NOISE-INDUCED STABILIZATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. The phenomenon of noise-induced stabilization occurs when an unstable deterministic system of ordinary differential equations is stabilized by the addition of randomness into the system. Noise-induced stabilization is quite an intriguing and surprising phenomenon as one's first intuition is often that noise will only serve to further destabilize the system. In this paper, we investigate under what conditions one-dimensional, autonomous stochastic differential equations are stable, where we take the notion of stability to be that of global stochastic boundedness. Specifically, we find the minimum amount of noise necessary for noise-induced stabilization to occur when the drift and noise coefficients are power, exponential, or logarithmic functions.

1. Introduction

This work is motivated by the intriguing phenomenon of noise-induced stabilization. Noise-induced stabilization occurs when the addition of randomness to an unstable deterministic system of ordinary differential equations (ODEs) results in a stable system of stochastic differential equations (SDEs). We are particularly motivated to find the minimum amount of noise required for the phenomenon of noise-induced stabilization to occur for one-dimensional, autonomous SDEs of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)\ .$$

Here $b(x)$ is the drift coefficient, which pushes the solution deterministically in some direction, $\sigma(x)$ is the noise coefficient, which controls the strength of the noise, and $B(t)$ is Brownian motion, a stochastic process where for each fixed t, $B(t)$ is a normally distributed random variable with mean 0 and variance t. We assume that $b(x)$ and $\sigma(x)$ are

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continuous functions and that there exists $\ell_0 \geq 0$ such that $\sigma(x) \neq 0$ for all $|x| \geq \ell_0$.

Our sense of stability comes from that of global stochastic boundedness, which is defined more formally below [3].

Definition 1. *$X(t)$ is stable if for all initial conditions and all $\epsilon > 0$, there exists some bound R such that*

$$P(|X(t)| \leq R) > 1 - \epsilon$$

for all $t \geq 0$.

In the deterministic setting where $X(t)$ is a solution to an ODE, the definition of stability reduces to that of boundedness. Hence, the unstable deterministic systems that we consider either blow up in finite time or wander off to infinity. We say that noise-induced stabilization occurs when the addition of noise to an unstable ODE results in a stable SDE.

Work by Scheutzow [5] has shown sufficient conditions for the occurrence of noise-induced stabilization. In this paper, we find *necessary* and sufficient conditions for noise-induced stabilization to occur when the drift and noise coefficients are restricted to certain forms. Section 2 discusses useful background information on the techniques that we use to prove noise-induced stabilization. Sections 3, 4, 5, and 6 present and prove our results concerning noise-induced stabilization for when the drift and noise coefficients are general power functions, polynomials, exponential functions, and logarithmic functions, respectively. Section 7 discusses the problem of the minimum amount of noise required to stabilize an unstable system when the noise coefficient is not restricted to a particular form.

2. Background

In this section, we discuss preliminary information on the methods used to find our results. In particular, our work uses a well-known result from [2] to determine the stability of SDEs, and specifically whether noise-induced stabilization occurs. For ease, we will refer to this result as the "Stochastic Stability Theorem."

Stochastic Stability Theorem. *Consider the general form for autonomous first-order SDEs:*

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where $b(x)$ and $\sigma(x)$ are continuous functions and there exists $\ell_0 \geq 0$ such that $\sigma(x) \neq 0$ for all $|x| \geq \ell_0$.

Define the following quantities:

- $s(x) = \begin{cases} \exp\left[\int_\ell^x \frac{-2b(z)}{\sigma^2(z)} dz\right] & \textit{when } x > \ell \\ \exp\left[\int_{-\ell}^x \frac{-2b(z)}{\sigma^2(z)} dz\right] & \textit{when } x < -\ell \end{cases}$
- $S(x) = \begin{cases} \int_\ell^x s(y) dy & \textit{for } x > \ell \\ \int_{-\ell}^x s(y) dy & \textit{for } x < -\ell \end{cases}$
- $m(x) = \dfrac{1}{s(x)\sigma^2(x)}$
- $M(x) = \begin{cases} \int_\ell^x m(y) dy & \textit{for } x > \ell \\ \int_{-\ell}^x m(y) dy & \textit{for } x < -\ell \end{cases}$

The SDE is stable if any only if there exists $\ell \geq \ell_0$ such that

$$S(\infty) = \infty, \quad S(-\infty) = -\infty, \quad |M(\infty)| < \infty, \quad |M(-\infty)| < \infty.$$

Note that since the lower limit of integration does not affect the convergence or divergence of these integrals, ℓ can be any real number greater than or equal to ℓ_0.

Due to the terms defined in the "Stochastic Stability Theorem," there are several reoccurring, non-trivial integrals in our proofs. Hence, we prove below the convergence or divergence of these types of integrals to use as a reference.

Lemma 1. *Consider the integral*

$$\int_1^\infty e^{cx^n} dx$$

where n is any real number.

1. *If $c \geq 0$, then the integral is infinite.*
2. *If $c < 0$ and $n \leq 0$, then the integral is infinite.*
3. *If $c < 0$ and $n > 0$, then the integral is finite*

Proof. (1) If $c \geq 0$, then $cx^n \geq 0$ for all $x \geq 1$, which implies $e^{cx^n} \geq 1$ for all $x \geq 1$. Thus,

$$\int_1^\infty e^{cx^n} dx \geq \int_1^\infty dx = \infty.$$

(2) If $c < 0$ and $n \leq 0$, then $x^n \leq 1$ for all $x \geq 1$, which implies $cx^n \geq c$ for all $x \geq 1$, which in turn implies that $e^{cx^n} \geq e^c$ for all $x \geq 1$. Thus

$$\int_1^\infty e^{cx^n} dx \geq \int_1^\infty e^c dx = \infty.$$

(3) By the Taylor series expansion for e^y, $e^y \geq \frac{y^k}{k!}$ for any $y > 0$ and any nonnegative integer k. Hence, $\frac{1}{e^y} \leq \frac{k!}{y^k}$ for any $y > 0$ and any nonnegative integer k. If $c < 0$ and $n > 0$, then $e^{cx^n} = \frac{1}{e^{|c|x^n}}$ where $|c|x^n > 0$ for all $x > 0$. Hence, $e^{cx^n} \leq \frac{k!}{(|c|x^n)^k} = \frac{k!}{|c|^k x^{nk}}$ for any nonnegative integer k for all $x > 0$. Since it holds for any nonnegative integer k and $n > 0$, we can choose k_0 such that $nk_0 > 1$. Thus,

$$\begin{aligned}\int_1^\infty e^{cx^n} dx &\leq \int_1^\infty \frac{k_0!}{|c|^{k_0} x^{nk_0}} dx \\ &= \left.\frac{-k_0!}{|c|^{k_0}(nk_0 - 1)x^{nk_0 - 1}}\right|_1^\infty \\ &= \frac{k_0!}{|c|^{k_0}(nk_0 - 1)} < \infty.\end{aligned}$$

□

Consider the integral

$$\int_{-\infty}^{-1} e^{cx^n} dx.$$

This integral is well-defined only when n is an integer, so we will now restrict to this case.

Corollary 1. *Consider the integral*

$$\int_{-\infty}^{-1} e^{cx^n} dx,$$

where n is an even integer.

(1) *If $c \geq 0$, then the integral is infinite for any n.*
(2) *If $c < 0$ and $n \leq 0$, then the integral is infinite.*
(3) *If $c < 0$ and $n > 0$, then the integral is finite*

Proof. If n is an even integer, then $e^{cx^n} = e^{c(-x)^n}$ and

$$\int_{-\infty}^{-1} e^{cx^n} dx = \int_1^\infty e^{cx^n} dx.$$

Thus, we get the same classification of the integral being finite versus infinite as in the previous lemma. $\square$

Corollary 2. *Consider the integral*

$$\int_{-\infty}^{-1} e^{cx^n}\,dx,$$

where n is an even integer.

(1) *If $c \leq 0$, then the integral is infinite for any n.*
(2) *If $c > 0$ and $n \leq 0$, then the integral is infinite.*
(3) *If $c > 0$ and $n > 0$, then the integral is finite*

Proof. If n is an odd integer, then $e^{cx^n} = e^{-c(-x)^n}$ and

$$\int_{-\infty}^{-1} e^{cx^n}\,dx = \int_{1}^{\infty} e^{-cx^n}\,dx.$$

Thus, we get almost the same classification of the integral being finite versus infinite as in the previous lemma, with just the sign of c reversed. $\square$

3. Power Function Stabilization

In this section, we consider ODE's where the drift coefficient is a general power function i.e.

$$dX(t) = r|X(t)|^q dt.$$

These ODE's are unstable for any $r \neq 0$.

Theorem 1. *Consider the SDE*

$$dX(t) = b(X(t))dt + \sigma(X(t))B(t)$$

where $b(X(t)) = a|X(t)|^q$ if $q \geq 0$ and $\sigma(X(t)) = r|X(t)|^p$ if $p \geq 0$. If $q < 0$ then the drift coefficient takes the following form:

$$b(X(t)) = \begin{cases} r|X(t)|^q & \text{for } |X(t)| \geq 1 \\ r & \text{for } |X(t)| < 1. \end{cases}$$

Likewise, if $p < 0$, then the noise coefficient takes the following form:

$$\sigma(X(t)) = \begin{cases} a|X(t)|^p & \text{for } |X(t)| \geq 1 \\ a & \text{for } |X(t)| < 1. \end{cases}$$

Also, r, a, q, and p are any real numbers. Noise-induced stabilization occurs if and only if $r \neq 0$ and one of the following sets of conditions is met:

- $p > \max\left\{\frac{1}{2}, \frac{q+1}{2}\right\}$ *for any $a \neq 0$ or*

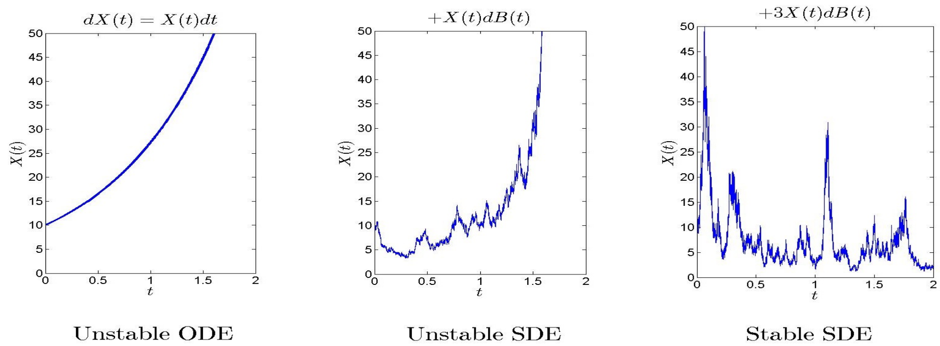

FIGURE 1. This image depicts an unstable power ODE that is stabilized with the sufficient amount of noise.

- $p = \dfrac{q+1}{2}$ *and* $\begin{cases} a^2 q > 2|r| & \textit{for } 0 < q \leq 1 \\ a^2 \geq 2|r| & \textit{for } q > 1. \end{cases}$

Figure 1 shows three separate graphs depicting the phenomenon of noise-induced stabilization. The graph on the far left shows an ODE that diverges off to infinity for all positive initial values; therefore, the ODE is unstable. The middle graph depicts the noise value of $X(t)dB(t)$ added to the previously unstable ODE. This specific amount of noise added does not stabilize the system as $a^2 < 2r$. The final image depicts $3X(t)dB(t)$ added to the unstable ODE. It can be seen that the SDE converges. This occurs since $a^2 > 2r$.

Proof. The "Stochastic Stability Theorem" will be used to show under what conditions noise-induced stabilization occurs. With $\ell = 1$ evaluation of the $s(x)$ term gives

$$s(x) = \begin{cases} \exp\left[\int_1^x \frac{-2r|z|^q}{a^2|z|^{2p}} dz\right] & \text{when } x > 1 \\ \exp\left[\int_{-1}^x \frac{-2r|z|^q}{a^2|z|^{2p}} dz\right] & \text{when } x < -1 \end{cases}$$
$$= \begin{cases} \exp\left[\frac{-2r}{a^2} \int_1^x z^{q-2p} dz\right] & \text{when } x > 1 \\ \exp\left[\frac{-2r}{a^2} \int_{-1}^x (-z)^{q-2p} dz\right] & \text{when } x < -1 \end{cases}$$

where the integration depends on the value of $q - 2p$. The non-critical case where $p \neq \frac{q+1}{2}$ will be considered first.

Simplification of Proof: $Y(t) = -X(t)$ must have the same exact stability as $X(t)$ since they have the same magnitude.

$$\begin{aligned}
dY(t) &= -dX(t) \\
&= -r|X(t)|^q dt - a|X(t)|^p dB(t) \\
&= -r|-Y(t)|^q dt - a|-Y(t)|^p dB(t) \\
&= -r|Y(t)|^q dt - a|Y(t)|^p dB(t)
\end{aligned}$$

Hence, the stability of $X(t)$ with $-r$, q, $-a$, and p must be equivalent to the stability with r, q, p, and a. Thus, when proving Theorem 3, it suffices to just prove the case with $r > 0$.

Non-Critical Case: Assume $q - 2p \neq -1$ and let $c = \frac{-2r}{a^2(q-2p+1)}$ and $n = q - 2p + 1$. Then

$$\begin{aligned}
s(x) &= \begin{cases} \exp\left[c(x^n - 1)\right] & \text{when } x > 1 \\ \exp\left[-c((-x)^n - 1)\right] & \text{when } x < -1 \end{cases} \\
&= \begin{cases} \exp\left[-c\right] \exp\left[cx^n\right] & \text{when } x > 1 \\ \exp\left[c\right] \exp\left[-c(-x)^n\right] & \text{when } x < -1 \end{cases}
\end{aligned}$$

and therefore

$$S(x) = \begin{cases} \exp\left[-c\right] \int_1^x \exp\left[cy^n\right] dy & \text{when } x > 1 \\ \exp\left[c\right] \int_{-1}^x \exp\left[-c(-y)^n\right] dy & \text{when } x < -1. \end{cases}$$

Evaluation of the $m(x)$ term gives

$$m(x) = \begin{cases} \frac{1}{a^2} \exp\left[c\right] x^{-2p} \exp\left[-cx^n\right] & \text{when } x > 1 \\ \frac{1}{a^2} \exp\left[-c\right] (-x)^{-2p} \exp\left[c(-x)^n\right] & \text{when } x < -1 \end{cases}$$

and then

$$M(x) = \begin{cases} \frac{1}{a^2} \exp\left[c\right] \int_1^x y^{-2p} \exp\left[-cy^n\right] dy & \text{when } x > 1 \\ \frac{1}{a^2} \exp\left[-c\right] \int_{-1}^x (-y)^{-2p} \exp\left[c(-y)^n\right] dy & \text{when } x < -1 \end{cases}$$

Letting $x = \infty$ gives

$$S(\infty) = \exp\left[-c\right] \int_1^\infty \exp\left[cy^n\right] dy \tag{1}$$

and

$$M(\infty) = \frac{1}{a^2} \exp\left[c\right] \int_1^\infty y^{-2p} \exp\left[-cy^n\right] dy \tag{2}$$

and letting $x = -\infty$ gives

$$\begin{aligned} S(-\infty) &= \exp[c] \int_{-1}^{-\infty} \exp[-c(-y)^n]\, dy \\ &= -\exp[c] \int_{-\infty}^{-1} \exp[-c(-y)^n]\, dy \\ &= -\exp[c] \int_{1}^{\infty} \exp[-cy^n]\, dy \end{aligned} \tag{3}$$

and

$$\begin{aligned} M(-\infty) &= \frac{1}{a^2} \exp[-c] \int_{-1}^{-\infty} (-y)^{-2p} \exp[c(-y)^n]\, dy \\ &= -\frac{1}{a^2} \exp[-c] \int_{-\infty}^{-1} (-y)^{-2p} \exp[c(-y)^n]\, dy \\ &= -\frac{1}{a^2} \exp[-c] \int_{1}^{\infty} y^{-2p} \exp[cy^n]\, dy \end{aligned} \tag{4}$$

Non-Critical Case 1 Assume $r > 0$, $p > \frac{q+1}{2}$, and $p > \frac{1}{2}$ which means $n = q - 2p + 1 < 0$ and $c = \frac{-2r}{a^2(q-2p+1)} > 0$. It follows that $c \geq cy^n > 0$ and therefore $\exp[c] \geq exp\,[cy^n] > 1$ on the interval from 1 to ∞. The lower bound of $\exp[cy^n]$ allows for the following comparison of (1):

$$S(\infty) > \exp[-c] \int_1^\infty dy = \infty.$$

Therefore, $S(\infty) = \infty$. The upper bound of $\exp[cy^n]$ allows for the following comparison of (4):

$$\begin{aligned} M(-\infty) &\geq -\frac{1}{a^2} \exp[-c] \int_1^\infty y^{-2p} \exp[c] dy \\ &= -\frac{1}{a^2} \int_1^\infty y^{-2p} dy \\ &> -\infty \end{aligned}$$

After cancellation of the exponential terms, the integral takes the form $\int_1^\infty y^{-2p} dy$. This integral converges if $-2p < -1$ which is the case since it was assumed that $p > \frac{1}{2}$. Therefore, $|M(-\infty)| < \infty$.

The constants n and $-c$ are both negative so $0 > -cy^n \geq -c$ on the interval. Then $1 > \exp[-cy^n] \geq \exp[-c]$. The lower bound of

$\exp[-cy^n]$ allows for the comparison

$$S(-\infty) \leq -\exp[c]\int_1^\infty \exp[-c]dy = -\int_1^\infty dy = -\infty$$

of (2) and the conclusion that $S(-\infty) = -\infty$. Therefore, $S(-\infty) = -\infty$. The upper bound of $\exp[-cy^n]$ enables the comparison

$$M(\infty) < \frac{1}{a^2}\exp[c]\int_1^\infty \frac{1}{y^{2p}}dy < \infty$$

of (3). Once again, the integral $\int_1^\infty y^{-2p}dy$ converges since $p > \frac{1}{2}$. Therefore, $|M(\infty)| < 0$.

