Mathematics 3

Differential Calculus

A derivative function defines the slope described by the original function.

Example 1 (FEIM):

Given: $y(x) = 3x^3 - 2x^2 + 7$. What is the slope of the function $y(x)$ at $x = 4$?

$$y'(x) = 9x^2 - 4x$$

$$y'(x) = (9)(4)^2 - (4)(4)$$

$$= 128$$

Example 2 (FEIM):

Given: $y_1' = \left(\frac{1}{2}\right)(1 + 4x - 7 + 2k)$. What is the value of k such that y_1 is perpendicular to the curve $y_2 = 2x$ at (1, 2)?

Perpendicular implies that $m_1 m_2 = -1$

Since $y_2'(1) = 2$, then

$$y_1'(1) = -\frac{1}{2} = \left(\frac{1}{2}\right)(1 + (4)(1) - 7 + 2k)$$

$$k = 1/2$$

Differential Calculus

Maxima

$$f'(a) = 0$$

$$f''(a) < 0 \qquad 7.3$$

Minima

$$f'(a) = 0$$

$$f''(a) > 0 \qquad 7.4$$

Example (FEIM): (maxima)

What is the maximum of the function $y = -x^3 + 3x$ for $x \geq -1$?

$$y' = -3x^2 + 3$$

$$y'' = -6x$$

When $y' = 0 = -3x^2 + 3$

$$x^2 = 1; x = \pm 1$$

$y''(1) = -6 < 0;$ therefore, this is a maximum.

$y''(-1) = 6 > 0;$ therefore, this is a minimum.

$$y(1) = -(1)^3 + 3 = 2$$

Differential Calculus

Inflection Point

$$f''(a) = 0 \qquad 7.5$$

$f''(a)$ changes sign about $x = a$

Example (FEIM):

What is the point of inflection of the function $y = -x^3 + 3x - 2$?

$y' = -3x^2 + 3$

$y'' = -6x$

$y'' = 0$ when $x = 0$ **and** $y'' > 0$ for $x < 0$; $y'' < 0$ for $x > 0$

Therefore this is an inflection point.

$y(0) = -(0)^3 + (3)(0) - 2 = -2$

Differential Calculus

Partial Derivative

- A derivative taken with respect to only one independent variable at a time.

Example (FEIM):
What is the partial derivative of $P(R, S, T)$ taken with respect to T?

$$P = 2R^3S^2T^{1/2} + R^{3/4}S\cos 2T$$

$$P = 2R^3S^2(T^{1/2}) + R^{3/4}S(\cos 2T)$$

$$\frac{\partial P}{\partial T} = 2R^3S^2(\tfrac{1}{2}T^{-1/2}) + R^{3/4}S(-2\sin 2T)$$

$$= R^3S^2T^{-1/2} - 2R^{3/4}S\sin 2T$$

Differential Calculus

Curvature

$$K = \frac{y''}{[1 + (y')^2]^{\frac{3}{2}}} \qquad 7.7$$

$$K = \frac{-x''}{[1 + (x')^2]^{\frac{3}{2}}} \qquad \left[x' = \frac{dx}{dy}\right] \qquad 7.8$$

Radius of Curvature $R = \frac{1}{|K|} = \frac{[1 + (y')^2]^{\frac{3}{2}}}{|y''|}$ 7.9

Example (FEIM):

What is the curvature of $y = -x^3 + 3x$ for $x = -1$?

(A) –2

(B) –1

(C) 0

(D) 6

$$y' = -3x^2 + 3 \qquad y'' = -6x$$

$$y'(-1) = 0 \qquad y''(-1) = 6$$

$$K = \frac{y''}{(1 + (y')^2)^{3/2}} = \frac{6}{(1 + (0)^2)^{3/2}} = 6$$

Therefore, (D) is correct.

Differential Calculus

Limits

Look at what the function does as it approaches the limit.
If the limit goes to plus or minus infinity:

- look for constants that become irrelevant
- look for functions that blow up fast: a factorial, an exponential

If the limit goes to a finite number:

- look at what happens at both plus and minus a small number

For $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right)$, $\left(\frac{f(a)}{g(a)}\right) = \frac{0}{0}$ or $= \frac{\infty}{\infty}$:

- Use L'Hôpital's rule $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a}\left(\frac{f^k(x)}{g^k(x)}\right)$ 7.11

NOTE: Use L'Hôpital's rule only when the next derivative of $f(x)$ and $g(x)$ exist.

