NOISE-INDUCED STABILIZATION OF PERTURBED HAMILTONIAN SYSTEMS

ANTHONY CONIGLIO[1], SARAH SPARKS[2], DANIEL WEITHERS[3], AND ADVISOR: DR. TIFFANY KOLBA[4]

ABSTRACT. Noise-induced stabilization is the phenomenon in which the addition of randomness to an unstable deterministic system of ordinary differential equations (ODEs) results in a stable system of stochastic differential equations (SDEs). A Hamiltonian system is a two-dimensional system of ODEs defined by a Hamiltonian function, which is constant along each solution curve. With stability defined as global stochastic boundedness, Hamiltonian systems cannot be stabilized by the addition of noise that is constant in space. Therefore we studied ways to deterministically perturb different Hamiltonian systems in such a way so that the qualitative behavior of solutions is preserved but noise-induced stabilization becomes possible. We provide a systematic framework for methods of perturbing these systems and proving noise-induced stabilization.

TABLE OF CONTENTS

[1]Indiana University, Bloomington, IN
[2]Frostburg State University, Frostburg, MD
[3]Carleton College, Northfield, MN
[4]Department of Mathematics and Statistics, Valparaiso University, Valparaiso, IN 46383: (tiffany.kolba@valpo.edu)

1. Introduction

Hamiltonian systems are a class of differential equations which were initially developed as a means of modeling closed physical systems. From a purely mathematical standpoint, this paper will examine how minor perturbations to Hamiltonian systems can fundamentally alter the behavior of the systems. Our goals of this paper are to provide a systematic framework for perturbing unstable Hamiltonian systems so that noise-induced stabilization can occur and to prove that the resulting systems are indeed stable. We will demonstrate several instances in which modified Hamiltonian systems are stabilized by the addition of white noise.

This paper will consist of five sections. In the first, we will define key terms and explain concepts which are used throughout the paper. In the second, third, and fourth, we examine three classes of perturbed Hamiltonian systems and prove their stability. In the fifth, we summarize our results and discuss future directions that this research could be taken.

1.1. Background & Definitions.

We begin with the definition of a Hamiltonian system.

Definition 1 (Hamiltonian System)**.** *A **Hamiltonian system** is a two-dimensional system of differential equations defined by a Hamiltonian function $H(x, y)$ given by*

$$\begin{aligned} \frac{dx_t}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dy_t}{dt} &= -\frac{\partial H}{\partial x}. \end{aligned}$$

Due to this structure, the value of the Hamiltonian function is constant along any solution curve of the system. To see why this is true, note that

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} \\ &= 0. \end{aligned}$$

This property is critical to the nature of the problem of noise-induced stabilization; to elaborate further, we must first discuss stochastic differential equations and stability.

We construct a system of stochastic differential equations (SDEs) by adding noise terms onto a deterministic system. If the original deterministic system is Hamiltonian, then we get a stochastic system of the following form:

$$\begin{aligned} \frac{dX_t}{dt} &= \frac{\partial H}{\partial y}(X_t, Y_t) + \epsilon_1 \frac{dB_1(t)}{dt} \\ \frac{dY_t}{dt} &= -\frac{\partial H}{\partial x}(X_t, Y_t) + \epsilon_2 \frac{dB_2(t)}{dt}. \end{aligned}$$

Here $B_1(t)$ and $B_2(t)$ are *independent* standard Brownian motions, and ϵ_1 and ϵ_2 are constants which control the strength of the noise in the $x-$ and $y-$ directions, respectively.

We are concerned with stability in the sense of global stochastic boundedness. In particular, we define stability in the following way.

Definition 2 (Stability)**.** *We say that a system is* ***stable*** *if, for all initial conditions and for all* $\delta > 0$*, there exists a bound* M*, such that* $P(|(X_t, Y_t)| \leq M) > 1 - \delta$ *for all* t*.*

In the deterministic setting, when $\epsilon_1 = \epsilon_2 = 0$, this definition of stability reduces to the requirement that all of its solution curves be bounded.