Through application of the "Stochastic Stability Theorem", it has been shown that noise-induced stabilization occurs when $r > 0$, $p > \frac{q+1}{2}$, and $p > \frac{1}{2}$. Recalling the proof simplification, it can also be concluded that noise-induced stabilization occurs when $-r > 0$, $p > \frac{q+1}{2}$, and $p > \frac{1}{2}$.

Non-Critical Case 2: Assume $r > 0$ and $p < \frac{q+1}{2}$. Then $n = q-2p+1 > 0$, $c = \frac{-2r}{a^2(q-2p+1)} < 0$, and $-c > 0$. Therefore, the integral contained in (1) is of the form $\int_1^\infty \exp[cy^n]dy$ where c is negative and n is positive. Lemma 1.3 and 1.4 indicates that integrals of this form converge. Therefore, $S(\infty) < \infty$. Application of the "Stochastic Stability Theorem" shows that noise-induced stabilization does not occur when $r > 0$ and $p < \frac{q+1}{2}$. The proof simplification indicates that noise-induced stabilization also does not occur when $-r > 0$ and $p < \frac{q+1}{2}$.

Non-Critical Case 3: Assume $r > 0$ and $\frac{1}{2} > p > \frac{q+1}{2}$. Then n is a negative constant and c is a positive constant. Therefore, $\exp[cy^n] > 1$ on the interval from 1 to ∞. This allows for the following comparison of (4):

$$M(\infty) < -\frac{1}{a^2}\exp\left[-c\right]\int_1^\infty y^{-2p}dy.$$

Since $p < \frac{1}{2}$, the integral $\int_1^\infty y^{-2p}dy$ must diverge. Therefore, $|M(-\infty)| = \infty$ so noise-induced stabilization does not occur when $r > 0$ and $\frac{1}{2} > p > \frac{q+1}{2}$. The simplification of the proof indicates that stabilization also does not occur when $-r > 0$ and $\frac{1}{2} > p > \frac{q+1}{2}$.

Critical Case: Assume $q - 2p = -1$. Then the $s(x)$ term takes the form

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x z^{-1}dz\right] & \text{when } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x (-z)^{-1}dz\right] & \text{when } x < -1 \end{cases}$$
$$= \begin{cases} \exp\left[\frac{-2r}{a^2}\ln(x)\right] & \text{when } x > 1 \\ \exp\left[\frac{2r}{a^2}\ln(-x)\right] & \text{when } x < -1 \end{cases}$$
$$= \begin{cases} x^{\frac{-2r}{a^2}} & \text{for } x > 1 \\ (-x)^{\frac{2r}{a^2}} & \text{for } x < -1 \end{cases}$$

and

$$S(x) = \begin{cases} \int_1^x s(y)dy & \text{for } x > 1 \\ \int_{-1}^x s(y)dy & \text{for } x < -1 \end{cases}$$
$$= \begin{cases} \int_1^x y^{\frac{-2r}{a^2}}dy & \text{for } x > 1 \\ \int_{-1}^x (-y)^{\frac{2r}{a^2}}dy & \text{for } x < -1. \end{cases}$$

Evaluation of the $m(x)$ term gives

$$m(x) = \begin{cases} \frac{1}{a^2}x^{\frac{2r}{a^2}-2p} & \text{for } x > 1 \\ \frac{1}{a^2}(-x)^{\frac{-2r}{a^2}-2p} & \text{for } x < -1 \end{cases}$$

and

$$M(x) = \begin{cases} \frac{1}{a^2}\int_1^x y^{\frac{2r}{a^2}-2p}dy & \text{for } x > 1 \\ \frac{1}{a^2}\int_{-1}^x (-y)^{\frac{-2r}{a^2}-2p}dy & \text{for } x < -1. \end{cases}$$

Evaluation of the terms at $x = \infty$ and $x = -\infty$ gives

$$S(\infty) = \int_1^\infty y^{\frac{-2r}{a^2}}dy, \tag{5}$$

$$\begin{aligned} S(-\infty) &= \int_{-1}^{-\infty} (-y)^{\frac{2r}{a^2}}dy \\ &= -\int_{-\infty}^{-1} (-y)^{\frac{2r}{a^2}}dy \\ &= -\int_1^\infty y^{\frac{2r}{a^2}}dy, \end{aligned} \tag{6}$$

$$M(\infty) = \frac{1}{a^2}\int_1^\infty y^{\frac{2r}{a^2}-2p}dy, \tag{7}$$

and

$$\begin{aligned} M(-\infty) &= \frac{1}{a^2}\int_{-1}^{-\infty}(-y)^{\frac{-2r}{a^2}-2p}dy \\ &= -\frac{1}{a^2}\int_{-\infty}^{-1}(-y)^{\frac{-2r}{a^2}-2p}dy \\ &= -\frac{1}{a^2}\int_{1}^{\infty}y^{\frac{-2r}{a^2}-2p}dy. \end{aligned} \tag{8}$$

Critical Case 1: Assume $r > 0$, $p = \frac{q+1}{2} > \frac{1}{2}$, $a^2 \geq 2r$ and $a^2q > 2r$. It follows from the assumptions that $0 < \frac{2r}{a^2} \leq 1$ for $q > 1$. Therefore, $-1 \leq \frac{-2r}{a^2} < \frac{2r}{a^2}$which means the integrals in both (5)

$$S(\infty) = \int_{1}^{\infty} y^{\frac{-2r}{a^2}}dy$$

and (6)

$$S(-\infty) = -\int_{1}^{\infty} y^{\frac{2r}{a^2}}dy$$

must diverge. Likewise, if $q \leq 1$, it must be the case that $-1 \leq -q < -\frac{2r}{a^2} < \frac{2r}{a^2} < q \leq 1$ so that the integrals must still diverge. Therefore, $S(\infty) = \infty$ and $S(-\infty) = -\infty$. Note, that in the case where r is negative, $\frac{-2r}{a^2} > \frac{2r}{a^2}$. This is why the magnitude of r must be considered in the stabilization conditions.

The assumption that $2r < a^2q$ and $p = \frac{q+1}{2}$ is equivalent to the assumption that $\frac{2r}{a^2} - 2p < -1$ for $q \leq 1$. Since r is positive it can further be implied that $-\frac{2r}{a^2} - 2p < -1$. The fact that both of these terms are strictly less than negative 1 means the integrals in both (7)

$$M(\infty) = \frac{1}{a^2}\int_{1}^{\infty} y^{\frac{2r}{a^2}-2p}dy$$

and (8)

$$M(-\infty) = -\frac{1}{a^2}\int_{1}^{\infty} y^{\frac{-2r}{a^2}-2p}dy$$

must converge. For $q > 1$, it must be the case that $q > 1 \geq \frac{2r}{a^2}$ which implies $\frac{-2r}{a^2} - 2p < \frac{2r}{a^2} - 2p < -1$ so the integrals once again converge. Therefore, $|M(\infty)|$ and $|M(-\infty)|$ are both finite.

Through application of the "Stochastic Stability Theorem", it has been shown that noise-induced stabilization occurs when $r > 0$, $p = \frac{q+1}{2} > \frac{1}{2}$, $a^2 \geq 2r$ and $a^2 q > 2r$. Recollection of the proof simplification motivates the conclusion that stabilization also occurs when $-r > 0$, $p = \frac{q+1}{2} > \frac{1}{2}$, $a^2 \geq -2r$ and $a^2 q > -2r$.

Critical Case 2: Assume $r > 0$, $p = \frac{q+1}{2}$, and $a^2 < 2r$. Then it must be the case that $\frac{-2r}{a^2} < -1$ which implies the integral contained in (5) must converge. Therefore, $S(\infty) < \infty$ so noise-induced stabilization does not occur. Likewise, if $r < 0$, $p = \frac{q+1}{2}$ and $a^2 < 2|r|$, then $\frac{-2r}{a^2} > 1$ and $\frac{2r}{a^2} < -1$ so the integral contained in (6) must converge. Therefore, $S(-\infty) > -\infty$ and noise-induced stabilization does not occur when $r > 0$, $p = \frac{q+1}{2}$, and $a^2 < 2r$. The proof simplification allows for the additional conclusion that stabilization does not occur when $-r > 0$, $p = \frac{q+1}{2}$, and $a^2 < -2r$.

Critical Case 3: Assume $r > 0$, $p = \frac{q+1}{2}$ and $a^2 q \leq 2r$. Then it must be the case that $\frac{2r}{a^2} - 2p \geq -1$ which implies the integral contained in (7) diverges. Therefore, $|M(\infty)| = \infty$ and noise-induced stabilization does not occur. Now assume $r < 0$, $p = \frac{q+1}{2}$ and $a^2 q \leq 2|r|$. Then it follows that $\frac{-2r}{a^2} - 2p \geq -1$ which implies the integral contained in (8) diverges. Therefore, $|M(-\infty)| = -\infty$ and noise-induced stabilization does not occur when $r > 0$, $p = \frac{q+1}{2}$ and $a^2 q \leq 2r$. The proof simplification indicates that stabilization also does not occur when $-r > 0$, $p = \frac{q+1}{2}$ and $a^2 q \leq -2r$.

Critical Case 4: Assume $p = \frac{q+1}{2} \leq \frac{1}{2}$ and assume $a^2 q > 2|r|$. These assumptions imply $\frac{2|r|}{a^2} < q \leq 0$. However, this is not possible. Therefore, there are no cases where both $p = \frac{q+1}{2} \leq \frac{1}{2}$ and $a^2 q > 2|r|$. $\square$

4. Polynomial Stabilization

In this section, we investigate stabilizing

$$dX(t) = b(X(t))dt,$$

where $b(x)$ is any polynomial of degree q. If r is the coefficient of the highest degree term, this ODE is unstable when q is even for any $r \neq 0$ and when q is odd for any $r > 0$.

The lemma below, as well as the following corollaries, describe general results about polynomials which will be used in the proof of noise-induced stabilization for when the drift and noise coefficients are polynomials.

Lemma 2. *Let $f(x)$ be any polynomial of degree q, where the coefficient of the highest degree term is $r > 0$. Then, there exists $k \geq 0$ such that*

$$r(x-k)^q \leq f(x) \leq r(x+k)^q$$

for all $x \geq k$.

Proof. If $q = 0$, then $f(x) = r$ and the claim is true for $k = 0$. Hence, assume $q \geq 1$. Since $f(x)$ is a polynomial of degree q, it has the following form:

$$\begin{aligned} f(x) &= rx^q + r_1 x^{q-1} + r_2 x^{q-2} + ... + r_{q-1}x + r_q \\ &= r\left(x^q + \frac{r_1}{r}x^{q-1} + \frac{r_2}{r}x^{q-2} + ... + \frac{r_{q-1}}{r}x + \frac{r_q}{r}\right) \end{aligned}$$

where $r_1, r_2, ..., r_{q-1}, r_q$ are any real constants. Let

$$r_{\max} = \max\{|\frac{r_1}{r}|, |\frac{r_2}{r}|, ..., |\frac{r_{q-1}}{r}|, |\frac{r_q}{r}|\}$$

and $k = \max\{q \cdot r_{\max}, 1\}$. Then for $x \geq k$,

$$rx^q - kx^{q-1} \leq f(x) \leq rx^q + kx^{q-1}$$

and factoring gives

$$rx^{q-1}(x-k) \leq f(x) \leq rx^{q-1}(x+k).$$

Hence for $x \geq k$,

$$r(x-k)^q \leq f(x) \leq r(x+k)^q.$$

□

Corollary 3. *Let $f(x)$ be any polynomial of degree q, where the coefficient of the highest degree term is $r < 0$. Then, there exists $k \geq 0$ such that*

$$r(x+k)^q \leq f(x) \leq r(x-k)^q$$

for all $x \geq k$.

Corollary 4. *Let $f(x)$ be any polynomial of degree q, where the coefficient of the highest degree term is $r > 0$. Then, there exists $k \geq 0$ such that if q is even,*

$$r(x+k)^q \leq f(x) \leq r(x-k)^q$$

and if q is odd,

$$r(x-k)^q \leq f(x) \leq r(x+k)^q$$

for all $x \leq -k$.

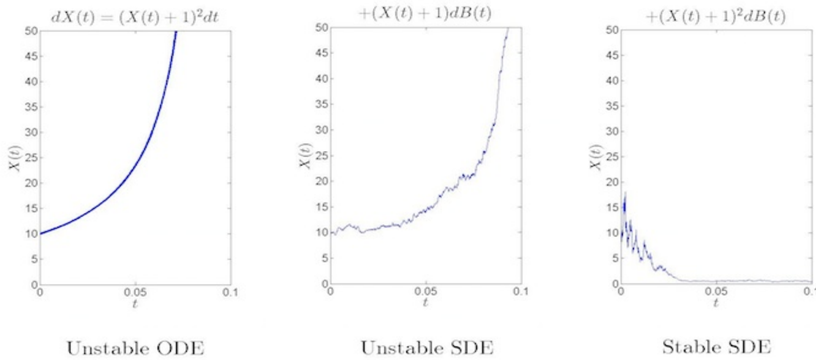

FIGURE 2. This image depicts an unstable polynomial ODE that is stabilized with the sufficient amount of noise.

Corollary 5. *Let $f(x)$ be any polynomial of degree q, where the coefficient of the highest degree term is $r < 0$. Then, there exists $k \geq 0$ such that if q is even,*

$$r(x-k)^q \leq f(x) \leq r(x+k)^q$$

and if q is odd,

$$r(x+k)^q \leq f(x) \leq r(x-k)^q$$

for all $x \leq -k$.

Theorem 2. *Let $b(x)$ be any polynomial of degree q, where the coefficient of the highest degree term is r, and let $\sigma(x)$ be any polynomial of degree p, where the coefficient of the highest degree term is $a \neq 0$. Then noise-induced stabilization of the SDE $dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$ occurs if and only if the corresponding ODE is unstable and one of the following conditions is met:*

- $p > \frac{q+1}{2}$ *or*
- $p = \frac{q+1}{2}$ *and* $\begin{cases} a^2 > 2r & \text{for } q = 1 \\ a^2 \geq 2r & \text{for } q \geq 3 \end{cases}$.

Figure 2 shows three separate graphs depicting the phenomenon of noise-induced stabilization where the drift and noise coefficients are polynomials. The graph on the far left shows an ODE that diverges off to infinity for all positive initial values; therefore, the ODE is unstable. The middle graph depicts the noise value of $(X(t) + 1)dB(t)$ added to the previously unstable ODE. This specific amount of noise added does not stabilize the system as $p < \frac{q+1}{2}$. The final image depicts

$(X(t)+1)^2 dB(t)$ added to the unstable ODE. It can be seen that the SDE converges. This occurs since $p > \frac{q+1}{2}$.

Proof. Suppose $r > 0$. By Lemma 2 and Corollary 4, there exist some k_b and k_σ such that

$$s(x) \geq \begin{cases} \exp\left[\int_\ell^x \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] & \text{when } x > \ell \geq \max(k_b, k_\sigma) \\ \exp\left[\int_{-\ell}^x \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz\right] & \text{when } x < -\ell \leq -\max(k_b, k_\sigma) \end{cases} \tag{9}$$

and

$$s(x) \leq \begin{cases} \exp\left[\int_\ell^x \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz\right] & \text{when } x > \ell \geq \max(k_b, k_\sigma) \\ \exp\left[\int_{-\ell}^x \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] & \text{when } x < -\ell \leq -\max(k_b, k_\sigma) \end{cases} \tag{10}$$

for even q, and

$$s(x) \geq \begin{cases} \exp\left[\int_\ell^x \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] & \text{when } x > \ell \geq \max(k_b, k_\sigma) \\ \exp\left[\int_{-\ell}^x \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] & \text{when } x < -\ell \leq -\max(k_b, k_\sigma) \end{cases} \tag{11}$$

and

$$s(x) \leq \begin{cases} \exp\left[\int_\ell^x \frac{-2r(x-k_b)^q}{a^2(x+k_\sigma)^{2p}} dz\right] & \text{when } x > \ell \geq \max(k_b, k_\sigma) \\ \exp\left[\int_{-\ell}^x \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz\right] & \text{when } x < -\ell \leq -\max(k_b, k_\sigma) \end{cases} \tag{12}$$

for odd q. These bounds allow for simpler analysis of conditions set by the "Stochastic Stability Theorem." First, consider the case where q is even. Suppose $p > \frac{q+1}{2}$. By (9),

$$S(\infty) \geq \int_\ell^\infty \exp\left[\int_\ell^x \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] dx.$$

Let $y = z - k_\sigma$. Then, by Lemma 2, there exists a $\tilde{k}_1 > 0$ such that

$$S(\infty) \geq \int_\ell^\infty \exp\left[\int_\ell^x \frac{-2r(y^q + \tilde{k}_1 y^{q-1})}{a^2 y^{2p}} dy\right] dx.$$

By dividing through and integrating, it can be found that

$$S(\infty) \geq c_1 c_2 \int_\ell^\infty \exp\left[\frac{-2r}{a^2}\frac{x^{q-2p+1}}{q-2p+1}\right] \exp\left[\frac{-2r\tilde{k}_1}{a^2}\frac{x^{q-2p}}{q-2p}\right] dx$$

where $c_1 = \exp\left[\frac{2r}{a^2}\frac{\ell^{q-2p+1}}{q-2p+1}\right]$ and $c_2 = \exp\left[\frac{2r\tilde{k}_1}{a^2}\frac{\ell^{q-2p}}{q-2p}\right]$. Since $q-2p < -1$, and therefore $\frac{-2r\tilde{k}_1}{a^2(q-2p)} > 0$, the smallest $\exp\left[\frac{-2r\tilde{k}_1}{a^2}\frac{x^{q-2p}}{q-2p}\right]$ can be is 1.