Differential Calculus

Example 1 (FEIM):
What is the value of $\lim_{x\to\infty}\left(\frac{x+4}{x-4}\right)$?

(A) 0
(B) 1
(C) ∞
(D) undefined

Divide the numerator and denominator by *x*.

$$\lim_{x\to\infty}\left(\frac{x+4}{x-4}\right)=\lim_{x\to\infty}\left(\frac{1+\frac{4}{x}}{1-\frac{4}{x}}\right)=\frac{1+0}{1-0}=1$$

Therefore, (B) is correct.

Differential Calculus

Example 2 (FEIM):

What is the value of $\lim_{x \to 2}\left(\frac{x^2 - 4}{x - 2}\right)$?

(A) 0

(B) 2

(C) 4

(D) ∞

Factor out an (x – 2) term in the numerator.

$$\lim_{x \to 2}\left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2}\left(\frac{(x - 2)(x + 2)}{x - 2}\right) = \lim_{x \to 2}(x + 2) = 2 + 2 = 4$$

Therefore, (C) is correct.

Differential Calculus

Example 3 (FEIM):
What is the value of $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$?

(A) 0
(B) 1/4
(C) 1/2
(D) ∞

Both the numerator and denominator approach 0, so use L'Hôpital's rule.

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin x}{2x}\right)$$

Both the numerator and denominator are still approaching 0, so use L'Hôpital's rule again.

$$\lim_{x\to 0}\left(\frac{\sin x}{2x}\right)=\lim_{x\to 0}\left(\frac{\cos x}{2}\right)=\frac{\cos(0)}{2}=1/2$$

Therefore, (C) is correct.

Integral Calculus

Constant of Integration

- added to the integral to recognize a possible term

Example (FEIM):

What is the constant of integration for $y(x) = \int (e^{2x} + 2x)dx$ if $y = 1$ when $x = 1$?

(A) $2 - e^2$

(B) $-\frac{1}{2}e^2$

(C) $4 - e^2$

(D) $1 + 2e^2$

$$y(x) = \tfrac{1}{2}e^{2x} + x^2 + C$$

$$y(1) = \tfrac{1}{2}e^2 + 1 + C = 1$$

$$C = -\tfrac{1}{2}e^2$$

Therefore, (B) is correct.

Integral Calculus

Indefinite Integrals

1. Look for ways to simplify the formula with algebra before integrating.
2. Plug in initial value(s).
3. Solve for constant(s).
4. Indefinite integrals can be solved by differentiating the answers, but this is usually the hard way.

Integral Calculus

Method of Integration – Integration by Parts

$$\int f(x)dg(x) = f(x)g(x) - \int g(x)df(x) + C \quad 7.14$$

Example (FEIM):

Find $\int x^2 e^x dx$.

Let $g(x) = e^x$ and $f(x) = x^2$

so $dg(x) = e^x dx \int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$

From the NCEES Handbook: $\int xe^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1)$.

Therefore, $\int x^2 e^x dx = x^2 e^x - 2(xe^x - e^x) + C$

Notice that chosing $dg(x) = x^2\ dx$ and $f(x) = e^x$ does not improve the integral.

Integral Calculus

Method of Integration – Integration by Substitution

- Trigonometric Substitutions:
  - $\sqrt{a^2 - x^2}$: substitute $x = a \sin\theta$ *7.15*
  - $\sqrt{a^2 + x^2}$: substitute $x = a \tan\theta$ *7.16*
  - $\sqrt{x^2 - a^2}$: substitute $x = a \sec\theta$ *7.17*

Example (FEIM):

Find $\int (e^x + 2x)^2 (e^x + 2)dx$.

Let $u(x) = e^x + 2x$

so, $du = (e^x + 2)dx$

$$\int (e^x + 2x)^2 (e^x + 2)dx = \int u^2 du = \frac{u^3}{3} + C = \frac{1}{3}(e^x + 2x)^3 + C$$

Integral Calculus

Method of Integration – Partial Fractions

- Transforms a proper polynomial fraction of two polynomials into a sum of simpler expressions

Example 1 (FEIM):

Find $\int \frac{6x^2+9x-3}{x(x+3)(x-1)}\,dx$, using the partial fraction expression.

$$\frac{6x^2+9x-3}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1} = \frac{A(x+3)(x-1)}{x(x+3)(x-1)} + \frac{B(x)(x-1)}{x(x+3)(x-1)} + \frac{C(x)(x+3)}{x(x+3)(x-1)}$$

So, $6x^2 + 9x - 3 = A(x + 3)(x - 1) + B(x)(x - 1) + C(x)(x + 3)$

Solve using the three simultaneous equations:

$A + B + C = 6$

$2A - B + 3C = 9$

$-3A = -3$

$A = 1$, $B = 2$, and $C = 3$

$$\int \frac{6x^2+9x-3}{x(x+3)(x-1)}\,dx = \int \frac{1}{x}\,dx + \int \frac{2}{x+3} + \int \frac{3}{x-1}\,dx = \ln|x| + 2\ln|x+3| + 3\ln|x-1| + C$$

If the denominator has repeated roots, then the partial fraction expansion will have all the powers of that root.

