

“The game SET hides some deep and lovely mathematics. Come explore the surprising connections between combinatorics and SET.”

# SET, Affine Planes and Latin Squares

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This article is about one of the most interesting and fascinating card games we know; the game of SET. Well, sort of. Really, this article is about some of the beautiful mathematics one can find in the game. Of course you can play the game without knowing any of the stuff that follows, but why not have some extra fun?

Before continuing with this article you should be somewhat familiar with the types of cards and the types of sets in this game, so if you do not know SET please turn to the overview on page 43.

OK, now you know what SET is, but what does it have to do with mathematics? Some obvious questions involve counting and probability:

- 1) How many sets are there among the 81 cards?
- 2) What is the probability of having a set among the 12 cards on the table?
- 3) How many sets do you expect to see among 12 cards at the beginning of the game?
- 4) How many cards can you have without a set?

The answers to these questions are provided on page 38. There is a lot of material on the internet about connections of the game to probability, discrete mathematics and set theory. The official web site <http://www.setgame.com> has links to articles that also have the answers to most of the questions above. Moreover, on the same web site you can read about how SET was invented by the biologist Marsha Jean Falco while she was studying epilepsy in German shepherds.

A little known fact is that in addition to the above counting questions, SET hides some deep and lovely mathematics as well. In this article we consider some surprising connections between combinatorics and SET.

## A Magic Square in SET

Let's begin on a small scale by looking at the group of nine filled purple cards. We can set up the cards in a 3×3 square. The first row contains all the cards with one object, the second row contains all the cards with two objects, and the third row contains all the cards with three objects. The first column

contains all the diamonds, the second column contains all the ovals, and the third column contains all the squiggles. This arrangement has the interesting property that the three cards in any row, any column, and any diagonal form a set. We just created a magic square of sets! Here is a picture where each card is represented by a number:

1	2	3
4	5	6
7	8	9

Here are the 8 sets of the magic square:

Rows:	(123)	(456)	(789)
Columns:	(147)	(258)	(369)
Diagonals:	(159)	(357)	

However, there are more sets in this picture. In addition to the main diagonals, we can also look at forward and backward “broken” diagonals as below.

Forward Diagonals:	(159)	(267)	(348)
Backward Diagonals:	(357)	(249)	(168)

Each of these diagonals also forms a set giving us twelve sets among the nine filled purple cards.

One strategy in the game is to look for sets systematically. The observation that given any two cards, there is exactly one card that will complete a set comes in handy. Let's see why: You can examine each of the four attributes of two given cards one at a time. If the two cards agree in an attribute then the third card must also agree in that attribute. If the two cards differ in an attribute then the third card must also differ in that attribute. This means that every pair of cards can appear in at most one set. Now if you look at the twelve sets previously described, you can see that every pair of cards is in a set, which means that the previous arrangement gives us all the possible twelve sets among nine cards.

## Affine Planes

If you consider each card in the above arrangement as a point on a plane then this arrangement is also known as an affine plane.

An affine plane consists of a set  $P$  of points and a family  $L$  of subsets of  $P$  called lines such that:

- (1) Each pair of points lies together in exactly one line.
- (2) Each line has at least two points.
- (3) (Parallel lines) Given a line  $l$  and a point  $p$  not on  $l$  there is one and only one line  $l'$  in  $L$  such that  $p$  is contained in  $l'$  and the intersection of  $l$  and  $l'$  is empty.
- (4) There are (at least) three distinct points, not all on the same line.

These axioms give rise to some basic facts that we state here without proof.

### Theorem 1

- (i) For a finite affine plane there is an integer  $n$  such that all lines have  $n$  points. (Here,  $n$  is known as the *order* of the affine plane.)
- (ii) An affine plane of order  $n$  has  $n^2$  points.
- (iii) In an affine plane of order  $n$ , each point lies on  $n + 1$  distinct lines.
- (iv) An affine plane of order  $n$  has  $n^2 + n$  lines.
- (v) Two lines are called parallel if they have no common points. In an affine plane of order  $n$ , the relation *parallel or equal to* is an equivalence relation and there are  $n + 1$  equivalence classes each containing  $n$  parallel lines.

*Do the cards in SET form an affine plane?*

To answer this question we will try to understand the structure of the collection of cards and build the affine plane. If we succeed then we will know that the answer is “yes.” Let’s see what the five facts above would tell us about SET if it were an affine plane. The game of SET has 81 cards (or points). Using part (ii) of Theorem 1, we can determine that if SET was an affine plane it must have order 9. In turn, order 9 means that each line must have 9 points. Part (iii) states that each point must be on 10 distinct lines, and Part (iv) states that there must be 90 lines in this game. Finally, those 90 lines can be separated into 10 equivalence classes with 9 lines each, according to part (v).