Therefore,

$$S(\infty) \geq c_1 c_2 \int_{\ell}^{\infty} \exp\left[\frac{-2r}{a^2}\frac{x^{q-2p+1}}{q-2p+1}\right] dx.$$

Since $q-2p+1 < 0$, this integral diverges by Lemma 1 . By comparison, $S(\infty) = \infty$.
With these bounds on $s(x)$,

$$S(-\infty) \leq \int_{-\ell}^{-\infty} \exp\left[\int_{-\ell}^{x} \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] dx.$$

Let $y = z - k_\sigma$. Then, by Corollary 4, there exists a $\tilde{k}_2 > 0$ such that

$$S(-\infty) \leq \int_{-\ell}^{-\infty} \exp\left[\int_{-\ell}^{x} \frac{-2r(y^q + \tilde{k}_2 y^{q-1})}{a^2 y^{2p}} dy\right] dx.$$

By methods similar to those used previously, it can be found that

$$S(-\infty) \leq c_1 \int_{-\ell}^{-\infty} \exp\left[\frac{-2r}{a^2}\frac{x^{q-2p+1}}{q-2p+1}\right] dx$$

where $c_1 = \exp\left[\frac{2r}{a^2}\frac{(-\ell)^{q-2p+1}}{q-2p+1}\right]$ and $c_2 = \exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{(-\ell)^{q-2p}}{q-2p}\right]$. By Corollary 2, this integral diverges to $-\infty$. Thus, $S(-\infty) = -\infty$.
Consider $M(\infty)$. By Lemma 2,

$$M(\infty) \leq \int_{\ell}^{\infty} \frac{1}{a^2(x-k_\sigma)^{2p}} \exp\left[\int_{\ell}^{x} \frac{2r(z^q + \tilde{k}_1 z^{q-1})}{a^2 z^{2p}} dz\right] dx.$$

There largest the exponential term can be is 1, and therefore

$$M(\infty) \leq c_1 c_2 \int_{\ell}^{\infty} \frac{1}{a^2(x-k_\sigma)^{2p}} dx$$

where $c_1 = \exp\left[\frac{2r}{a^2}\frac{\ell^{q-2p+1}}{q-2p+1}\right]$ and $c_2 = \exp\left[\frac{2r\tilde{k}_1}{a^2}\frac{\ell^{q-2p}}{q-2p}\right]$. This integral converges for $p > \frac{1}{2}$, which is always true since $p > \frac{q+1}{2}$ and $q \geq 0$. Thus, $M(\infty) < \infty$. Proof that $|M(-\infty)| < \infty$ follows similarly. By Corollary 4,

$$|M(-\infty)| \leq \left|\int_{-\ell}^{-\infty} \frac{1}{a^2(x+k_\sigma)^{2p}} \exp\left[\int_{-\ell}^{x} \frac{2r(z^q + \tilde{k}_2 z^{q-1})}{a^2 z^{2p}} dz\right] dx\right|.$$

The largest the exponential term can be is $\frac{1}{c_1}$ where $c_1 = \exp\left[\frac{-2r}{a^2}\frac{(-\ell)^{q-2p+1}}{q-2p+1}\right]$.
Thus,

$$|M(-\infty)| \leq \left|c_2 \int_{-\ell}^{-\infty} \frac{1}{a^2(x+k_\sigma)^{2p}} dx\right|,$$

where and $c_2 = \exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{(-\ell)^{q-2p}}{q-2p}\right]$. This integral converges for $p > \frac{1}{2}$, which again is always true. Therefore $|M(-\infty)| < \infty$, and noise-induced stabilization does occur when $p > \frac{q+1}{2}$, $r > 0$, and q is even.
Now, assume q is odd. The proofs that $S(\infty) = \infty$ and $M(\infty) < \infty$ are identical to the case where q is even. Now consider $S(-\infty)$. By (12),

$$S(-\infty) \leq \int_{-\ell}^{-\infty} \exp\left[\int_{-\ell}^{x} \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}}dz\right] dx.$$

Let $y = z + k_\sigma$. Then, by Corollary 4, there exists a $\tilde{k}_2 > 0$ such that

$$S(-\infty) \leq \int_{-\ell}^{-\infty} \exp\left[\int_{-\ell}^{x} \frac{-2r(y^q - \tilde{k}_2 y^{q-1})}{a^2 y^{2p}}dy\right] dx.$$

By dividing through and integrating, it can be found that

$$S(-\infty) \leq c_1 c_2 \int_{-\ell}^{-\infty} \exp\left[\frac{-2r}{a^2}\frac{x^{q-2p+1}}{q-2p+1}\right] \exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{x^{q-2p}}{q-2p}\right] dx$$

where $c_1 = \exp\left[\frac{2r}{a^2}\frac{(-\ell)^{q-2p+1}}{q-2p+1}\right]$ and $c_2 = \exp\left[\frac{-2r\tilde{k}_2}{a^2}\frac{(-\ell)^{q-2p}}{q-2p}\right]$. Since $q-2p$ is odd, $\exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{x^{q-2p}}{q-2p}\right] \geq 1$. Therefore,

$$S(-\infty) \leq c_1 c_2 \int_{-\ell}^{-\infty} \exp\left[\frac{-2r}{a^2}\frac{x^{q-2p+1}}{q-2p+1}\right] dx.$$

By Corollary 1, this integral diverges to negative infinity, and therefore $S(-\infty) = -\infty$. Now consider $M(-\infty)$. By Corollary 4,

$$|M(-\infty)| \leq \left|\int_{-\ell}^{-\infty} \frac{1}{a^2(x+k_\sigma)^{2p}} \exp\left[\int_{-\ell}^{x} \frac{2r(z^q - \tilde{k}_2 z^{q-1})}{a^2 z^{2p}}dz\right] dx\right|.$$

The largest the exponential term can be is 1. Therefore,

$$|M(-\infty)| \leq \left|c_1 c_2 \int_{-\ell}^{-\infty} \frac{1}{a^2(x+k_\sigma)^{2p}}dx\right|,$$

where $c_1 = \exp\left[\frac{-2r}{a^2}\frac{(-\ell)^{q-2p+1}}{q-2p+1}\right]$ and $c_2 = \exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{(-\ell)^{q-2p}}{q-2p}\right]$. Again, this integral converges and therefore $|M(-\infty)| < \infty$. Now, it is sufficient to say that noise-induced stabilization occurs when $p > \frac{q+1}{2}$, $r > 0$, and q is odd.

Suppose $p = \frac{q}{2}$. By (10), $S(\infty)$ can be bounded such that

$$S(\infty) \leq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz\right] dx.$$

Let $y = z + k_\sigma$, and, by Lemma 2, there exists a $\tilde{k}_2 > 0$ such that

$$S(\infty) \leq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r(y^q - \tilde{k}_2 y^{q-1})}{a^2 y^{2p}} dy\right] dx.$$

By dividing through and integrating, it can be found that

$$\begin{aligned} S(\infty) &\leq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r}{a^2} dz\right] \exp\left[\int_{\ell}^{x} \frac{2r\tilde{k}_2}{a^2} \frac{1}{z} dz\right] dx \\ &= c_1 c_2 \int_{\ell}^{\infty} \exp\left[\frac{-2rx}{a^2}\right] |x|^{\frac{2r\tilde{k}_2}{a^2}} dx, \end{aligned}$$

where $c_1 = \exp\left[\frac{2r\ell}{a^2}\right]$ and $c_2 = |\ell|^{\frac{-2r\tilde{k}_2}{a^2}}$. This integral converges, and therefore $S(\infty) < \infty$.
Suppose $p < \frac{q}{2}$. By the same bounds on $s(x)$,

$$S(\infty) \leq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz\right] dx.$$

Let $y = z + k_\sigma$. By Lemma 2, there exists a $\tilde{k}_2 > 0$ such that

$$\begin{aligned} S(\infty) &\leq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r(y^q - \tilde{k}_2 y^{q-1})}{a^2 y^{2p}} dy\right] dx \\ &= c_1 c_2 \int_{\ell}^{\infty} \exp\left[\frac{-2r}{a^2} \frac{x^{q-2p+1}}{q-2p+1}\right] \exp\left[\frac{2r\tilde{k}_2}{a^2} \frac{x^{q-2p}}{q-2p}\right] dx, \end{aligned}$$

where $c_1 = \exp\left[\frac{2r}{a^2}\frac{\ell^{q-2p+1}}{q-2p+1}\right]$ and $c_2 = \exp\left[\frac{-2r\tilde{k}_2}{a^2}\frac{\ell^{q-2p}}{q-2p}\right]$. For sufficiently large x,

$$\begin{aligned} S(\infty) &\leq c_1 c_2 \int_{\ell}^{\infty} \exp\left[\frac{-2r}{a^2} \frac{x^{q-2p+1}}{q-2p+1}\right] \exp\left[\frac{r}{a^2} \frac{x^{q-2p+1}}{q-2p+1}\right] dx \\ &= c_1 c_2 \int_{\ell}^{\infty} \exp\left[\frac{-r}{a^2} \frac{x^{q-2p+1}}{q-2p+1}\right] dx. \end{aligned}$$

By Lemma 1 this integral converges, and thus $S(\infty) < \infty$. Therefore noise-induced stabilization does not occur when $p < \frac{q+1}{2}$ and $r > 0$.

Suppose $p = \frac{q+1}{2}$, and suppose $a^2 > 2r$ when $q = 1$ and $a^2 \geq 2r$ when $q \geq 3$. By (9),

$$S(\infty) \geq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r(z+k_b)^q}{a^2(z-k_\sigma)^{2p}} dz\right] dx.$$

Let $y = z - k_\sigma$. By Lemma 2, there exists a $\tilde{k}_1 > 0$ such that

$$\begin{aligned} S(\infty) &\geq \int_{\ell}^{\infty} \exp\left[\int_{\ell}^{x} \frac{-2r(y^q + \tilde{k}_1 z^{q-1})}{a^2 y^{2p}} dy\right] dx \\ &= c_1 c_2 \int_{\ell}^{\infty} |x|^{\frac{-2r}{a^2}} \exp\left[\frac{2r\tilde{k}_1}{a^2}\frac{1}{x}\right] dx, \end{aligned}$$

where $c_1 = |\ell|^{\frac{2r}{a^2}}$ and $c_2 = \exp[\frac{-2r\tilde{k}_1}{a^2}\frac{1}{\ell}]$. This integral diverges for $a^2 \geq 2r$, and therefore $S(\infty) = \infty$. Now, consider $S(-\infty)$. Note that since $p = \frac{q+1}{2}$, and since p is an integer, q must be odd. By (12),

$$S(-\infty) \leq \int_{-\ell}^{-\infty} \exp\left[\int_{-\ell}^{x} \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz\right] dx.$$

Let $y = z + k_\sigma$. By Corollary 4, there exists a $\tilde{k}_2 > 0$ such that

$$\begin{aligned} S(-\infty) &\leq \int_{-\ell}^{-\infty} \exp\left[\int_{-\ell}^{x} \frac{-2r(z^q - \tilde{k}_2 z^{q-1})}{a^2 z^{2p}} dz\right] dx \\ &\leq c_1 c_2 \int_{-\ell}^{-\infty} |x|^{\frac{-2r}{a^2}} \exp\left[\frac{-2r\tilde{k}_2}{a^2}\frac{1}{x}\right] dx, \end{aligned}$$

where $c_1 = |-\ell|^{\frac{2r}{a^2}}$ and $c_2 = \exp[\frac{-2r\tilde{k}_2}{a^2}\frac{1}{\ell}]$. This integral diverges to negative infinity for $a^2 \geq 2r$, and therefore $S(-\infty) = -\infty$. Consider $M(\infty)$. By Lemma 2,

$$M(\infty) \leq \int_{\ell}^{\infty} \frac{1}{a^2(x-k_\sigma)^{2p}} \exp\left[\int_{\ell}^{x} \frac{2r(z^q + \tilde{k}_1 z^{q-1})}{a^2 z^{2p}} dz\right] dx.$$

The exponential term can be partially bounded such that,

$$M(\infty) \leq c_1 \int_{\ell}^{\infty} \frac{1}{a^2(x-k_\sigma)^{2p}} |x|^{\frac{2r}{a^2}} dx,$$

where $c_1 = |\ell|^{\frac{-2r}{a^2}}$. For sufficiently large x, $\frac{1}{a^2(x-k_\sigma)^{2p}} \leq \frac{2^{2p}}{a^2x^{2p}}$. Therefore,

$$\begin{aligned} M(\infty) &\leq c_1 \int_\ell^\infty \frac{2^{2p}}{a^2x^{2p}} |x|^{\frac{2r}{a^2}} dx \\ &= c_1 \int_\ell^\infty \frac{2^{2p}}{a^2} x^{\frac{2r}{a^2}-2p} dx. \end{aligned}$$

This integral converges for $a^2q > 2r$, and therefore $M(\infty) < \infty$.
Consider $M(-\infty)$. By Corollary 4,

$$\begin{aligned} |M(-\infty)| &\leq \left| \int_{-\ell}^{-\infty} \frac{1}{a^2(x+k_\sigma)^{2p}} \exp\left[\int_{-\ell}^{x} \frac{2r(z^q - \tilde{k}_2 z^{q-1})}{a^2z^{2p}} dz \right] dx \right| \\ &= \left| c_1c_2 \int_{-\ell}^{-\infty} \frac{1}{a^2(x+k_\sigma)^{2p}} |x|^{\frac{2r}{a^2}} \exp\left[\frac{2r\tilde{k}_2}{a^2} \frac{1}{x} \right] dx \right|, \end{aligned}$$

where $c_1 = |-\ell|^{\frac{-2r}{a^2}}$ and $c_2 = \exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{1}{\ell}\right]$. The largest $\exp\left[\frac{2r\tilde{k}_2}{a^2}\frac{1}{x}\right]$ can be is 1. Also, for sufficiently large x,

$$\begin{aligned} |M(-\infty)| &\leq \left| c_1c_2 \int_{-\ell}^{-\infty} \frac{2^{2p}}{a^2x^{2p}} |x|^{\frac{2r}{a^2}} dx \right| \\ &= \left| c_1c_2 \int_{-\ell}^{-\infty} \frac{2^{2p}}{a^2} |x|^{\frac{2r}{a^2}-2p} dx \right| \end{aligned}$$

This integral converges, and therefore $|M(-\infty)| < \infty$.
Now, suppose $a^2 \leq 2r$ when $q = 1$ and $a^2 < 2r$ when $q \geq 3$. Consider $S(\infty)$. By the bounds on $s(x)$,

$$S(\infty) \leq \int_\ell^\infty \exp\left[\int_\ell^x \frac{-2r(z-k_b)^q}{a^2(z+k_\sigma)^{2p}} dz \right] dx.$$

Let $y = z + k_\sigma$. By Lemma 2, there exists a $\tilde{k}_2 > 0$ such that

$$\begin{aligned} S(\infty) &\leq \int_\ell^\infty \exp\left[\int_\ell^x \frac{-2r(y^q - \tilde{k}_2 z^{q-1})}{a^2y^{2p}} dy \right] dx \\ &= c_1c_2 \int_\ell^\infty |x|^{\frac{-2r}{a^2}} \exp\left[\frac{-2r\tilde{k}_2}{a^2} \frac{1}{x} \right] dx, \end{aligned}$$

where $c_1 = |\ell|^{\frac{2r}{a^2}}$ and $c_2 = \exp[\frac{2r\tilde{k}_1}{a^2}\frac{1}{\ell}]$. Since $\exp\left[\frac{-2r\tilde{k}_2}{a^2}\frac{1}{x}\right] \leq 1$,

$$S(\infty) \leq c_1c_2 \int_\ell^\infty |x|^{\frac{-2r}{a^2}} dx.$$

This integral converges when $a^2 \leq 2r$, and therefore $S(\infty) < \infty$. Thus, when $p = \frac{q+1}{2}$, and $r > 0$, noise-induced stabilization occurs if and only if $a^2 > 2r$ when $q = 1$ and $a^2 \geq 2r$ when $q \geq 3$.

Let $Y(t) = -X(t)$, and suppose q is even. Since $Y(t)$ is bounded if and only if $X(t)$ is bounded, $Y(t)$ must have the exact same stability as $X(t)$. It follows that

$$\begin{aligned} dY(t) &= -dX(t) \\ &= -\left[b(X(t)dt + \sigma(X(t))dB(t)\right] \\ &= -\left[b(-Y(t)dt + \sigma(-Y(t))dB(t)\right]. \end{aligned}$$

Since the highest degree term of the drift coefficient determines the stability, and since q is even, the stability of $X(t)$ with leading coefficient $-r$ is equivalent to its stability with leading coefficient r. Since the noise coefficient, $\sigma(x)$, is always squared in the terms defined by the "Stochastic Stability Theorem", the sign of the coefficient plays no role in the stability of the SDE. Note that this only holds for even q since the original ODE is stable for $r < 0$ and q odd.

$\square$

5. Exponential Function Stabilization

This section considers ODE's where the drift coefficient is an exponential function, i.e.

$$dX(t) = r(\exp[X(t)])^q dt.$$

where r and q are any real number. The ODE is unstable for any $r \neq 0$.