Example 2 (FEIM):
Find the partial fraction expansion of $\frac{4x-9}{(x-3)^2}$.

$$\frac{4x-9}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A(x-3)}{x-3(x-3)} + \frac{B}{(x-3)^2}$$

$$4x - 9 = Ax - 3A + B$$

Solve using the two simultaneous equations.

$A = 4$

$-9 = -3A + B$

$A = 4$ and $B = 3$

Therefore, $\frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2}$

Integral Calculus

Definite Integrals

1. Solve the indefinite integral (without the constant of integration).
2. Evaluate at upper and lower bounds.
3. Subtract lower bound value from upper bound value.

Example (FEIM):
Find the integral between π/3 and π/4 of *f*(*x*) = cos *x*.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos x dx = -\cos\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\cos\frac{\pi}{3} - \left(-\cos\frac{\pi}{4}\right)$$

$$= -0.5 + 0.707 = 0.207$$

Integral Calculus

Average Value

$$\text{Average} = \frac{1}{b-a}\int_a^b f(x)dx$$

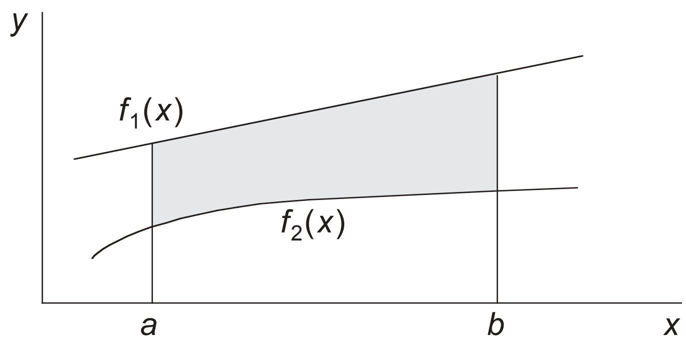
Example (FEIM):

What is the average value of $y(x) = 2x + 4$ between $x = 0$ and $x = 4$?

$$\text{Average} = \frac{1}{4-0}\int_0^4 (2x+4)dx = \left(\frac{1}{4}\right)\left(\frac{2x^2}{2}\right)\Bigg|_0^4 = \frac{1}{4}\left(4^2+(4)(4)\right) = 8$$

Integral Calculus

Area Problems



$$\text{area} = \int_a^b (f_1(x) - f_2(x))dx$$

Example (FEIM):
What is the area between y_1 = (1/4)x + 3 and y_2 = 6x – 1 between x = 0 and x = 1/2?

$$\text{Area} = \int_0^{\frac{1}{2}}\left(\left(\frac{1}{4}x+3\right)-(6x-1)\right)dx = \int_0^{\frac{1}{2}}\left(-\frac{23}{4}x+4\right)dx$$

$$= \left(-\frac{23}{8}x^2+4x\right)\Bigg|_0^{\frac{1}{2}} = \left(-\frac{23}{8}\right)\left(\frac{1}{2}\right)^2+\frac{4}{2} = \frac{41}{32}$$

Integral Calculus

Centroid

$$x_c = \int \frac{x\,dA}{A} \qquad 7.23$$

$$y_c = \int \frac{y\,dA}{A} \qquad 7.24$$

First Moment of Area

$$M_y = \int x\,dA = x_c A \qquad 7.27$$

$$M_x = \int y\,dA = y_c A \qquad 7.28$$

Moment of Inertia

$$I_x = \int y^2 dA \qquad 7.29$$

$$I_y = \int x^2 dA \qquad 7.30$$

Differential Equations

First-Order Homogeneous Equations

General form: $$y' + ay = 0 \quad 8.6$$

General solution: $$y(x) = Ce^{-ax} \quad 8.7$$

Initial condition: usually $y(b)$ = constant or $y'(b)$ = constant

$$C = \frac{y(b)}{e^{-ab}} \quad \text{or} \quad C = \frac{y'(b)}{e^{-ab}}$$

Differential Equations

Example (FEIM):
Find the solution to the differential equation $y = 4y'$ if $y(0) = 1$.

