{\lambda_3 S^*}{\mu_1+\delta+\epsilon}\right]-1>0$$

which is equivalent to

$$\frac{\lambda_3 S^*}{\mu_2+\eta}>(\mu_1+\delta+\epsilon)\left[\frac{1}{\lambda_1 R_s}-\frac{1}{\mu_2+\eta}\right]$$

To summarize, we obtained two thresholds for q and λ_3 representing the situation when the quarantine causes an endemic situation:

$$q>\frac{(\mu_1+\delta+\epsilon)(1-\mathcal{R}_0)}{\frac{\lambda_1 R_s^*}{\mu_2+\eta}\left[1+\frac{\lambda_3 S^*}{\mu_1+\delta+\epsilon}\right]-1}$$

and

$$\frac{\lambda_3 S^*}{\mu_2+\eta}>\max\left\{1,(\mu_1+\delta+\epsilon)\left[\frac{1}{\lambda_1 R_s}-\frac{1}{\mu_2+\eta}\right]\right\}$$

4. Conclusion

The main result can be summarized as follows: assuming the carrier (the rat) cannot be controlled or contained and provided that the infection rate between quarantined infected people and healthy rats is high enough, the quarantine measure has the net effect of increasing the epidemic in the rat population. This in turns leads to an increase of the epidemic in the healthy humans. In other words, the quarantined establishments become sources of new infections. Further research is necessary to investigate whether the threshold values when this effect happens are biologically realistic.

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