5.1. **General Exponential Stabilization.**

Theorem 3. *Consider the SDE*

$$dX(t) = r(\exp[X(t)])^q dt + a(\exp[X(t)])^p dB(t)$$

where $r, a, q,$ and p are real numbers. Noise-induced stabilization occurs if and only if $r \neq 0$, $a \neq 0$ and one of the following sets of conditions is met:

- $r > 0$ *and* $p > \max\{0, \frac{q}{2}\}$ *or*
- $r < 0$ *and* $-p > \max\{0, \frac{-q}{2}\}$.

Figure 3 shows three separate graphs depicting the phenomenon of noise-induced stabilization. The graph on the far left shows an ODE that diverges off to infinity for all positive initial values; therefore, the ODE is unstable. The middle graph depicts the noise value of $10\exp\left[\frac{1}{4}X(t)\right]dB(t)$ added to the previously unstable ODE. This specific amount of noise added does not stabilize the system as $p < \frac{1}{2}$. The

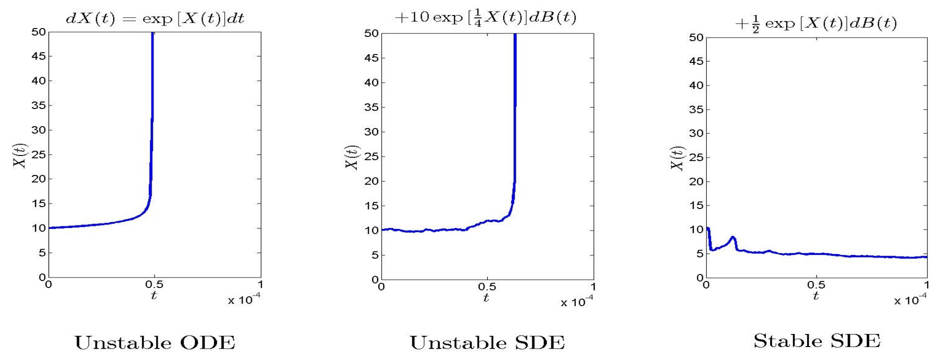

Figure 3. This image depicts an unstable exponential ODE that is stabilized with the sufficient amount of noise.

final image depicts $\frac{1}{2}\exp[X(t)]\,dB(t)$ added to the unstable ODE. It can be seen that the SDE converges. This occurs since $p > \frac{1}{2}$.

Proof. The "Stochastic Stability Theorem" will be used to show under what conditions noise-induced stabilization occurs.

Simplification of Proof: $Y(t) = -X(t)$ must have the same exact stability as $X(t)$ since they have the same magnitude.

$$\begin{aligned}
dY(t) &= -dX(t)\\
&= -r(\exp[X(t)])^q dt - a(\exp[X(t)])^p dB(t)\\
&= -r(\exp[-Y(t)])^q dt - a(\exp[-Y(t)])^p dB(t)\\
&= -r(\exp[Y(t)])^{-q} dt - a(\exp[Y(t)])^{-p} dB(t)
\end{aligned}$$

Hence, the stability of $X(t)$ with $-r$ must be equivalent to the stability with r, but with $-q, -p$, and $-a$ substituted for q, p, and a. Thus, when proving Theorem 3, it suffices to just prove the case with $r > 0$.

With $\ell = 1$, the $s(x)$ term takes the form

$$s(x) = \begin{cases} \exp\left[\int_1^x \frac{-2r\exp[qz]}{a^2\exp[2pz]}dz\right] & \text{for } x > 1 \\ \exp\left[\int_{-1}^x \frac{-2r\exp[qz]}{a^2\exp[2pz]}dz\right] & \text{for } x < -1 \end{cases}$$

$$= \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x \exp[(q-2p)z]dz\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x \exp[(q-2p)z]dz\right] & \text{for } x < -1 \end{cases}$$

$$= \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x \exp[kz]dz\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x \exp[kz]dz\right] & \text{for } x < -1 \end{cases}$$

where $k = q - 2p$. The integration depends on the value of k. The non-critical case where $k \neq 0$ will first be considered.

Non-Critical Case: Assume $k \neq 0$.

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2k}(\exp[kx] - \exp[k])\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2k}(\exp[kx] - \exp[-k])\right] & \text{for } x < -1 \end{cases}$$

$$= \begin{cases} \exp\left[\frac{2r\exp[k]}{a^2k}\right]\exp\left[\frac{-2r}{a^2k}\exp[kx]\right] & \text{for } x > 1 \\ \exp\left[\frac{2r\exp[-k]}{a^2k}\right]\exp\left[\frac{-2r}{a^2k}\exp[kx]\right] & \text{for } x < -1 \end{cases}$$

and

$$S(x) = \begin{cases} \exp\left[\frac{2r\exp[k]}{a^2k}\right]\int_1^x \exp\left[\frac{-2r}{a^2k}\exp[ky]\right]dy & \text{for } x > 1 \\ \exp\left[\frac{2r\exp[-k]}{a^2k}\right]\int_{-1}^x \exp\left[\frac{-2r}{a^2k}\exp[ky]\right]dy & \text{for } x < -1 \end{cases}$$

$$= \begin{cases} c_1\int_1^x \exp\left[\frac{-2r}{a^2k}\exp[ky]\right]dy & \text{for } x > 1 \\ c_2\int_{-1}^x \exp\left[\frac{-2r}{a^2k}\exp[ky]\right]dy & \text{for } x < -1 \end{cases}$$

where $c_1 = \exp\left[\frac{2r\exp[k]}{a^2k}\right]$ and $c_2 = \exp\left[\frac{2r\exp[-k]}{a^2k}\right]$. Moving on to the $m(x)$ term gives

$$m(x) = \begin{cases} \frac{1}{a^2c_1}\exp\left[\frac{2r}{a^2k}\exp[kx]\right]\exp[-2px] & \text{for } x > 1 \\ \frac{1}{a^2c_2}\exp\left[\frac{2r}{a^2k}\exp[kx]\right]\exp[-2px] & \text{for } x < -1 \end{cases}$$

and

$$M(x) = \begin{cases} \frac{1}{a^2c_1}\int_1^x \exp\left[\frac{2r}{a^2k}\exp[ky]\right]\exp[-2py]dy & \text{for } x > 1 \\ \frac{1}{a^2c_2}\int_{-1}^x \exp\left[\frac{2r}{a^2k}\exp[ky]\right]\exp[-2py]dy & \text{for } x < -1 \end{cases}$$

$$= \begin{cases} \frac{1}{a^2c_1}\int_1^x \exp\left[\frac{2r}{a^2k}\exp[ky] - 2py\right]dy & \text{for } x > 1 \\ \frac{1}{a^2c_2}\int_{-1}^x \exp\left[\frac{2r}{a^2k}\exp[ky] - 2py\right]dy & \text{for } x < -1. \end{cases}$$

Setting x equal to positive infinity gives

$$S(\infty) = c_1 \int_1^{\infty} \exp\left[\frac{-2r}{a^2k}\exp[ky]\right] dy \tag{13}$$

and

$$M(\infty) = \frac{1}{a^2c_1}\int_1^{\infty} \exp\left[\frac{2r}{a^2k}\exp[ky] - 2py\right] dy. \tag{14}$$

Setting x equal to negative infinity gives

$$\begin{aligned} S(-\infty) &= c_2 \int_{-1}^{-\infty} \exp\left[\frac{-2r}{a^2k}\exp[ky]\right] dy \\ &= -c_2 \int_{-\infty}^{-1} \exp\left[\frac{-2r}{a^2k}\exp[ky]\right] dy \\ &= -c_2 \int_1^{\infty} \exp\left[\frac{-2r}{a^2k}\exp[-ky]\right] dy \end{aligned} \tag{15}$$

and

$$\begin{aligned} M(-\infty) =& \frac{1}{a^2c_2}\int_{-1}^{-\infty} \exp\left[\frac{2r}{a^2k}\exp[ky] - 2py\right] dy \\ =& -\frac{1}{a^2c_2}\int_{-\infty}^{-1} \exp\left[\frac{2r}{a^2k}\exp[ky] - 2py\right] dy \\ =& -\frac{1}{a^2c_2}\int_1^{\infty} \exp\left[\frac{2r}{a^2k}\exp[-ky] + 2py\right] dy \\ =& -\frac{1}{a^2c_2}\int_1^{\infty} \exp\left[\frac{2r}{a^2k}\left(\exp[-ky] + \frac{a^2kp}{r}y\right)\right] dy. \end{aligned} \tag{16}$$

Non-Critical Case 1: Assume $r > 0$ and $p > \max\{0, \frac{q}{2}\}$. Since $p > \max\{0, \frac{q}{2}\}$, it must be the case that $k = q - 2p < 0$ and $p > 0$. Since k is negative and r is positive, it follows that the constant $\frac{-2r}{a^2k}$ is positive and therefore the exponential terms $\exp\left[\frac{-2r}{a^2k}\exp[ky]\right]$ and $\exp\left[\frac{-2r}{a^2k}\exp[-ky]\right]$ must be greater than 1 on the interval from 1 to ∞. This allows for the comparison

$$S(\infty) \geq c_1 \int_1^{\infty} dy = \infty$$

and

$$S(-\infty) \leq -c_2 \int_1^{\infty} dy = -\infty$$

and the conclusion that (13) and (15) diverge. Therefore $S(\infty) = \infty$ and $S(-\infty) = -\infty$. The exponential decay term $\exp[ky]$ bounds the exponential term $\exp\left[\frac{2r}{a^2k}\exp[ky]\right]$ between $\exp\left[\frac{2r}{a^2k}\right]$ (lower bound) and 1 (upper bound). This allows for the following comparison of (14) to a smaller integral:

$$\begin{aligned} M(\infty) &\leq \frac{1}{a^2c_1}\int_1^{\infty}\exp\left[-2py\right]dy \\ &< \infty \end{aligned}$$

The smaller integral has the form $\int_1^{\infty}\exp[-2py]dy$ with where the constant $-p$ is negative. Then Lemma 1 indicates that the integral converges and $|M(\infty)| < \infty$.

Since $-k$ is positive, the Taylor series expansion indicates $\exp[-ky] > -ky$ on the interval from 1 to ∞. Now, assuming $-k + \frac{a^2pk}{r} > 0$, the following comparison of (16) to a smaller magnitude integral can be made:

$$\begin{aligned} M(-\infty) &\geq -\frac{1}{a^2c_2}\int_1^{\infty}\exp\left[\frac{2r}{a^2k}\left(-ky + \frac{a^2pk}{r}y\right)\right] \\ &= -\frac{1}{a^2c_2}\int_1^{\infty}\exp\left[\frac{2r}{a^2k}\left(-k + \frac{a^2pk}{r}\right)y\right] \\ &> -\infty. \end{aligned}$$

Since it was assumed that $-k + \frac{a^2pk}{r} > 0$, the smaller magnitude integral takes the form $\int_1^{\infty}\exp[cy^n]dy$ where c is negative and $n = 1$. Then Lemma 1 indicates that integrals of this form converge so $|M(-\infty)| < \infty$ if $-k + \frac{a^2pk}{r} > 0$.

Now, assume $-k + \frac{a^2pk}{r} \leq 0$ which is equivalent to the assumption that $-k \leq \frac{-a^2pk}{r}$. This inequality allows for the comparison $\exp[-ky] \leq \exp\left[-\frac{a^2pk}{r}\right]$ and the following comparison of (16) to a smaller magnitude integral:

$$M(-\infty) \geq -\frac{1}{a^2c_2}\int_1^{\infty}\exp\left[\frac{2r}{a^2k}\left(\exp\left[-\frac{a^2pk}{r}y\right] + \frac{a^2pk}{r}y\right)\right].$$

Recalling that $-\frac{a^2pk}{r} > 0$ and considering the Taylor Series Expansion, the comparison $\exp\left[-\frac{a^2pk}{r}y\right] > \left(-\frac{a^2pk}{r}\right)y + \frac{1}{2}\left(-\frac{a^2pk}{r}\right)^2 y^2$ can be made. This allows for the following comparison of the smaller magnitude integral:

$$
\begin{aligned}
M(-\infty) \geq & -\frac{1}{a^2c_2} \\
& \int_1^\infty \exp\left[\frac{2r}{a^2k}\left(-\frac{a^2pk}{r}y + \frac{1}{2}\left(-\frac{a^2pk}{r}\right)^2 y^2 + \frac{a^2pk}{r}y\right)\right] \\
= & -\frac{1}{a^2c_2}\int_1^\infty \exp\left[\frac{r}{a^2k}\left(-\frac{a^2pk}{r}\right)^2 y^2\right] \\
> & -\infty.
\end{aligned}
$$

The final integral takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{r}{a^2k}\left(-\frac{a^2pk}{r}\right)^2$ is a negative constant and $n = 2$ is a positive constant. Then Lemma 1 indicates that integrals of this form converge so then $|M(-\infty)| < \infty$.

Application of the "Stochastic Stability Theorem" indicates that noise-induced stabilization occurs when $r > 0$ and $p > \max\{0, \frac{q}{2}\}$. The simplification of the proof indicates that noise-induced stabilization also occurs when $-r > 0$ and $-p > \max\{0, \frac{-q}{2}\}$.

Non-Critical Case 2: Assume $r > 0$ and $q > 2p$. Since $q > 2p$, it must be the case that $k = q - 2p > 0$ and $\frac{-2r}{a^2k} < 0$. The Taylor series expansion indicates that $\exp[ky] > ky$ on the interval which allows for the following comparison of (13) to a smaller magnitude integral:

$$
\begin{aligned}
S(\infty) &\leq c_1 \int_1^\infty \exp\left[\frac{-2r}{a^2k}ky\right] dy \\
&< \infty
\end{aligned}
$$

The smaller magnitude integral takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{-2r}{a^2}$ is a negative constant and $n = 1$. Then Lemma 1 indicates that integrals of this form converge so $S(\infty) < \infty$.

Application of the "Stochastic Stability Theorem" indicates that noise-induced stabilization does not occur when $r > 0$ and $p < \frac{q}{2}$. The simplification of proofs indicates that noise-induced stabilization also does not occur when $-r > 0$ and $-p < \frac{-q}{2}$.

Non-Critical Case 3: Assume $r > 0$ and $\frac{q}{2} < p \leq 0$. It follows from the assumptions that $k = q - 2p < 0$ which implies $\frac{2r}{a^2k} < 0$. The exponential decay term $\exp[ky]$ bounds $\exp\left[\frac{2r}{a^2k}\exp[ky]\right]$ between $\exp\left[\frac{2r}{a^2k}\right]$

and 1 which allows for the following comparison of (14) to a smaller magnitude integral:

$$M(\infty) \geq \frac{1}{a^2 c_1} \int_1^\infty \exp\left[\frac{2r}{a^2 k}\right] \exp[-2py] dy.$$

Since $-2p$ is a non-negative constant, it must be the case that $\exp[-2py] \geq 1$ on the interval which allows for the second comparison

$$M(\infty) \geq \frac{1}{a^2 c_1} \exp\left[\frac{2r}{a^2 k}\right] \int_1^\infty dy$$

where (14) is being compared to an even smaller magnitude integral which diverges. Therefore, $|M(\infty)| = \infty$.

Application of the "Stochastic Stability Theorem" indicates that noise-induced stabilization does not occur when $r > 0$ and $\frac{q}{2} < p \leq 0$. The simplification of proof indicates that noise-induced stabilization does not occur when $-r > 0$ and $\frac{-q}{2} < -p \leq 0$ which can also be stated as when $r < 0$ and $\frac{q}{2} > p \geq 0$.

Critical Case: Assume $q = 2p$ so $k = 0$ and consider when $x > 1$. Then the $s(x)$ term takes the form

$$\begin{aligned} s(x) &= \exp\left[\frac{-2r}{a^2} \int_1^x \exp[0] dz\right] \\ &= \exp\left[\frac{-2r}{a^2}(x-1)\right] \\ &= \exp\left[\frac{2r}{a^2}\right] \exp\left[\frac{-2rx}{a^2}\right] \end{aligned}$$

so then the $S(x)$ term takes the form

$$\begin{aligned} S(x) &= \exp\left[\frac{2r}{a^2}\right] \int_1^x \exp\left[\frac{-2ry}{a^2}\right] dy \\ &= \exp\left[\frac{2r}{a^2}\right] \frac{-a^2}{2r} \left(\exp\left[\frac{-2rx}{a^2}\right] - \exp\left[\frac{-2r}{a^2}\right]\right) \\ &= \frac{-a^2}{2r} \exp\left[\frac{2r}{a^2}\right] \exp\left[\frac{-2rx}{a^2}\right] + \frac{a^2}{2r} \end{aligned}$$

Considering when $r < 0$ and setting $x = \infty$ gives

$$\begin{aligned} S(\infty) &= \frac{-a^2}{2r} \exp\left[\frac{2r}{a^2}\right] \exp\left[\frac{-2r\infty}{a^2}\right] + \frac{a^2}{2r} \\ &= \frac{a^2}{2r} < \infty. \end{aligned}$$

The "Stochastic Stability Theorem" indicates that noise-induced stabilization does not occur when $r < 0$ and $p = \frac{q}{2}$. The simplification of proof indicates that noise-induced stabilization also does not occur when $-r < 0$ and $p = \frac{q}{2}$. $\square$

The general exponential stabilization theorem shows that sufficiently large exponential growth can stabilize any exponential drift coefficient. Recognizing the difference between the drift coefficient experiencing exponential growth as opposed to exponential decay, the following sections consider them separately.

5.2. **Exponential Growth Stabilization.** In this section, stabilization of exponential growth in the drift coefficient is considered which means r and q have the same sign.