(A) $4e^{-4t}$

(B) $1/4e^{-1/4t}$

(C) $e^{-1/4t}$

(D) $e^{1/4t}$

Rearrange in the standard form.

$$4y' - y = 0$$

$$y' - \tfrac{1}{4}y = 0$$

General solution, $y = Ce^{-at}$

$$C = \frac{y(b)}{e^{-ab}} = \frac{y(0)}{e^{(1/4)(0)}} = 1$$

Since $a = -1/4$ and $C = 1$, then $y = e^{1/4t}$.

Therefore, (D) is correct.

Differential Equations

Separable Equations – integrating both sides
$m(x)dx = n(y)dy$

Example (FEIM):

Reduce $y' + 3(2y - \sin x) - (x\sin x + 6y) = 0$ to a separable equation.

$$y' + (3)(2)y - 6y - 3\sin x - x\sin x = 0$$

$$\frac{dy}{dx} = 3\sin x + x\sin x$$

$$dy = (3\sin x + x\sin x)dx$$

Then both sides can be integrated.
$y = -3\cos x + (\sin x - x\cos x) + C$

Differential Equations

Second-Order Homogeneous Equations

General form: $y'' + 2ay' + by = 0$ *8.8*

Characteristic equation: $r^2 + 2ar + b = 0$ *8.9*

Roots: $r_{1,2} = -a \pm \sqrt{a^2 - b}$ *8.10*

General solutions

Real roots ($a^2 > b$): $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ [overdamped] *8.11*

Real and equal roots ($a^2 = b$): $y = (C_1 + C_2 x)e^{rx}$ [critically damped] *8.12*

Complex roots ($a^2 < b$): $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ [underdamped] *8.13*

$\alpha = -a$ *8.14*

$\beta = \sqrt{b - a^2}$ *8.15*

Initial conditions

Usually y(constant) = constant and y'(constant) = constant.

Results in two simultaneous equations and two unknowns.

Differential Equations

Example (FEIM): $y'' + 6y' + 5y = 0$

$$y(0) = 1$$

$$y'(0) = 0$$

Write the equation in the standard form.

$$y'' + (2)(3)y' + 5y = 0$$

The characteristic equation is $r^2 + (2)(3)r + 5 = 0$

The roots are $-3 \pm \sqrt{3^2 - 5} = -3 \pm 2 = -1, -5$

This is the overdamped case because there are two real roots, so the general solution is $y + C_1 e^{-1x} + C_2 e^{-5x}$

$$y(0) = 1 = C_1 + C_2$$

$$y'(0) = 0 = -C_1 - 5C_2$$

$$1 = -4C_2$$

$$C_2 = -\tfrac{1}{4}$$

$$C_1 = 1\tfrac{1}{4}$$

$$y = 1\tfrac{1}{4} e^{-x} - \tfrac{1}{4} e^{-5x}$$

Differential Equations

Nonhomogeneous Equations

General solution: $y(x) = y_h(x) + y_p(x)$ 8.16

To solve the particular solution:

- know the form of the solution
- differentiate and then plug into the original equation
- collect like terms

The coefficients of the like terms must sum to zero, giving simultaneous equations.
Solve the equations and determine the constant(s).

Differential Equations

Example (FEIM):

Find the particular solution for the differential equation $y'' - y' - 2y = 10\cos x$.

From the table in the NCEES Handbook, the particular solution has the form:

$$y_p = B_1 \cos x + B_2 \sin x$$

$$y_p' = -B_1 \sin x + B_2 \cos x$$

$$y_p'' = -B_1 \cos x - B_2 \sin x$$

Substituting gives

$$-B_1 \cos x - B_2 \sin x - (-B_1 \sin x + B_2 \cos x) - 2(B_1 \cos x + B_2 \sin x) = 10 \cos x$$

$$(-3\,B_1 - B_2) \cos x + (B_1 - 3\,B_2) \sin x = 10 \cos x$$

Isolating the sin and cos coefficients, we get the following simultaneous equations.

$$-3B_1 - B_2 = 10$$

$$B_1 - 3B_2 = 0$$

$$B_1 = -3$$

$$B_2 = -1$$

$$y_p = -3\cos x - \sin x$$

Differential Equations

Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)] \quad 8.17$$

$$\omega = \frac{2\pi}{\tau} \quad 8.18$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} f(t)\cos(n\omega t)\,dt \quad 8.19$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} f(t)\sin(n\omega t)\,dt \quad 8.20$$