This is where things start to get really interesting. It turns out that each line contains 9 points and is isomorphic to the “magic square” on the 9 filled purple cards. Thus, we know that each line contains 12 sets. If we could find the 90 lines, finding all  $90 \times 12 = 1080$  sets of the game would be relatively easy.

Unfortunately, finding the corresponding structure on 81 elements is easier said than done. Let’s start by laying out all 81 cards on a table using the following set up.

Let the first three rows be all of the purple cards, first filled, then empty, then shaded. Rows 4–6 are the red cards in the same order, and rows 7–9 are the green cards.

Let the first three columns be all of the diamonds, first single, then double, and then triple. Columns 4–6 contain the ovals in a similar order, and finally, columns 7–9 contain the squiggles.

Each column and each row is a line giving us 18 lines. These lines also give us two equivalence classes of parallel lines. The lines in each equivalence class have the properties that they do not intersect and they cover all the points of the plane.

We still have 72 more lines, or 8 parallel classes to discover. We need to find 8 different card arrangements, 9 lines in each arrangement keeping in mind that any two cards can appear together in a line only once. For example, no two cards that appear in the same row or column in our original set up could be together in any row or column of any of the other arrangements. But this is just the idea behind orthogonal Latin squares!

### Latin Squares

Latin squares are square tables of objects where each object appears exactly once in every row and every column. An example of a  $9 \times 9$  Latin square is given below.

1	2	3	4	5	6	7	8	9
9	1	2	3	4	5	6	7	8
8	9	1	2	3	4	5	6	7
7	8	9	1	2	3	4	5	6
6	7	8	9	1	2	3	4	5
5	6	7	8	9	1	2	3	4
4	5	6	7	8	9	1	2	3
3	4	5	6	7	8	9	1	2
2	3	4	5	6	7	8	9	1

**Figure 1.** A  $9 \times 9$  Latin square

Let’s superimpose the above Latin square on the 81 cards on the table. Consider a number in the Latin square. The number appears 9 times and the corresponding 9 cards on the table will form a line in our affine plane. Using each of the 9 numbers that appear in the Latin square we can find 9 lines. Since we are looking for 72 lines we would need to use 8 Latin squares that never produce the same 2 points on a line. Fortunately, so-called orthogonal Latin squares will do the trick!

Given two Latin squares,  $A$  (whose entries are  $a_{ij}$ ) and  $B$  (whose entries are  $b_{ij}$ ), we say that  $A$  and  $B$  are *orthogonal* if the ordered pair  $(a_{ij}, b_{ij})$  is different for every  $i$  and  $j$ . The Latin

square in Figure 2 is orthogonal to the one given in Figure 1.

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
3	4	5	6	7	8	9	1	2
4	5	6	7	8	9	1	2	3
5	6	7	8	9	1	2	3	4
6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6
8	9	1	2	3	4	5	6	7
9	1	2	3	4	5	6	7	8

**Figure 2.** Another 9×9 Latin square, orthogonal to the one in Figure 1.

The table below shows the ordered pairs that we get from the above two Latin squares; you can easily check that each pair is unique.

1,1	2,2	3,3	4,4	5,5	6,6	7,7	8,8	9,9
9,2	1,3	2,4	3,5	4,6	5,7	6,8	7,9	8,1
8,3	9,4	1,5	2,6	3,7	4,8	5,9	6,1	7,2
7,4	8,5	9,6	1,7	2,8	3,9	4,1	5,2	6,3
6,5	7,6	8,7	9,8	1,9	2,1	3,2	4,3	5,4
5,6	6,7	7,8	8,9	9,1	1,2	2,3	3,4	4,5
4,7	5,8	6,9	7,1	8,2	9,3	1,4	2,5	3,6
3,8	4,9	5,1	6,2	7,3	8,4	9,5	1,6	2,7
2,9	3,1	4,2	5,3	6,4	7,5	8,6	9,7	1,8

**Figure 3.** Ordered pairs created by superimposing elements from Figures 1 and 2.

It turns out there are eight mutually orthogonal 9×9 Latin squares. We can repeat the process of finding 9 lines for each of the 8 mutually orthogonal Latin squares in order to find 72 lines. Because the Latin squares we use are mutually orthogonal there cannot be more than one common point on any two of these lines, and thus the cards in SET do form an affine plane.

We hope you enjoyed this excursion into combinatorics, but now it's time to play a round of SET, or at least to solve the daily puzzle on <http://www.setgame.com>. Have fun!

**Answers to boxed questions from page 36.**

- 1) 1080
- 2) 94%
- 3) 220/79
- 4) 20