Theorem 4. *Consider the SDE $dX(t) = r \cdot \exp[qX(t)]\, dt + \sigma(X(t))dB(t)$ where*

$$\sigma(X(t)) = \begin{cases} a \cdot |X(t)|^m \cdot \exp[pX(t)] & \text{for } sgn(r)x \geq 1 \\ a \cdot \exp[pX(t)] & \text{for } 0 \leq sgn(r)x < 1 \\ a \cdot \exp[nX(t)] & \text{for } sgn(r)x < 0 \end{cases}$$

and $r, a, q, p, m, n \in \mathbb{R}$. More specifically, consider when r, q and n have the same sign and $p = \frac{q}{2}$. Noise-induced stabilization occurs if and only if $r \neq 0$, $a \neq 0$, and one of the following sets of conditions is met:

- $m > \frac{1}{2}$, $|n| > |\frac{q}{2}|$,
- $m = \frac{1}{2}$, $|n| > |\frac{q}{2}|$, *and* $a^2 \geq 2|r|$,
- $m > \frac{1}{2}$, $|n| = |\frac{q}{2}|$ *and* $a^2 n < r$ *or*
- $m = \frac{1}{2}$, $|n| = |\frac{q}{2}| < \frac{1}{2}$, *and* $a^2 \geq 2|r|$.

Simplification of Proof: $Y(t) = -X(t)$ must have the same exact stability as $X(t)$ since they have the same magnitude.

$$dY(t) = -dX(t)$$
$$= \begin{cases} -r(\exp[X(t)])^q dt - a|X(t)|^m(\exp[X(t)])^p dB(t) \\ \text{ for } \operatorname{sgn}(r)X(t) > 1 \\ -r(\exp[X(t)])^q dt - a(\exp[X(t)])^p dB(t) \\ \text{ for } 0 \leq \operatorname{sgn}(r)X(t) < 1 \\ -r(\exp[X(t)])^q dt - a(\exp[X(t)])^n dB(t) \\ \text{ for } \operatorname{sgn}(r)X(t) < -1 \end{cases}$$
$$= \begin{cases} -r(\exp[-Y(t)])^q dt - a|-Y(t)|^m(\exp[-Y(t)])^p dB(t) \\ \text{ for } \operatorname{sgn}(r) - Y(t) > 1 \\ -r(\exp[-Y(t)])^q dt - a(\exp[-Y(t)])^p dB(t) \\ \text{ for } 0 \leq \operatorname{sgn}(r) - Y(t) < 1 \\ -r(\exp[-Y(t)])^q dt - a(\exp[-Y(t)])^n dB(t) \\ \text{ for } \operatorname{sgn}(r) - Y(t) < -1 \end{cases}$$
$$= \begin{cases} -r(\exp[Y(t)])^{-q} dt - a|Y(t)|^m(\exp[Y(t)])^{-p} dB(t) \\ \text{ for } \operatorname{sgn}(-r)Y(t) > 1 \\ -r(\exp[Y(t)])^{-q} dt - a(\exp[Y(t)])^{-p} dB(t) \\ \text{ for } 0 \leq \operatorname{sgn}(-r)Y(t) < 1 \\ -r(\exp[Y(t)])^{-q} dt - a(\exp[Y(t)])^{-n} dB(t) \\ \text{ for } \operatorname{sgn}(-r)Y(t) < -1 \end{cases}$$

Hence, the stability of $X(t)$ with $-r$ $-q$, m, $-a$, $-p$, and $-n$ must be equivalent to the stability with r, q, a, m, p, and n. Thus, when proving Theorem 3, it suffices to just prove the case with $r > 0$.

Proof. The "Stochastic Stability Theorem" will be used to show the conditions for noise-induced stabilization.
With $\ell = 1$ and $r > 0$, the $s(x)$ term takes the form

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x z^{-2m} dz\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x \exp\left[(q-2n)z\right] dz\right] & \text{for } x < -1 \end{cases}$$

where the integration depends on the value of m.

Non-Critical Case: Assuming $m \neq \frac{1}{2}$ and $n \neq \frac{q}{2}$ gives

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2(1-2m)}(x^{1-2m}-1)\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2(q-2n)}(\exp[(q-2n)x] - \exp[2n-q]\right] & \text{for } x < -1 \end{cases}$$
$$= \begin{cases} \exp\left[\frac{2r}{a^2(1-2m)}\right]\exp\left[\frac{-2r}{a^2(1-2m)}\cdot x^{1-2k}\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r\exp[2n-q]}{a^2(q-2n)}\right]\exp\left[\frac{-2r}{a^2(q-2n)}\exp[(q-2n)x]\right] & \text{for } x < -1 \end{cases}$$
$$= \begin{cases} c_1\exp\left[\frac{-2r}{a^2(1-2m)}\cdot x^{1-2m}\right] & \text{for } x > 1 \\ c_2\exp\left[\frac{-2r}{a^2(q-2n)}\exp[(q-2n)x]\right] & \text{for } x < -1 \end{cases}$$

and

$$S(x) = \begin{cases} c_1\int_\ell^x \exp\left[\frac{-2r}{a^2(1-2m)}\cdot y^{1-2m}\right]dy & \text{for } x > 1 \\ c_2\int_{-\ell}^x \exp\left[\frac{-2r}{a^2(q-2n)}\exp[(q-2n)y]\right]dy & \text{for } x < -1 \end{cases}$$

where $c_1 = \exp\left[\frac{2r}{a^2(1-2m)}\right]$ and $c_2 = \exp\left[\frac{-2r\exp[2n-q]}{a^2(q-2n}\right]$. Evaluation of the $m(x)$ term gives

$$m(x) = \begin{cases} \frac{1}{c_1a^2}x^{-2m}\exp\left[\frac{2r}{a^2(1-2m)}\cdot x^{1-2m} - 2px\right] & \text{for } x > 1 \\ \frac{1}{c_2a^2}\exp\left[\frac{2r}{a^2(q-2n)}\exp[(q-2n)x] - 2nx\right] & \text{for } x < -1 \end{cases}$$

and

$$M(x) = \begin{cases} \frac{1}{c_1a^2}\int_\ell^x y^{-2m}\exp\left[\frac{2r}{a^2(1-2m)}\cdot y^{1-2m} - 2py\right]dy & \text{for } x > 1 \\ \frac{1}{c_2a^2}\int_{-\ell}^x \exp\left[\frac{-2r}{a^2(2n-q)}\exp[(q-2n)y] - 2ny\right]dy & \text{for } x < -1. \end{cases}$$

Letting $x = \infty$ gives

$$S(\infty) = c_1\int_\ell^\infty \exp\left[\frac{-2r}{a^2(1-2m)}\cdot y^{1-2m}\right]dy \tag{17}$$

and

$$M(\infty) = \frac{1}{c_1a^2}\int_\ell^\infty y^{-2m}\exp\left[\frac{2r}{a^2(1-2m)}\cdot y^{1-2m}\right]\exp\left[-2py\right]dy. \tag{18}$$

Likewise, letting $x = -\infty$ gives

$$\begin{aligned} S(-\infty) &= c_2 \int_{-\ell}^{-\infty} \exp\left[\frac{-2r}{a^2(q-2n)} \exp[(q-2n)y]\right] dy \\ &= -c_2 \int_{-\infty}^{-\ell} \exp\left[\frac{2r}{a^2(2n-q)} \exp[(q-2n)y]\right] dy \\ &= -c_2 \int_{\ell}^{\infty} \exp\left[\frac{2r}{a^2(2n-q)} \exp[(2n-q)y]\right] dy \qquad (19) \end{aligned}$$

and

$$\begin{aligned} M(-\infty) &= \frac{1}{c_2 a^2} \int_{-\ell}^{-\infty} \exp\left[\frac{-2r}{a^2(2n-q)} \exp[(q-2n)y] - 2ny\right] dy \\ &= -\frac{1}{c_2 a^2} \int_{-\infty}^{-\ell} \exp\left[\frac{-2r}{a^2(2n-q)} \exp[(q-2n)y] - 2ny\right] dy \\ &= -\frac{1}{c_2 a^2} \int_{\ell}^{\infty} \exp\left[\frac{-2r}{a^2(2n-q)} \exp[(2n-q)y] + 2ny\right] dy. \qquad (20) \end{aligned}$$

Non-Critical Case 1: Assume $r > 0$, $q > 0$, $n > 0$, and $m > \frac{1}{2}$ and $n > \frac{q}{2}$. The integral in (17) takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{-2r}{a^2(1-2m)}$ is a positive constant so the integral must diverge by Lemma 1. Therefore, $S(\infty) = \infty$. Since $m > \frac{1}{2}$, then $1 - 2m < 0$ so the term y^{1-2m} bounds $\exp\left[\frac{2r}{a^2(1-2m)} \cdot y^{1-2m}\right]$ between $\exp\left[\frac{2r}{a^2(1-2m)} \cdot y^{1-2m}\right]$ and 1. This allows for the following comparison of (18) to the smaller magnitude integral:

$$M(\infty) \leq \frac{1}{c_1 a^2} \exp\left[\frac{2r}{a^2(1-2m)}\right] \int_{\ell}^{\infty} y^{-2m} \exp\left[-2py\right].$$

The exponential decay term $\exp[-2py]$ allows for the further comparison of (18) to an even smaller magnitude integral:

$$M(\infty) \leq \frac{1}{c_1 a^2} \exp\left[\frac{2r}{a^2(1-2m)}\right] \int_{\ell}^{\infty} y^{-2m} dy.$$

Once again, since $m > \frac{1}{2}$, it must be the case that this final integral converges. Therefore, $|M(\infty)| < \infty$. The conditions for $x < -1$ are the same as for *Case 1* of the general exponential theorem. Therefore, $S(-\infty) = -\infty$ and $|M(-\infty)| < \infty$. Application of the "Stochastic Stability Theorem" indicates that noise-induced stabilization occurs when $r > 0$, $q > 0$, $m > \frac{1}{2}$ and $n > \frac{q}{2}$. The proof simplification indicates that stabilization also occurs when $-r > 0$, $-q > 0$, $m > \frac{1}{2}$, and $-n > \frac{-q}{2}$.

Non-Critical Case 2: Assume $r > 0$, $q > 0$, and $m < \frac{1}{2}$. Then the integral in (17) takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{-2r}{a^2(1-2m)}$ is a negative constant and $n = 1 - 2m$ is a positive constant. Then Lemma 1 indicates that integrals of this form converge so $S(\infty) < \infty$ and noise-induced stabilization does not occur when $r > 0$, $q > 0$, and $m < \frac{1}{2}$. Then the proof simplification allows for the conclusion that noise-induced stabilization also does not occur when $-r > 0$, $-q > 0$, and $m < \frac{1}{2}$.

Non-Critical Case 3: Assume $r > 0$ and $0 < n < \frac{q}{2}$. It follows from the assumption that $2n - q$ is a negative constant. Therefore, the exponential decay term $\exp[(2n-q)y]$ bounds the exponential term

$$\exp\left[\frac{-2r}{a^2(2n-q)}\exp[(2n-q)y]\right]$$

between $\exp\left[\frac{-2r}{a^2(2n-q)}\right]$ (upper bound) and 1 lower bound on the interval from 1 to ∞. This fact allows for the following comparison of (20)

$$M(-\infty) \leq -\frac{1}{c_2a^2}\int_\ell^\infty \exp\left[2ny\right]dy$$

The integral (20) is compared to has the form $\int_1^\infty \exp[cy^n]dy$ where $c = 2n$ and $n = 1$. Then Lemma 1 indicates that integrals of this form diverge so $|M(-\infty)| = \infty$ and noise-induced stabilization does not occur when $r > 0$ and $0 < n < \frac{q}{2}$. Additionally, the proof simplification indicates that noise-induced stabilization also does not occur when $-r > 0$ and $0 < -n < \frac{-q}{2}$.

Critical Case: Now, considering when $m = \frac{1}{2}$ and $x \geq 1$, the $s(x)$ term takes the form

$$\begin{aligned} s(x) &= \exp\left[\frac{-2r}{a^2}\int_1^x z^{-1}dz\right] \\ &= \exp\left[\frac{-2r}{a^2}\ln(x)\right] \\ &= x^{\frac{-2r}{a^2}} \end{aligned}$$

and then

$$S(x) = \int_1^x y^{\frac{-2r}{a^2}}dy.$$

Evaluation of $m(x)$ gives

$$m(x) = \frac{1}{a^2} x^{\frac{2r}{a^2}-1} \exp[-2px]$$

and

$$M(x) = \frac{1}{a^2} \int_1^x y^{\frac{2r}{a^2}-1} \exp[-2py] dy.$$

Letting $x = \infty$ gives

$$S(\infty) = \int_1^\infty y^{\frac{-2r}{a^2}} dy \tag{21}$$

and

$$M(\infty) = \frac{1}{a^2} \int_\ell^\infty y^{\frac{2r}{a^2}-1} \exp[-2py] dy. \tag{22}$$

Critical Case 1: Assume $r > 0$, $m = \frac{1}{2}$, $n > \frac{q}{2} > 0$, and $a^2 \geq 2r$. Once again, the conditions for $x < -1$ are the same as for *Case 1* of the previous theorem. Therefore, $S(-\infty) = -\infty$ and $|M(-\infty)| < \infty$. The assumption that $a^2 \geq 2r$ is equivalent to the assumption that $\frac{-2r}{a^2} \geq -1$. Then the integral in (21) diverges and $S(\infty) = \infty$. The assumptions also imply that $\frac{2r}{a^2} - 1 \leq 0$. Then $y^{\frac{2r}{a^2}-1} \leq 1$ on the interval which allows for the following comparison of (22) to a larger magnitude integral.

$$M(\infty) \leq \frac{1}{a^2} \int_\ell^\infty \exp[-2py] dy$$

The smaller magnitude integral has the form $\int_1^\infty \exp[cy^n]dy$ where $c = -2p$ is a negative constant and $n = 1$. Then Lemma 1 indicates that integrals of this form converge so $|M(\infty)| < \infty$. Application of the "Stochastic Stability Theorem" indicates that noise-induced stabilization occurs when $r > 0$, $m = \frac{1}{2}$, $n > \frac{q}{2} > 0$, and $a^2 \geq 2r$. The proof simplification further concludes that stabilization also occurs $-r > 0$, $m = \frac{1}{2}$, $-n > \frac{-q}{2} > 0$, and $a^2 \geq -2r$.

Critical Case 2: Assume $r > 0$, $m = \frac{1}{2}$, and $a^2 < 2r$. It follows from the assumptions that $\frac{-2r}{a^2} < -1$. Therefore, the integral contained in (21) converges. So $S(\infty) = \infty$ and noise-induced stabilization does not occur when $r > 0$, $m = \frac{1}{2}$, and $a^2 < 2r$. The proof simplification indicates that noise-induced stabilization also does not occur when $-r > 0$, $m = \frac{1}{2}$, and $a^2 < -2r$.

Semi-Critical Case: Consider when $r > 0$ and $n = \frac{q}{2}$. Then for $x < -1$, the $s(x)$ term takes the form

$$\begin{aligned} s(x) &= \exp\left[\frac{-2r}{a^2}\int_{-1}^{x} dz\right] \\ &= \exp\left[\frac{2r}{a^2}\right]\exp\left[\frac{-2r}{a^2}x\right], \end{aligned}$$

and

$$S(x) = \exp\left[\frac{2r}{a^2}\right]\int_{-1}^{x}\exp\left[\frac{-2r}{a^2}y\right]dy.$$

Evaluation of the $m(x)$ term gives

$$m(x) = \frac{1}{a^2}\exp\left[\frac{-2r}{a^2}\right]\exp\left[\left(\frac{2r}{a^2} - 2n\right)x\right]$$

and

$$M(x) = \frac{1}{a^2}\exp\left[\frac{-2r}{a^2}\right]\int_{-1}^{x}\exp\left[\left(\frac{2r}{a^2} - 2n\right)y\right]dy.$$

Setting $x = -\infty$ gives

$$S(-\infty) = -\exp\left[\frac{2r}{a^2}\right]\int_{1}^{\infty}\exp\left[\frac{2r}{a^2}y\right]dy. \tag{23}$$

and

$$M(-\infty) = -\frac{1}{a^2}\exp\left[\frac{-2r}{a^2}\right]\int_{1}^{\infty}\exp\left[\left(2n - \frac{2r}{a^2}\right)y\right]dy. \tag{24}$$

Semi-Critical Case 1: Assume $r > 0$, $m > \frac{1}{2}$, $n = \frac{q}{2} > 0$ and $a^2 n < r$. It was proven in *Non-Critical Case 1* that $S(\infty) = \infty$ and $|M(\infty)| < \infty$ under these conditions. The integral in (23) has the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{2r}{a^2}$ is positive and $n = 1$. Then Lemma 1 indicates that integrals of this form diverge. Therefore, $S(-\infty) = -\infty$. The assumption that $a^2 n < r$ is equivalent to the assumption that $2n - \frac{2r}{a^2} < 0$. Therefore, the integral contained in (24) takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = 2n - \frac{2r}{a^2}$ is a negative constant and $n = 1$. Then Lemma 1 indicates that integrals of this form converge so $|M(\infty)| < \infty$. Application of the "Stochastic Stability Theorem indicates that noise-induced stabilization occurs when $r > 0$, $m > \frac{1}{2}$, $n = \frac{q}{2} > 0$ and $a^2 n < r$. The proof simplification allows for the conclusion that stabilization also occurs when $-r > 0$, $m > \frac{1}{2}$, and $-n > \frac{-q}{2} > 0$, and $-a^2 n < r$.

