

# $k$ -Equitable Labeling of Graphs

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## Abstract

A labeling of a graph is a function  $L$  from the vertex set to some subset of the natural numbers. The image of a vertex is called its label. We assign the label  $|L(u) - L(v)|$  to the edge incident with vertices  $u$  and  $v$ . In a  $k$ -equitable labeling the image of  $L$  is the set  $\{0, 1, 2, \dots, k-1\}$ . We require both the vertex labels and the edge labels to be as equally distributed as possible. We examine 4-equitable labelings of caterpillars.

## Definitions

**Graph** A graph is a collection of points, also known as vertices, and lines, also known as edges, which are pairs of the points.

**Adjacent** Two vertices are adjacent if there is an edge between them.

**Incident** A vertex and an edge are incident if they meet.

**Degree** The degree of a vertex is the number of edges incident to that vertex.

**End Vertex** An end vertex is a vertex of degree 1.

**Path** A path is an alternating sequence of incident vertices and edges where no vertex or edge occurs twice.

**Empty Graph** An empty graph is a graph without edges. The empty graph is a path.

**Connected Graph** A graph is connected if there is a path between every pair of vertices.

**Cycle** A cycle is almost a path except the first vertex is the same as the last vertex, so every vertex has a degree of 2.

**Tree** A tree is a connected graph without cycles.

**Caterpillar** A caterpillar is a tree with the property that the removal of its end vertices leaves a path.

**Spine** A spine is the path that results from the removal of all end vertices from a caterpillar.

**Centipede** An  $n$ -centipede is a caterpillar such that for every vertex on the spine of the caterpillar there are  $n$  adjacent end vertices.

**Star** An  $n$ -star is a caterpillar on  $n + 1$  vertices with a 1-vertex spine having degree  $n$ .

**Labeling** A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

**$k$ -Equitable Labeling** A  $k$ -equitable labeling of a graph is a function  $L$  from the vertex set to the set  $\{0, 1, 2, \dots, k - 1\}$ . The image of a vertex is called its label. We assign the label  $|L(u) - L(v)|$  to the edge incident with vertices  $u$  and  $v$ . We require both the vertex labels and the edge labels to be as equally distributed as possible; that is, if  $v_i$  denotes the number of vertices labeled  $i$  and  $e_i$  denotes the number of edges labeled  $i$ , we require  $|v_i - v_j| \leq 1$  and  $|e_i - e_j| \leq 1$  for every  $i, j$  in the image set.

## Proofs

### Labelings

#### Labeling a Path

**Theorem.** *Every path is  $k$ -equitable for any  $k$ .*

*Proof.* Given  $k$  labels  $\{0, 1, \dots, k - 1\}$ , start by labeling the first vertex (of degree 1)  $k - 1$ ; to yield an edge label of ' $k - 1$ ', two adjacent vertices must be  $k - 1$  and 0, so label the proceeding vertex 0. The next vertex will then be labeled  $k - 2$ , given that there are only two means of yielding an edge labeled  $k - 2$ , with vertices  $k - 1$  and 1 or  $k - 2$  and 0; because  $k - 1$  has already been used, the remaining choice of labels will be  $k - 2$ . Likewise, the next vertex will be labeled 2, followed by  $k - 3$ , and so on until one reaches vertex label  $\frac{k-2}{2}$  for any even  $k$  and  $\frac{k-1}{2}$  for any odd  $k$ . If the path is not yet fully labeled, then the proceeding vertex must be labeled  $\frac{k-2}{2}$  for any even  $k$ , or  $\frac{k-1}{2}$  for any odd  $k$ . This is the only way to yield an edge label 0, having exhausted all other choices of labels. The pattern for the labeling is then reversed; any time one reaches  $k - 1$ , or  $\frac{k-2}{2}$  for any even  $k$  or  $\frac{k-1}{2}$  for any odd  $k$ , the proceeding label will be the same for the aforementioned reason.  $\square$

#### Labeling a Star

**Theorem.** *Every star is  $k$ -equitable for any  $k$ .*

*Proof.* Given  $k$  labels  $\{0, 1, \dots, k - 1\}$ , begin by labeling the center vertex  $k - 1$  or 0. If the center vertex is  $k - 1$ , cycle through all other vertices with labels

$k - 2, k - 3, \dots, 1, 0$ ; if the center vertex is 0, cycle through all other vertices with labels  $1, 2, \dots, k - 2, k - 1$ . If all vertices are not yet labeled, begin labeling vertices starting with the same label as the center vertex, then repeat the cycling through the vertices as previously determined by the center vertex.  $\square$

### Labeling a 1-centipede

**Theorem.** *Every 1-centipede is  $k$ -equitable for any  $k$ .*

*Proof.* Given  $k$  labels  $\{0, 1, \dots, k - 1\}$ , a set of  $2n$  vertices, with path of  $n$  vertices, with  $j = 1$  edge off of each vertex to another distinct vertex, begin by labeling the path  $0, k - 2, 2, k - 4, \dots$ , and the vertices off of the path  $k - 1, 1, k - 3, 3, \dots$  respectively. Given that the resulting edge set of the path will be of labels  $k - 2, k - 4, k - 6, \dots, 0$ , and the edge set of the edges off of the path will be  $k - 1, k - 3, k - 5, \dots, 1$ , all labels are accounted for. When you have exhausted all vertices, simply begin labeling the path with the reverse of the path labeling if  $k$  is even, and with the reverse of the extensions labeling if  $k$  is odd.  $\square$

## Methods

### Generating a labeling given another distinct labeling

**Theorem.** *Given a  $k$ -equitable labeling of a graph generated by  $L : v \rightarrow 0, 1, \dots, k - 1$ , then  $L' : v \rightarrow 0, 1, \dots, k - 1$  with  $L'(v) = k - 1 - L(v)$  is also a  $k$ -equitable labeling.*

*Proof.* For any two adjacent vertices  $u$  and  $v$ , consider  $L(u)$  and  $L(v)$ . By the methods of labeling, it follows that  $L(uv) = |L(u) - L(v)|$ . Now, consider  $L'(u) = k - 1 - L(u)$  and  $L'(v) = k - 1 - L(v)$ :

$$\begin{aligned}
L'(uv) &= |L'(u) - L'(v)| \\
&= |(k - 1 - L(u)) - (k - 1 - L(v))| \\
&= |(k - 1) - L(u) - (k - 1) + L(v)| \\
&= |(k - 1) - L(u) - (k - 1) + L(v)| \\
&= |-L(u) + L(v)| \\
&= |-1(L(u) - L(v))| \\
&= |L(u) - L(v)|
\end{aligned}$$

Therefore, another distinct labeling is generated.  $\square$

### 'Inverting' a Path

**Corollary.** *Consider a  $k$ -equitable labeling a path on  $2k$  vertices by the method detailed in 'Labeling a Path.' It follows that the  $k^{\text{th}}$  and  $k + 1^{\text{st}}$  vertices share*

*the same label, so the path on  $2k$  vertices is made up of two paths on  $k$  vertices, with their identically labeled end vertices adjacent. We then say the method of inversion is that we continue a labeling as if we were starting anew with a different labeling, such that we yield a  $k$ -equitable result.*