Example (FEIM):
Find the Fourier coefficients for a square wave function $f(t)$ with a period of 2π.

$$f(t) = -2 \text{ when} -\pi < x < 0$$

$$f(t) = 2 \text{ when } 0 < x < \pi$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} F(t)\cos(nt)dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} -2\cos nx dx + \int_0^{\pi} 2\cos nx dx\right) = 0$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} F(t)\sin(nt)dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} -2\sin nx dx + \int_0^{\pi} 2\sin nx dx\right)$$

$$= \frac{(2)(2)}{n\pi}(1-\cos n\pi)$$

Differential Equations

Laplace Transforms

To solve differential equations with Laplace transforms:

1. Put the equation in standard form: $y'' + b_1 y' + b_2 y = f(t)$ *8.28*
2. Take the Laplace transform of both sides: $\mathcal{L}(y'') + b_1\mathcal{L}(y') + b_2\mathcal{L}(y) = \mathcal{L}(f(t))$ *8.29*
3. Expand terms, using these relationships: $\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$ *8.30*

   $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$ *8.31*
4. Use algebra to solve for $L(y)$.
5. Plug in the initial conditions: $y(0) = c$; $y'(0) = k$.
6. Take the inverse transform: $y(t) = \mathcal{L}^{-1}(\mathcal{L}(y))$ *8.32*

Differential Equations

Example (FEIM):
Solve by Laplace transform:
$y'' + 4y' + 3y = 0, y(0) = 3, y'(0) = 1$

The equation is already in standard form.

Take the Laplace transform of both sides.
$s^2 y - sy(0) - y'(0) + 4(sy - y(0)) + 3(y) = 0$

Plug in initial conditions and rearrange.
$s^2 y + 4sy + 3y = 3s + 1 + (3)(4)$

$(s+3)(s+1)y = 3s + 13$

Solve for *y* and separate by partial fractions.

$$y = \frac{3s+13}{(s+3)(s+1)}$$

Partial fraction expansion

$$\frac{3s+13}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

$$= \frac{A(s+1) + B(s+3)}{(s+3)(s+1)}$$

$$= \frac{(A+B)s + (A+3B)}{(s+3)(s+1)}$$

$A + B = 3$

$A + 3B = 13$

$A = -2;\ B = 5$

$$y = \frac{-2}{s+3} + \frac{5}{s+1}$$

Take the inverse Laplace transform of *y*. $y(t) = -2e^{-3t} + 5e^{-t}$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+\alpha)}\right) = e^{-\alpha t}$$

Difference Equations

First-order: balance on a loan

$P_k = P_{k-1}(1 + i) - A$

Second-order: Fibonacci number sequence

$y(k) = y(k-1) + y(k-2)$

where $y(-1) = 1$ and $y(-2) = 1$

or

$f(k+2) = f(k+1) + f(k)$

where $f(0) = 1$ and $f(1) = 1$

Difference Equations

Example (FEIM):

What is a solution to the linear difference equation $y(k + 1) = 15y(k)$?

(A) $y(k) = 15/(1 + 15^k)$

(B) $y(k) = 15^{k/16}$

(C) $y(k) = C + 15^k$, C is a constant

(D) $y(k) = 15^k$

Try (D) by plugging in a $(k + 1)$ for every k.

$y(k + 1) = 15^{k+1}$

$y(k + 1) = 15(15^k)$

$y(k + 1) = 15y(^k)$

so $y(k) = 15^k$

Therefore, (D) is correct.

Difference Equations

z-Transforms

To solve difference equations using the z-transform:

1. Convert to standard form: $y(k + 1) = ay(k)$.
2. Take the z-transform of both sides of the equation.
3. Expand terms.
4. Plug in terms: $y(0)$, $y(1)$, $y(-1)$, etc.
5. Manipulate into a form that has an inverse transform.
6. Take the inverse transform.

Difference Equations

Example (FEIM):
Solve the linear difference equation $y(k + 1) = 15y(k)$ by z-transform, given that $y(0) = 1$.

Convert to standard form.

$$y(k+1) - 15y(k) = 0$$

Take the z-transform, expand the terms, and plug in the terms.

$$zY(z) - zy(0) - 15Y(z) = 0$$

$$Y(z)(z - 15) = z$$

$$Y(z) = \frac{z}{z-15} = \frac{1}{1-15z^{-1}}$$

Take the inverse transform.

$y(k) = 15^k$