Semi-Critical Case 2: Assume $r > 0$, $m = \frac{1}{2}$, $0 < n = \frac{q}{2} < \frac{1}{2}$, and $a^2 \geq 2r$. It was proven in *Critical Case 1* that $S(\infty) = \infty$ and

$|M(\infty)| < \infty$ under these conditions. It was proven in *Semi-Critical Case 1* that $S(-\infty) = -\infty$ under these conditions. Since $a^2 \geq 2r$, it must be the case that $\frac{2r}{a^2} \leq 1$. Then the assumption that $n < \frac{1}{2}$ implies $2n - \frac{2r}{a^2} < 0$. The integral contained in (24) has the form $\int_1^\infty \exp[cy^n]dy$ where $c = 2n - \frac{2r}{a^2}$ is negative and $n - 1$. Then Lemma 1 indicates that integrals of this form converge. Therefore $|M(-\infty)| < \infty$ and the "Stochastic Stability Theorem" indicates that noise-induced stabilization occurs when $r > 0$, $m = \frac{1}{2}$, $0 < n = \frac{q}{2} < \frac{1}{2}$, and $a^2 \geq 2r$. The proof simplification allows for the conclusion that stabilization also occurs when $-r > 0$, $m = \frac{1}{2}$, $0 < -n = \frac{-q}{2} < \frac{1}{2}$, and $a^2 \geq -2r$

Semi-Critical Case 3: Assume $r > 0$, $n = \frac{q}{2} > 0$ and $a^2 n \geq r$. Then $2n - \frac{2r}{a^2} \geq 0$. The integral contained in (24) takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = 2n - \frac{2r}{a^2} \geq 0$. Then Lemma 1 indicates that integrals of this form diverge so $|M(-\infty)| = \infty$ and noise-induced stabilization does not occur when $r > 0$, $n = \frac{q}{2} > 0$ and $a^2 n \geq r$. Then the proof simplification indicates that stabilization also does not occur when $-r > 0$, $-n = \frac{-q}{2} > 0$, and $-a^2 n \geq -r$.

Semi-Critical Case 4: Assume $r > 0$, $m = \frac{1}{2}$, $a^2 \geq 2r$, and $\frac{1}{2} \leq n = \frac{q}{2}$. It follows that $\frac{2r}{a^2} \leq 1$ and since $n \geq \frac{1}{2}$, it must be the case that $2n - \frac{2r}{a^2} \geq 0$. Since (24) takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = 2n - \frac{2r}{a^2} \geq 0$, the integral must diverge by Lemma 1. Therefore, noise-induced stabilization must not occur when $r > 0$, $m = \frac{1}{2}$, $a^2 \geq 2r$, and $\frac{1}{2} \leq n = \frac{q}{2}$. Then stabilization also does not occur when $-r > 0$, $m = \frac{1}{2}$, $a^2 \geq -2r$, and $\frac{1}{2} \leq -n = \frac{-q}{2}$. □

The exponential growth theorem expands on the general exponential theorem by defining the noise coefficient piece-wise and therefore allowing for the critical case where $p = \frac{q}{2}$.

5.3. **Exponential Decay Stabilization.** In this section, stabilization of exponential decay in the drift coefficient is considered which means r and q have the opposite sign.

Theorem 5. *Consider the SDE* $dX(t) = r \cdot \exp[qX(t)]\,dt + a \cdot |X(t)|^p dB(t)$ *where* r *and* q *have opposite signs. Noise-induced stabilization occurs if and only if* $r \neq 0$, $a \neq 0$, *and* $p > \frac{1}{2}$.

Proof. The "Stochastic Stability Theorem" will be used to show under what conditions noise-induced stabilization occurs.

Simplification of Proof: $Y(t) = -X(t)$ must have the same exact stability as $X(t)$ since they have the same magnitude.

$$\begin{aligned} dY(t) &= -dX(t) \\ &= -r(\exp[X(t)])^q dt - a|X(t)|^p dB(t) \\ &= -r(\exp[-Y(t)])^q dt - a|-Y(t)|^p dB(t) \\ &= -r(\exp[Y(t)])^{-q} dt - a|Y(t)|^p dB(t) \end{aligned}$$

Hence, the stability of $X(t)$ with $-r$, $-q$, $-a$, and p must be equivalent to the stability with r, q, p, and a. Thus, when proving Theorem 3, it suffices to just prove the case with $r > 0$.

The $s(x)$ term takes the form

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x z^{-2p}\exp[qz]dz\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x (-z)^{-2p}\exp[qz]dz\right] & \text{for } x < -1 \end{cases}$$

and

$$S(x) = \begin{cases} \int_1^x \exp\left[\frac{-2r}{a^2}\int_1^y z^{-2p}\exp[qz]dz\right] dy & \text{for } x > 1 \\ \int_{-1}^x \exp\left[\frac{-2r}{a^2}\int_{-1}^y (-z)^{-2p}\exp[qz]dz\right] dy & \text{for } x < -1. \end{cases}$$

The $m(x)$ term then takes the form

$$m(x) = \begin{cases} \frac{1}{a^2}x^{-2p}\exp\left[\frac{2r}{a^2}\int_1^x z^{-2p}\exp[qz]dz\right] & \text{for } x > 1 \\ \frac{1}{a^2}(-x)^{-2p}\exp\left[\frac{2r}{a^2}\int_{-1}^x (-z)^{-2p}\exp[qz]dz\right] & \text{for } x < -1 \end{cases}$$

and

$$M(x) = \begin{cases} \frac{1}{a^2}\int_1^x y^{-2p}\exp\left[\frac{2r}{a^2}\int_1^y z^{-2p}\exp[qz]dz\right] dy & \text{for } x > 1 \\ \frac{1}{a^2}\int_{-1}^x (-y)^{-2p}\exp\left[\frac{2r}{a^2}\int_{-1}^y (-z)^{-2p}\exp[qz]dz\right] dy & \text{for } x < -1. \end{cases}$$

If $x = \infty$, then

$$S(\infty) = \int_1^\infty \exp\left[\frac{-2r}{a^2}\int_1^y z^{-2p}\exp[qz]dz\right] dy \tag{25}$$

and

$$M(\infty) = \frac{1}{a^2}\int_1^\infty y^{-2p}\exp\left[\frac{2r}{a^2}\int_1^y z^{-2p}\exp[qz]dz\right] dy. \tag{26}$$

If $x = -\infty$, then

$$S(-\infty) = \int_{-1}^{-\infty} \exp\left[\frac{-2r}{a^2}\int_{-1}^{y}(-z)^{-2p}\exp[qz]dz\right]dy$$
$$= -\int_{1}^{\infty} \exp\left[\frac{2r}{a^2}\int_{1}^{y} z^{-2p}\exp[-qz]dz\right]dy \tag{27}$$

and

$$M(-\infty) = \frac{1}{a^2}\int_{-1}^{-\infty}(-y)^{-2p}\exp\left[\frac{2r}{a^2}\int_{-1}^{y}(-z)^{-2p}\exp[qz]dz\right]dy$$
$$= -\frac{1}{a^2}\int_{1}^{\infty} y^{-2p}\exp\left[\frac{-2r}{a^2}\int_{1}^{y} z^{-2p}\exp[-qz]dz\right]dy \tag{28}$$

Case 1: Assume $r > 0$, $q < 0$, $p > \frac{1}{2}$. Then $\frac{-2r}{a^2}$ is a negative constant and since $\exp[qz]$ is an exponential decay term, the following comparison of (25) can be made:

$$S(\infty) \geq \int_{1}^{\infty} \exp\left[\frac{-2r}{a^2}\int_{1}^{y} z^{-2p}dz\right]dy$$

With the assumption that $p > \frac{1}{2}$, the comparison takes the form

$$S(\infty) \geq \exp\left[\frac{2r}{a^2(1-2p)}\right]\int_{1}^{\infty}\exp\left[\frac{-2r}{a^2(1-2p)}y^{1-2p}\right]dy.$$

Now the integral takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{-2r}{a^2(1-2p)}$ is a positive constant. Then Lemma 1 indicates that integrals of this form diverge so $S(\infty) = \infty$. Since $\frac{2r}{a^2} > 0$ and $\exp[-qy]$ is an exponential growth term, the following comparison holds for (27):

$$S(-\infty) \leq -\int_{1}^{\infty} \exp\left[\frac{2r}{a^2}\int_{1}^{y} z^{-2p}dz\right]dy$$
$$= -\exp\left[\frac{-2r}{a^2(1-2p)}\right]\int_{1}^{\infty}\exp\left[\frac{2r}{a^2(1-2p)}y^{1-2p}\right]dy$$

The final comparison integral takes the form $\int_1^\infty \exp[cy^n]dy$ where $c = \frac{2r}{a^2(1-2p)}$ is a negative constant and $n = 1-2p$ is a negative constant. Then Lemma 1 indicates that integrals of this form diverge so $S(-\infty) = -\infty$. Using the fact that $\frac{2r}{a^2} > 0$ and $\exp[qz]$ is an exponential decay term, the following comparison of (26) can be made:

$$M(\infty) \leq \frac{1}{a^2}\int_{1}^{\infty} y^{-2p}\exp\left[\frac{2r}{a^2}\int_{1}^{y} z^{-2p}dz\right]dy$$
$$= \frac{1}{a^2}\exp\left[\frac{-2r}{a^2(1-2p)}\right]\int_{1}^{\infty} y^{-2p}\exp\left[\frac{2r}{a^2(1-2p)}y^{1-2p}\right]dy.$$

The exponential term

$$\exp\left[\frac{2r}{a^2(1-2p)}y^{1-2p}\right]$$

must be less than or equal to 1 on the interval which allows for the following comparison of (26):

$$M(\infty) \leq \frac{1}{a^2}\exp\left[\frac{-2r}{a^2(1-2p)}\right]\int_1^\infty y^{-2p}dy.$$

This final integral must converge since $-2p$ must be less than -1. Therefore, $|M(\infty)| < \infty$. Once again using the fact that $\frac{-2r}{a^2}$ is a negative constant and $\exp[-qz]$ is an exponential growth term, the following comparison of (28) can be made:

$$\begin{aligned} M(-\infty) &\geq -\frac{1}{a^2}\int_1^\infty y^{-2p}\exp\left[\frac{-2r}{a^2}\int_1^y z^{-2p}dz\right]dy \\ &= -\frac{1}{a^2}\exp\left[\frac{2r}{a^2(1-2p)}\right]\int_1^\infty y^{-2p}\exp\left[\frac{-2r}{a^2(1-2p)}y^{1-2p}\right]dy. \end{aligned}$$

Now, the exponential term

$$\exp\left[\frac{-2r}{a^2(1-2p)}y^{1-2p}\right]$$

is bounded between $\exp\left[\frac{-2r}{a^2(1-2p)}\right]$ (upper bound) and 1 (lower bound) due to $1-2p$ being a negative power. This allows for the following comparison:

$$\begin{aligned} M(-\infty) &\geq -\frac{1}{a^2}\exp\left[\frac{2r}{a^2(1-2p)}\right]\int_1^\infty y^{-2p}\exp\left[\frac{-2r}{a^2(1-2p)}\right]dy \\ &= -\frac{1}{a^2}\int_1^\infty y^{-2p}dy \\ &> -\infty. \end{aligned}$$

Once again, this final integral must converge since $-2p$ is strictly less than -1. Therefore, $|M(-\infty)| < \infty$. Application of the "Stochastic Stability Theorem" indicates that noise-induced stabilization occurs when $r > 0$, $q < 0$, $p > \frac{1}{2}$. Then stabilization also occurs when $-r > 0$, $-q < 0$, and $p > \frac{1}{2}$.

Case 2: Assume $r > 0$, $q < 0$, $p \leq \frac{1}{2}$. Since $\frac{2r}{a^2}$ is a positive constant, and $\exp[qz]$ is an exponential decay term, the following comparison of

(26) holds:

$$
\begin{aligned}
M(\infty) &\geq \frac{1}{a^2} \int_1^\infty y^{-2p} \exp\left[\frac{2r}{a^2}\int_1^y z^{-2p} dz\right] dy \\
&= \frac{1}{a^2} \exp\left[\frac{-2r}{a^2(1-2p)}\right] \int_1^\infty y^{-2p} \exp\left[\frac{2r}{a^2(1-2p)} y^{1-2p}\right] dy.
\end{aligned}
$$

Since $p \leq \frac{1}{2}$, the constant $\frac{2r}{a^2(1-2p)}$ must be positive so that the exponential term

$$
\exp\left[\frac{2r}{a^2(1-2p)} y^{1-2p}\right]
$$

is greater than or equal to 1 on the interval. This fact allows for the additional comparison of (26) to an even smaller magnitude integral:

$$
M(\infty) \geq \frac{1}{a^2} \exp\left[\frac{-2r}{a^2(1-2p)}\right] \int_1^\infty y^{-2p} dy.
$$

Since $-2p \geq -1$, this final integral must diverge. Therefore, $|M(\infty)| = \infty$ and noise-induced stabilization does not occur when $r > 0$, $q < 0$, $p \leq \frac{1}{2}$. Then it also does not occur when $-r > 0$, $-q < 0$, and $p \leq \frac{1}{2}$. $\square$

The exponential decay theorem shows that an exponential noise-coefficient is not required for stabilization of all exponential drift coefficients. In this theorem, a power noise coefficient was used to stabilize the exponential decay term.

6. Logarithmic Function Stabilization

In this section we will investigate the stabilization of the ODE

$$
dX(t) = r[\ln(|X(t)|)]^q dt,
$$

where r and q are real numbers with $q \geq 0$. This ODE is unstable if and only if $r \neq 0$.

Theorem 6. *Consider the SDE*

$$
dX(t) = r[\ln(|X(t)|)]^q dt + \begin{cases} a|X(t)|^m [\ln(|X(t)|)]^p dB(t) & \text{for } x > 2 \\ a(2^m)[\ln(2)]^p dB(t) & \text{for } x \leq 2 \end{cases}
$$

where r, a, q, p, and m are real numbers and $q, p, m \geq 0$. Noise-induced stabilization occurs if and only if the ODE is unstable, and one of the following conditions is met:

- $m > \frac{1}{2}$,
- $m = \frac{1}{2}$ *and* $p > \frac{q+1}{2}$, *or*
- $m = \frac{1}{2}$ *and* $p = \frac{q+1}{2}$ *and* $a^2 q > 2|r|$.

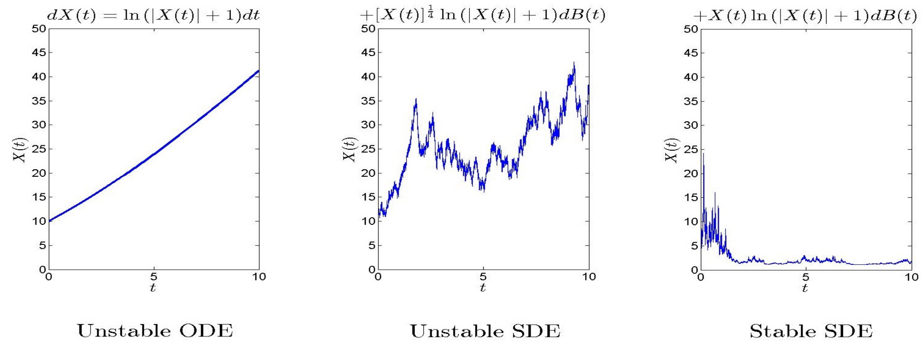

FIGURE 4. This image depicts an unstable logarithmic ODE that is stabilized with the sufficient amount of noise.

Figure 4 shows three separate graphs depicting the phenomenon of noise-induced stabilization. The graph on the far left shows an ODE that diverges off to infinity for all positive initial values; therefore, the ODE is unstable. The middle graph depicts the noise value of $[X(t)]^{\frac{1}{4}}\ln(|X(t)|+1)dB(t)$ added to the previously unstable ODE. This specific amount of noise added does not stabilize the system as $m < 12$. The final image depicts $X(t)\ln(|X(t)|+1)$ added to the unstable ODE. It can be seen that the SDE converges. This occurs since $m > \frac{1}{2}$.

Proof. For this SDE we are looking at $b(x) = r\ln(|x|)^q$ and $\sigma(x) = a|x|^m\ln(|x|)^p$ where $r > 0$. Assume there exists some bound ℓ such that $\sigma(x)$ is not undefined, then by the Stochastic Stability Theorem

$$s(x) = \begin{cases} \exp\left[\int_\ell^x \frac{-2r\ln(|z|)^q}{a^2|z|^{2m}\ln(|z|)^{2p}}dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^x \frac{-2r\ln(|z|)^q}{a^2|z|^{2m}\ln(|z|)^{2p}}dz\right] & \text{for } x < -\ell. \end{cases}$$

After further simplification,

$$s(x) = \begin{cases} \exp\left[\int_\ell^x \frac{-2r\ln(|z|)^{q-2p}}{a^2|z|^{2m}}dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^x \frac{-2r\ln(|z|)^{q-2p}}{a^2|z|^{2m}}dz\right] & \text{for } x < -\ell. \end{cases} \tag{29}$$

In order to find where noise-induced stabilization will occur, we must separate the value of m into three different possibilities: $m > \frac{1}{2}$, $m < \frac{1}{2}$, and $m = \frac{1}{2}$.

Case 1: $m > \frac{1}{2}$

To show that when $m > \frac{1}{2}$ all values of r, a, q, p, and m there is noise-induced stabilization we want $s(x)$ to be greater than or equal to some integral that diverges. Since $\ln(|z|) \geq \ln(2)$ for $z \geq 2$. Thus,

$$s(x) \geq \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2r\ln(2)^{(q-2p)}}{a^2|z|^{2m}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2r\ln(2)^{(q-2p)}}{a^2|z|^{2m}} dz\right] & \text{for } x < -\ell. \end{cases}$$

After integrating,

$$s(x) \geq \begin{cases} \exp\left[\frac{-2r\ln(2)^{(q-2p)}|z|^{(-2m+1)}}{a^2(-2m+1)}\Big|_{\ell}^{x}\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r\ln(2)^{(q-2p)}|z|^{(-2m+1)}}{a^2(-2m+1)}\Big|_{-\ell}^{x}\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} c_1 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x < -\ell \end{cases} \tag{30}$$

and

$$m(x) \leq \begin{cases} \dfrac{\exp\left[\frac{2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_1 a^2 x^{2m} \ln(|x|)^{2p}} & \text{for } x > \ell \\ \dfrac{\exp\left[\frac{2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_2 a^2 x^{2m} \ln(|x|)^{2p}} & \text{for } x < -\ell \end{cases} \tag{31}$$

where $c_1 = \exp\left[\frac{2r\ln(2)^{(q-2p)}|\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$ and $c_2 = \exp\left[\frac{2r\ln(2)^{(q-2p)}|-\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$. First assume $q - 2p < 0$. Using (30) and (31) we know

$$S(\infty) \geq \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|y|^{(-2m+1)}}{a^2(-2m+1)}\right] dy,$$

$$M(\infty) \leq \int_{\ell}^{\infty} \frac{\exp\left[\frac{2r\ln(2)^{(q-2p)}|y|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_1 a^2 y^{2m} \ln(|y|)^{2p}} dy,$$

$$S(-\infty) \geq \int_{-\ell}^{-\infty} c_2 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|y|^{(-2m+1)}}{a^2(-2m+1)}\right] dy, \text{ and}$$

$$M(-\infty) \leq \int_{-\ell}^{-\infty} \frac{\exp\left[\frac{2r\ln(2)^{(q-2p)}|y|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_2 a^2 y^{2m} \ln(|y|)^{2p}} dy.$$

Looking at $S(\infty)$ using Lemma 1 allows us to conclude that $S(\infty) \geq \infty$. For $M(\infty)$ the exponential will converge since the coefficient is negative; therefore, as a whole $M(\infty) < \infty$. For $S(-\infty)$ the largest value the exponential function can be is 1; therefore, $S(-\infty) \geq -\int_{-\infty}^{-\ell} c_2 dy$, which equals $-\infty$. From this it can be concluded that $S(-\infty)$ is infinite. Then $M(-\infty) < \infty$ since the exponential function is converging from its coefficient being negative. Now assume $q - 2p > 0$. To show $s(x)$ is greater than something that diverges we use the comparison $\ln(|z|)^{q-2p} \leq |z|^{m-\frac{1}{2}}$. Then,

$$s(x) \geq \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2r|z|^{(m-\frac{1}{2})}}{a^2|z|^{2m}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2r|z|^{(m-\frac{1}{2})}}{a^2|z|^{2m}} dz\right] & \text{for } x < -\ell, \end{cases}$$

which simplifies to

$$s(x) \geq \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2r}{a^2|z|^{(m+\frac{1}{2})}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2r}{a^2|z|^{(m+\frac{1}{2})}} dz\right] & \text{for } x < -\ell. \end{cases}$$

Then,

$$s(x) \geq \begin{cases} \exp\left[\left.\frac{-2r|z|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right|_{\ell}^{x}\right] & \text{for } x > \ell \\ \exp\left[\left.\frac{-2r|z|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right|_{-\ell}^{x}\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} c_1 \exp\left[\frac{-2r|x|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r|x|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] & \text{for } x < -\ell \end{cases} \tag{32}$$

and

$$m(x) \leq \begin{cases} \frac{\exp\left[\frac{2r|x|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]}{c_1 a^2|x|^{2m}\ln(|x|)^{2p}} & \text{for } x > \ell \\ \frac{\exp\left[\frac{2r|x|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]}{c_2 a^2|x|^{2m}\ln(|x|)^{2p}} & \text{for } x < -\ell \end{cases} \tag{33}$$

where $c_1 = \exp\left[\frac{2r|\ell|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]$ and $c_1 = \exp\left[\frac{2r|-\ell|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]$. Using (32) and (33), and the Stochastic Stability Theorem it can be proven that

$$S(\infty) \geq \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r|y|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] dy,$$

$$|M(\infty)| \leq \int_{\ell}^{\infty} \frac{\exp\left[\frac{2r|y|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]}{c_1 a^2 |y|^{2m} \ln(|y|)^{2p}} dy,$$

$$S(-\infty) \geq \int_{-\ell}^{-\infty} c_2 \exp\left[\frac{-2r|y|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] dy, \text{ and}$$

$$|M(-\infty)| \leq \int_{-\ell}^{-\infty} \frac{\exp\left[\frac{2r|y|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]}{c_2 a^2 |y|^{2m} \ln(|y|)^{2p}} dy.$$

When evaluating $S(\infty)$ the coefficient in the exponential is positive; therefore, by the integral comparison test it can be concluded that $S(\infty)$ diverges. For $M(\infty)$ the coefficient of the exponential function is negative, which causes it to converge. Since the exponential is converging, and m and $p \geq 0$, then the integral as a whole is also converging. In conclusion, $M(\infty) < \infty$. As for $S(-\infty)$, the smallest value the exponential function can be is 1; therefore $S(-\infty) \geq -\int_{-\infty}^{-\ell} c_2 dy$. This integral diverges to $-\infty$, thus $S(-\infty) \geq -\infty$. For similar reasons as $M(\infty)$, $M(-\infty)$ converges as well. Finally, assume $q - 2p = 0$ when $m > \frac{1}{2}$. Then,

$$s(x) = \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2r}{a^2|z|^{2m}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2r}{a^2|z|^{2m}} dz\right] & \text{for } x < -\ell \end{cases}$$

After integrating, it can be said that

$$s(x) = \begin{cases} \exp\left[\left.\frac{-2r|z|^{(-2m+1)}}{a^2(-2m+1)}\right|_{\ell}^{x}\right] & \text{for } x > \ell \\ \exp\left[\left.\frac{-2r|z|^{(-2m+1)}}{a^2(-2m+1)}\right|_{-\ell}^{x}\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} c_1 \exp\left[\frac{-2r|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x < -\ell \end{cases} \tag{34}$$

and

$$m(x) = \begin{cases} \frac{\exp\left[\frac{2r|x|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_1 a^2 |x|^{2m} \ln(|x|)^{2p}} & \text{for } x > \ell \\ \frac{\exp\left[\frac{2r|x|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_2 a^2 |x|^{2m} \ln(|x|)^{2p}} & \text{for } x < -\ell, \end{cases} \tag{35}$$

where $c_1 = \exp\left[\frac{-2r|\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$ and $c_2 = \exp\left[\frac{-2r|-\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$. Again, using (34) and (35), and the Stochastic Stability Theorem it can be determined that

$$S(\infty) = \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r|y|^{(-2m+1)}}{a^2(-2m+1)}\right] dy,$$

$$|M(\infty)| = \int_{\ell}^{\infty} \frac{\exp\left[\frac{2r|y|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_1 a^2 |y|^{2m} \ln(|y|)^{2p}} dy,$$

$$S(-\infty) = \int_{-\ell}^{-\infty} c_2 \exp\left[\frac{-2r|y|^{(-2m+1)}}{a^2(-2m+1)}\right] dy, \text{ and}$$

$$|M(-\infty)| = \int_{-\ell}^{-\infty} \frac{\exp\left[\frac{2r|y|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_2 a^2 |y|^{2m} \ln(|y|)^{2p}} dy.$$

When evaluating the integral of $S(\infty)$, Lemma 1 says since the coefficient of the exponential function is positive, $S(\infty)$ is infinite. The coefficient of the exponential function for $M(\infty)$ is negative; therefore, the exponential function will converge. Since m and p are positive, and the exponential function is converging, the integral of those functions will converge as well. Thus, $M(\infty) < \infty$. As for $S(-\infty)$, the $\int_{-\ell}^{-\infty} c_2 \exp\left[\frac{-2ry^{(-2m+1)}}{a^2(-2m+1)}\right] dy = -\int_{-\infty}^{-\ell} c_2 \exp\left[\frac{-2ry^{(-2m+1)}}{a^2(-2m+1)}\right] dy$, which diverges to negative infinity. Therefore, $S(-\infty) = -\infty$. From $M(\infty)$ it is already known that the function in $M(-\infty)$ is converging. Thus, $|M(-\infty)| < \infty$.

In summary, when $m > \frac{1}{2}$ we found that noise-induced stabilization always occurs.

Case 2: $m < \frac{1}{2}$

Looking back at (29), in order to show that noise-induced stabilization does not occur for $m < \frac{1}{2}$, $s(x)$ must be less than or equal to some function that converges. In this case, we use the comparison $\ln(2)^{q-2p} \leq \ln(|z|)^{q-2p}$ for $z \geq 2$ to show

$$s(x) \leq \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2r\ln(2)^{(q-2p)}}{a^2|z|^{2m}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2r\ln(2)^{(q-2p)}}{a^2|z|^{2m}} dz\right] & \text{for } x < -\ell. \end{cases}$$

After integration,

$$s(x) \leq \begin{cases} \exp\left[\frac{-2r\ln(2)^{(q-2p)}|z|^{(-2m+1)}}{a^2(-2m+1)}\Bigg|_{\ell}^{x}\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r\ln(2)^{(q-2p)}|z|^{(-2m+1)}}{a^2(-2m+1)}\Bigg|_{-\ell}^{x}\right] & \text{for } x < -\ell \end{cases}$$

$$\leq \begin{cases} c_1 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x < -\ell \end{cases} \tag{36}$$

and

$$m(x) \geq \begin{cases} \frac{\exp\left[\frac{2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_1 a^2|x|^{2m}\ln(|x|)^{2p}} & \text{for } x > \ell \\ \frac{\exp\left[\frac{2r\ln(2)^{(q-2p)}|x|^{(-2m+1)}}{a^2(-2m+1)}\right]}{c_2 a^2|x|^{2m}\ln(|x|)^{2p}} & \text{for } x < -\ell \end{cases}$$

where $c_1 = \exp\left[\frac{2r\ln(2)^{(q-2p)}|\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$ and $c_2 = \exp\left[\frac{2r\ln(2)^{(q-2p)}|-\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$.

First suppose $q-2p > 0$, then referring back to (36) can be said that

$$S(\infty) \leq \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r\ln(2)^{(q-2p)}|y|^{(-2m+1)}}{a^2(-2m+1)}\right] dy.$$

Using Lemma 1, $S(\infty)$ is less than or equal to a converging integral; therefore, noise-induced stabilization does not occur for $m < \frac{1}{2}$ when $q-2p > 0$. Now assume $q-2p < 0$. Using the comparison $\ln(|z|)^{q-2p} \leq |z|^{m-\frac{1}{2}}$,

$$s(x) \leq \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2r|z|^{(m-\frac{1}{2})}}{a^2|z|^{2m}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2r|z|^{(m-\frac{1}{2})}}{a^2|z|^{2m}} dz\right] & \text{for } x < -\ell. \end{cases}$$

Which simplifies to

$$s(x) \leq \begin{cases} \exp\left[\int_\ell^x \frac{-2r}{a^2|z|^{(m+\frac{1}{2})}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^x \frac{-2r}{a^2|z|^{(m+\frac{1}{2})}} dz\right] & \text{for } x < -\ell. \end{cases}$$

After integration,

$$s(x) \leq \begin{cases} \exp\left[\left.\frac{-2r|z|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right|_\ell^x\right] & \text{for } x > \ell \\ \exp\left[\left.\frac{-2r|z|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right|_{-\ell}^x\right] & \text{for } x < -\ell \end{cases}$$

$$\leq \begin{cases} c_1 \exp\left[\frac{-2r|x|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r|x|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] & \text{for } x < -\ell \end{cases} \tag{37}$$

where $c_1 = \exp\left[\frac{2r|\ell|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]$ and $c_2 = \exp\left[\frac{2r|-\ell|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right]$. Then using (37) it can be shown that

$$S(\infty) \leq \int_\ell^\infty c_1 \exp\left[\frac{-2r|y|^{(-m+\frac{1}{2})}}{a^2(-m+\frac{1}{2})}\right] dy.$$

Lemma 1 allows us to conclude that this equation is finite; therefore, $S(\infty)$ is less than a finite number. From the stochastic stability theorem, since $S(\infty) < \infty$ then the SDE is unstable. Finally, suppose $q - 2p = 0$. Looking back at (29) it can be said that

$$s(x) = \begin{cases} \exp\left[\int_\ell^x \frac{-2r}{a^2|z|^{2m}} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^x \frac{-2r}{a^2|z|^{2m}} dz\right] & \text{for } x < -\ell \end{cases}$$

After integration,

$$s(x) = \begin{cases} \exp\left[\left.\frac{-2r|z|^{(-2m+1)}}{a^2(-2m+1)}\right|_\ell^x\right] & \text{for } x > \ell \\ \exp\left[\left.\frac{-2r|z|^{(-2m+1)}}{a^2(-2m+1)}\right|_{-\ell}^x\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} c_1 \exp\left[\frac{-2r|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r|x|^{(-2m+1)}}{a^2(-2m+1)}\right] & \text{for } x < -\ell \end{cases} \tag{38}$$

where $c_1 = \exp\left[\frac{2r|\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$ and $c_2 = \exp\left[\frac{2r|-\ell|^{(-2m+1)}}{a^2(-2m+1)}\right]$. Investigating $S(\infty)$ it can be determined that

$$S(\infty) = \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r|y|^{(-2m+1)}}{a^2(-2m+1)}\right] dy,$$

which is a finite integral. Therefore, $S(\infty) < \infty$ and noise-induced stabilization does not occur for $m < \frac{1}{2}$ when $q - 2p = 0$. Furthermore, when $m < \frac{1}{2}$, it can be concluded from these integrations that noise-induced stabilization never occurs.

Case 3: $m = \frac{1}{2}$
Referring back to (29) when $m = \frac{1}{2}$

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_{\ell}^{x} \frac{1}{|z|\ln(|z|)^{2p-q}}dz\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r}{a^2}\int_{-\ell}^{x} \frac{1}{|z|\ln(|z|)^{2p-q}}dz\right] & \text{for } x < -\ell \end{cases}$$

In order to solve this equation a u-substitution must be used. Let

$$u = \ln(|z|) \quad \text{and} \quad du = \frac{1}{|z|}dz \tag{39}$$

then,

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_{\ln(|\ell|)}^{\ln(|x|)} \frac{1}{u^{2p-q}}du\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r}{a^2}\int_{\ln(|-\ell|)}^{\ln(|x|)} \frac{1}{u^{2p-q}}du\right] & \text{for } x < -\ell \end{cases}$$

After integrating,

$$s(x) = \begin{cases} \exp\left[\frac{-2ru^{q-2p+1}}{a^2(q-2p+1)}\Big|_{\ln(|\ell|)}^{\ln(|x|)}\right] & \text{for } x > \ell \\ \exp\left[\frac{-2ru^{q-2p+1}}{a^2(q-2p+1)}\Big|_{\ln(|-\ell|)}^{\ln(|x|)}\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} c_1 \exp\left[\frac{-2r\ln(|x|)^{q-2p+1}}{a^2(q-2p+1)}\right] & \text{for } x > \ell \\ c_2 \exp\left[\frac{-2r\ln(|x|)^{q-2p+1}}{a^2(q-2p+1)}\right] & \text{for } x < -\ell \end{cases} \tag{40}$$

and

$$m(x) = \begin{cases} \frac{\exp\left[\frac{2r\ln(|x|)^{q-2p+1}}{a^2(q-2p+1)}\right]}{c_1 a^2|x|\ln(|x|)^{2p}} & \text{for } x > \ell \\ \frac{\exp\left[\frac{2r\ln(|x|)^{q-2p+1}}{a^2(q-2p+1)}\right]}{c_2 a^2|x|\ln(|x|)^{2p}} & \text{for } x < -\ell \end{cases} \tag{41}$$

where $c_1 = \exp\left[\frac{2r\ln(|\ell|)^{q-2p+1}}{a^2(q-2p+1)}\right]$ and $c_2 = \exp\left[\frac{2r\ln(|-\ell|)^{q-2p+1}}{a^2(q-2p+1)}\right]$. To continue investigating the stabilization of the SDE next we will look at

$S(\infty), M(\infty), S(-\infty)$, and $M(-\infty)$. In order to integrate these equations we must separate this into three different values of $q-2p+1$. First, assume that $q-2p+1>0$.

$$\begin{aligned} S(\infty) &= \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right] dy \\ &< \infty \end{aligned}$$

Since $S(\infty)$ is finite when $q-2p+1>0$, by Lemma 1, then by the stochastic stability theorem there is no noise-induced stabilization. Now, assume that $q-2p+1<0$.

$$\begin{aligned} S(\infty) &= \int_{\ell}^{\infty} c_1 \exp\left[\frac{-2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right] dy \\ &= \infty \end{aligned}$$

$$\begin{aligned} M(\infty) &= \int_{\ell}^{\infty} \frac{\exp\left[\frac{2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right]}{c_1 a^2 y\ln(|y|)^{2p}} dy \\ &\le \int_{\ell}^{\infty} \frac{1}{c_1 a^2 y\ln(|y|)^{2p}} dy. \end{aligned}$$

To solve for $M(\infty)$ we use the same u-substitution from (39). After doing so,

$$M(\infty) \le \frac{1}{c_1 a^2}\int_{\ln(|\ell|)}^{\ln(|\infty|)} \frac{1}{u^{2p}} du.$$

In order for $M(\infty)<\infty$ we must have $2p>1$. Now to look at $S(-\infty)$, and $M(-\infty)$,

$$\begin{aligned} S(-\infty) &= \int_{-\ell}^{-\infty} c_2 \exp\left[\frac{-2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right] dy \\ &= -\int_{-\infty}^{-\ell} c_2 \exp\left[\frac{-2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right] dy \\ &= -\int_{\ell}^{\infty} c_2 \exp\left[\frac{-2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right] dy \\ &= -\infty \end{aligned}$$

$$\begin{aligned} M(-\infty) &= \int_{-\ell}^{-\infty} \frac{\exp\left[\frac{2r\ln(|y|)^{q-2p+1}}{a^2(q-2p+1)}\right]}{c_2 a^2 y\ln(|y|)^{2p}} dy \\ &\le \int_{-\ell}^{-\infty} \frac{1}{c_2 a^2 y\ln(|y|)^{2p}} dy. \end{aligned}$$

Using (39) we know that

$$M(\infty) \leq \frac{1}{c_2 a^2} \int_{\ln(|-\ell|)}^{\ln(|-\infty|)} \frac{1}{u^{2p}} du.$$

With the absolute values around the negative numbers, $M(-\infty) = M(\infty)$; therefore, they have the same stability. Thus, when $m = \frac{1}{2}$ we find that for $q-2p+1 < 0$, which can also be written as $p > \frac{q+1}{2}$, noise-induced stabilization does occur. Finally, assume that $q-2p+1 = 0$. In order to further investigate the stability of the SDE when $q-2p+1 = 0$ we must re-evaluate (29) to make sure the integration is not undefined. When doing so we find

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2} \int_{\ln(|\ell|)}^{\ln(|x|)} u^{q-2p} du\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r}{a^2} \int_{\ln(|-\ell|)}^{\ln(|x|)} u^{q-2p} du\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} \exp\left[\frac{-2r}{a^2} \int_{\ln(|\ell|)}^{\ln(|x|)} \frac{1}{u} du\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r}{a^2} \int_{\ln(|-\ell|)}^{\ln(|x|)} \frac{1}{u} du\right] & \text{for } x < -\ell. \end{cases}$$

After integrating,

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2} \cdot \ln(u)\Big|_{\ln(|\ell|)}^{\ln(|x|)}\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r}{a^2} \cdot \ln(u)\Big|_{\ln(|-\ell|)}^{\ln(|x|)}\right] & \text{for } x < -\ell \end{cases}$$

$$= \begin{cases} \exp\left[\frac{-2r\ln(\ln(|x|))}{a^2}\right] \exp\left[\frac{2r\ln(\ln(|\ell|))}{a^2}\right] & \text{for } x > \ell \\ \exp\left[\frac{-2r\ln(\ln(|x|))}{a^2}\right] \exp\left[\frac{2r\ln(\ln(|-\ell|))}{a^2}\right] & \text{for } x < -\ell. \end{cases}$$

With cancellations between the exponentials and logarithms $s(x)$ simplifies down to

$$s(x) = \begin{cases} c_1[\ln(|x|)]^{\frac{-2r}{a^2}} & \text{for } x > \ell \\ c_2[\ln(|x|)]^{\frac{-2r}{a^2}} & \text{for } x < -\ell \end{cases} \tag{42}$$

and

$$m(x) = \begin{cases} \frac{[\ln(|x|)]^{\frac{2r}{a^2}-2p}}{c_1 a^2 |x|} & \text{for } x > \ell \\ \frac{[\ln(|x|)]^{\frac{2r}{a^2}-2p}}{c_2 a^2 |x|} & \text{for } x < -\ell, \end{cases} \tag{43}$$

where $c_1 = [\ln(|\ell|)]^{\frac{2r}{a^2}}$ and $c_2 = [\ln(|-\ell|)]^{\frac{2r}{a^2}}$. After finding the values of $s(x)$ and $m(x)$ the Stochastic Stability Theorem says,

$$S(\infty) = \int_{\ell}^{\infty} c_1[\ln(|y|)]^{\frac{-2r}{a^2}} dy,$$

which diverges for any value of $a \neq 0$, and

$$M(\infty) = \int_{\ell}^{\infty} \frac{[\ln(|y|)]^{\frac{2r}{a^2}-2p}}{c_1 a^2 |y|} dy.$$

Using (39),

$$M(\infty) = \int_{\ln(|\ell|)}^{\ln(|\infty|)} \frac{u^{\frac{2r}{a^2}-2p}}{c_1 a^2} du.$$

In order to get $M(\infty) < \infty$ we must have $\frac{2r}{a^2} - 2p < -1$; therefore, $a^2 q > 2r$. Referring back to the theorem,

$$S(-\infty) = \int_{-\ell}^{-\infty} c_2[\ln(|y|)]^{\frac{-2r}{a^2}} dy.$$

Similarly to $S(\infty)$, this integral goes to negative infinity for any value of $a \neq 0$. Using (39) for $M(\infty)$,

$$\begin{aligned} M(-\infty) &= \int_{\ln(|-\ell|)}^{\ln(|-\infty|)} \frac{u^{\frac{2r}{a^2}-2p}}{c_2 a^2} du \\ &= M(\infty). \end{aligned}$$

Since $M(-\infty) = M(\infty)$ then, stabilization for $M(-\infty)$ is the same as $M(\infty)$. Thus, when $m = \frac{1}{2}$, and $q - 2p + 1 = 0$ we will have noise-induced stabilization for $a^2 q > 2|r|$.

When looking at the cases where $r < 0$, let $Y(t) = -X(t)$.Since $Y(t)$ is bounded if and only if $X(t)$ is bounded, $Y(t)$ must have the exact same stability as $X(t)$. Suppose $X(t)$ satisfies

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where $b(x) = r[\ln(|x|)]^q$, $\sigma(x) = a|x|^m \ln(|x|)^p$, and $r < 0$. It follows that

$$\begin{aligned} dY(t) &= -dX(t) \\ &= -\left[r[\ln(|X(t)|)]^q dt + a|X(t)|^m \ln(|X(t)|)^p dB(t)\right] \\ &= -\left[r[\ln(|-Y(t)|)]^q dt + a|(-Y(t))|^m \ln(|(-Y(t))|)^p dB(t)\right]. \end{aligned}$$

Since there are absolute values within the logarithmic and power terms the stability of $X(t)$ with leading coefficient $-r$ is equivalent to its stability with leading coefficient r. $\square$

7. General Noise Minimum

In this section we explore the minimum amount of noise necessary for noise-induced stabilization to occur when the noise coefficient is not restricted to a particular form. The definitions below characterize what it means for a noise coefficient $\sigma(x)$ to be the minimum necessary to stabilize an ODE with drift coefficient $b(x)$.

Definition 2. *Let $b(x)$ and $\sigma(x)$ be continuous functions. We say that $\sigma(x)$ stabilizes $b(x)$ if the solution $x(t)$ to the ODE*

$$dx(t) = b(x(t))dt$$

is unstable, but the solution $X(t)$ to the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

is stable.

Definition 3. *Let $b(x)$ and $\sigma(x)$ be continuous functions. We say that $\sigma(x)$ is the minimum noise necessary to stabilize $b(x)$ if*

1. *$\sigma(x)$ stabilizes $b(x)$ and*
2. *for any continuous function $\tilde{\sigma}(x)$ such that*

$$\limsup_{x\to\pm\infty} \left|\frac{\tilde{\sigma}(x)}{\sigma(x)}\right| < 1 \ ,$$

$\tilde{\sigma}(x)$ does not stabilize $b(x)$.

Notes:

1. The "lim sup" appears above since the limit may not exist; e.g. an oscillatory function.
2. The minimum noise is not unique. There may be multiple functions that satisfy the definition of minimum noise and for any two of these, the limit of the absolute value of the ratio will always be one.

Lemma 3. *Let $b(x)$ and $\hat{\sigma}(x)$ be continuous functions and suppose $\hat{\sigma}(x)$ does not stabilize $b(x)$. Furthermore, suppose there exists $\ell_0 \geq 0$ such that $\hat{\sigma}(x) \neq 0$ and $b(x) \neq 0$ for all $|x| \geq \ell_0$, and additionally that $b(x)$ is either strictly positive for $x \geq \ell_0$ or that $b(x)$ is strictly negative for $x \leq -\ell_0$. Then for any continuous function $\tilde{\sigma}(x)$ such that*

$$\limsup_{x\to\pm\infty} \left|\frac{\tilde{\sigma}(x)}{\hat{\sigma}(x)}\right| < 1 \ ,$$

$\tilde{\sigma}(x)$ also does not stabilize $b(x)$.

Proof. Without loss of generality, we assume $b(x)$ is strictly positive for $x \geq \ell_0$ and that either $\hat{S}(\infty) < \infty$ or $\hat{M}(\infty) = \infty$, where $\hat{S}(x)$ is $S(x)$ from the "Stochastic Stability Theorem" with $\hat{\sigma}(x)$ substituted for $\sigma(x)$, and likewise with $\hat{M}(x)$ (and $\hat{s}(x)$ and $\hat{m}(x)$).

Let $\tilde{\sigma}(x)$ be any continuous function such that

$$\limsup_{x\to\pm\infty} \left|\frac{\tilde{\sigma}(x)}{\hat{\sigma}(x)}\right| < 1 \ .$$

Then there exists $\ell \geq \ell_0$ such that $\dfrac{\tilde{\sigma}^2(x)}{\hat{\sigma}^2(x)} < 1$ for all $|x| \geq \ell$. Define $\tilde{s}(x)$, $\tilde{m}(x)$, $\tilde{S}(x)$, and $\tilde{M}(x)$ to be the corresponding values from the "Stochastic Stability Theorem" with $\tilde{\sigma}(x)$ substituted for $\sigma(x)$.

Now since $b(x) > 0$ for all $x \geq \ell$, then $\dfrac{-2b(x)}{\tilde{\sigma}^2(x)} < \dfrac{-2b(x)}{\hat{\sigma}^2(x)}$ for all $x \geq \ell$, which implies $\tilde{s}(x) < \hat{s}(x)$ and $\tilde{m}(x) > \hat{m}(x)$ for all $x \geq \ell$. Thus, if $\hat{S}(\infty) < \infty$, $\tilde{S}(\infty) < \infty$ as well, or if $\hat{M}(\infty) = \infty$, $\tilde{M}(\infty) = \infty$ as well. Therefore, $\tilde{\sigma}(x)$ does not stabilize $b(x)$.

□

Lemma 4. *Let $b(x)$ and $\sigma(x)$ be continuous functions. Furthermore, suppose there exists $\ell_0 \geq 0$ such that $\sigma(x) \neq 0$ and $b(x) \neq 0$ for all $|x| \geq \ell_0$, and additionally that $b(x)$ is either strictly positive for $x \geq \ell_0$ or that $b(x)$ is strictly negative for $x \leq -\ell_0$. If $\sigma(x)$ stabilizes $b(x)$ and there exists a continuous function $\hat{\sigma}(x)$ such that*

$$\limsup_{x\to\pm\infty} \left|\frac{\hat{\sigma}(x)}{\sigma(x)}\right| = 1 \ ,$$

but $\hat{\sigma}(x)$ does not stabilize $b(x)$, then $\sigma(x)$ is the minimum noise necessary to stabilize $b(x)$.

Proof. Since $\hat{\sigma}(x)$ does not stabilize $b(x)$, any continuous function $\tilde{\sigma}(x)$ such that

$$\limsup_{x\to\pm\infty} \left|\frac{\tilde{\sigma}(x)}{\hat{\sigma}(x)}\right| < 1$$

also does not stabilize $b(x)$ by Lemma 3.

Now for any continuous function $\tilde{\sigma}(x)$,

$$\begin{aligned}\limsup_{x\to\pm\infty} \left|\frac{\tilde{\sigma}(x)}{\sigma(x)}\right| &= \limsup_{x\to\pm\infty} \left(\left|\frac{\tilde{\sigma}(x)}{\hat{\sigma}(x)}\right| \cdot \left|\frac{\hat{\sigma}(x)}{\sigma(x)}\right|\right) \\ &\leq \left(\limsup_{x\to\pm\infty} \left|\frac{\tilde{\sigma}(x)}{\hat{\sigma}(x)}\right|\right) \cdot \left(\limsup_{x\to\pm\infty} \left|\frac{\hat{\sigma}(x)}{\sigma(x)}\right|\right) < 1 \ .\end{aligned}$$

Thus, $\sigma(x)$ is the minimum noise necessary to stabilize $b(x)$. □

Building upon the above definitions and lemmas, we now prove the overall minimum noise necessary for stabilization when the drift coefficient is a general power function with power $q > 0$.

Theorem 7. *Suppose $b(x) = r|x|^q$ where $r \neq 0$ and $q > 1$. Then*

$$\sigma(x) = (\sqrt{2|r|})|x|^{\frac{q+1}{2}}$$

is the minimum noise necessary to stabilize $b(x)$.

Proof. We know that $\sigma(x)$ stabilizes $b(x)$ by the power function stabilization theorem 1. Consider

$$\hat{\sigma}(x) = (\sqrt{2|r|})|x|^{\frac{q+1}{2}} \left(\frac{1}{\sqrt{1 + \frac{2}{\ln(|x|)}}} \right)$$

for $|x| \geq 2$. $\hat{\sigma}(x)$ can be defined to be anything nonzero for $|x| < 2$ such that the function is continuous. Then

$$\limsup_{x \to \pm\infty} \left| \frac{\hat{\sigma}(x)}{\sigma(x)} \right| = 1 \ .$$

Without loss of generality, assume $r > 0$. Then for $x \geq 2$,

$$\frac{-2b(x)}{\hat{\sigma}^2(x)} = \frac{-1}{|x|} \left(1 + \frac{2}{\ln(|x|)} \right) = \frac{-1}{|x|} + \frac{-2}{|x| \ln(|x|)} \ .$$

Thus for $x \geq 2$,

$$\begin{aligned} \hat{s}(x) &= \exp \left[\int_2^x \frac{-1}{|z|} + \frac{-2}{|z| \ln(|z|)} dz \right] \\ &= \exp \left[-\ln(|x|) + \ln(2) - 2\ln(\ln(|x|)) + 2\ln(\ln(2)) \right] \\ &= \frac{2(\ln(2))^2}{|x|(\ln(|x|))^2} \ . \end{aligned}$$

Hence,

$$\hat{S}(\infty) = \int_2^\infty \frac{2(\ln(2))^2}{|x|(\ln(|x|))^2} dx = 2\ln(2) < \infty$$

which implies that $\hat{\sigma}(x)$ does not stabilize $b(x)$. Therefore, by Lemma 4, $\sigma(x)$ is the minimum noise necessary to stabilize $b(x)$.

□

Theorem 8. *Consider $b(x) = r|x|^q$ where $r \neq 0$ and $0 < q \leq 1$. Then*

$$\sigma(x) = \begin{cases} \left(\sqrt{\frac{2|r|}{q}}\right)|x|^{\frac{q+1}{2}}\left(\frac{1}{\sqrt{1-\frac{2}{q\ln(|x|)}}}\right) & \text{for } |x| \geq 2 \\ \left(\sqrt{\frac{2|r|}{q}}\right)|2|^{\frac{q+1}{2}}\left(\frac{1}{\sqrt{1-\frac{2}{q\ln(|2|)}}}\right) & \text{for } |x| < 2 \end{cases}$$

is the minimum noise necessary to stabilize $b(x)$.

Proof. By the power function stabilization theorem 1, we know that

$$\hat{\sigma}(x) = \left(\sqrt{\frac{2|r|}{q}}\right)|x|^{\frac{q+1}{2}}$$

does not stabilize $b(x)$. Thus, by Lemma 4, it suffices to show that $\sigma(x)$ does stabilize $b(x)$.

Without loss of generality, assume $r > 0$. Then for $x \geq 2$,

$$\frac{-2b(x)}{\sigma^2(x)} = \frac{-q}{x}\left(1 - \frac{2}{\ln(x)}\right) = \frac{-q}{x} + \frac{2}{x\ln(x)} \ .$$

Thus for $x \geq 2$,

$$\begin{aligned} s(x) &= \exp\left[\int_2^x \frac{-q}{z} + \frac{2q}{z\ln(z)}dz\right] \\ &= \exp\left[-q\ln(x) + q\ln(2) + 2\ln(\ln(x)) - 2\ln(\ln(2))\right] \\ &= \frac{2^q(\ln(x))^2}{x^q(\ln(2))^2} \ . \end{aligned}$$

Hence,

$$\begin{aligned} S(\infty) &= \int_2^\infty \frac{2^q(\ln(x))^2}{x^q(\ln(2))^2}dx \\ &\geq \int_2^\infty \frac{2^q}{x^q}dx \\ &= \infty \ . \end{aligned}$$

Now

$$\begin{aligned} M(\infty) &= \int_2^\infty \frac{x^q(\ln(2))^2}{2^q(\ln(x))^2} \cdot \frac{q}{2rx^{q+1}}\left(1 - \frac{2}{q\ln(x)}\right)dx \\ &\leq \frac{q(\ln(2))^2}{2^{q+1}r}\int_2^\infty \frac{1}{x(\ln(x))^2}dx \\ &< \infty \ . \end{aligned}$$

$S(-\infty) = -\infty$ and $|M(-\infty)| < \infty$ since $b(x) > 0$ for $x < 0$. Therefore, $\sigma(x)$ is the minimum noise necessary to stabilize $b(x)$. □

Note that it is still an open question whether or not an overall minimum noise coefficient exists for when the drift coefficient is a general power function with $q \leq 0$. The minimum noise necessary to stabilize a polynomial of degree $q \geq 2$ should be analogous to that stated in Theorem 7, and analogous to that stated in Theorem 8 for a polynomial of degree $q = 1$. As for when the drift coefficient is an exponential or logarithmic function, or more general, the existence of an overall minimum noise coefficient, and its form if it exists, remains an open problem.

8. Conclusion

Initially, the phenomenon of noise-induced stabilization might seem counterintuitive; however, in this paper we have shown stabilization of ODEs can occur with the addition of noise. In addition to stabilizing the ODEs, we have proven the precise minimum amount of noise necessary for stabilization when the drift and noise coefficients are general power functions, polynomials, or exponential or logarithmic functions.

Furthermore, we hope our results will motivate future work in investigating other forms of the noise and drift coefficients, and finding the precise minimum amount of noise needed to stabilize systems where the noise coefficient is not restricted to a specific form.

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