For any unstable Hamiltonian system, adding white noise is not sufficient to stabilize the system since Lebesgue measure is always invariant. This fact can be seen by observing the structure of the generator corresponding to a Hamiltonian system and computing its adjoint. For any arbitrary two-dimensional Hamiltonian system with Hamiltonian function $H(x, y)$ perturbed by white noise, the generator has the form

$$\mathcal{L}\phi = \frac{\partial H}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\epsilon_1^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{\epsilon_2^2}{2}\frac{\partial^2 \phi}{\partial y^2}.$$

The adjoint to the generator has the form

$$\begin{aligned}\mathcal{L}^*\phi &= -\frac{\partial^2 H}{\partial x \partial y}\phi - \frac{\partial H}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial^2 H}{\partial y \partial x}\phi + \frac{\partial H}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\epsilon_1^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{\epsilon_2^2}{2}\frac{\partial^2 \phi}{\partial y^2} \\ &= -\frac{\partial H}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\epsilon_1^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{\epsilon_2^2}{2}\frac{\partial^2 \phi}{\partial y^2}.\end{aligned}$$

From this form it can be seen that when ϕ is identically equal to a constant, it satisfies $\mathcal{L}^*\phi = 0$, which implies that Lebesgue measure is invariant.

In light of this fact about Hamiltonian systems, we must deterministically modify a unstable Hamiltonian system if noise-induced stabilization is to occur. However, we wish to modify a system in such a way so that the behavior of the deterministic modified system is qualitatively similar to that of the original Hamiltonian system. Most importantly, we want the modified system to still be unstable in the deterministic setting, and for any initial condition we want the limiting behavior to be the same as for the original Hamiltonian.

Instead of writing out each entire system, from here on we will refer to a perturbed system by its Hamiltonian function and by the modification functions $f(x, y)$ and $g(x, y)$. After perturbing the Hamiltonian systems, we then add noise to produce noise-induced stabilization.

Definition 3 (Lyapunov function)**.** *A function* $V(x, y)$ *is called a* ***Lyapunov function*** *if it satisfies the following three properties:*

$$(i) \quad V(x, y) \in C^\infty(\mathbb{R}^2)$$

$$(ii) \quad \lim_{r\to\infty}\left[\inf_{(x,y)\in B_r^c} V(x, y)\right] = \infty$$

$$(iii) \quad \lim_{r\to\infty}\left[\inf_{(x,y)\in B_r^c} [-(\mathcal{L}V)(x, y)]\right] = \infty \ ,$$

where $\mathcal{L}$ *is the generator corresponding to the system.*

For our modified stochastic systems, the generator $\mathcal{L}$ will have the following form:

$$\mathcal{L} = \Big[\frac{\partial H}{\partial y} - f\Big]\frac{\partial}{\partial x} + \Big[-\frac{\partial H}{\partial x} - g\Big]\frac{\partial}{\partial y} + \frac{\epsilon_1^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\epsilon_2^2}{2}\frac{\partial^2}{\partial y^2}.$$

It is well-known that the existence of a Lyapunov function satisfying the above definition implies that the corresponding system of SDEs is stable.

Now that we have defined the key terms and concepts, we give a brief outline of the proof technique we will use.

1.2. **Proof Method.** Three different Hamiltonian systems are examined in sections 2, 3, and 4 of this paper. Each section will begin with a brief discussion of the deterministic characteristics of the Hamiltonian system. This will be followed by an explanation of how the deterministic modifications for that system were chosen to promote noise-induced stabilization, and then a proof that the resulting system is indeed stable.

The method of proof is the same for all of the cases studied in this paper. First, the plane is divided into overlapping regions. Then, local Lyapunov functions are found for each of those regions. By local Lyapunov function, we mean a function that satisfies the three conditions in Definition 3 on some subset of the plane. In some regions, "natural" Lyapunov functions exist, such as the norm to some power. In other regions (namely, regions where the deterministic dynamics are unstable), local Lyapunov functions must be constructed using an algorithm that "propagates" the Lyapunov function from a neighboring region. This technique will later be discussed in greater detail. The final step of the proof is to patch together the local Lyapunov functions to create one global Lyapunov function, the existence of which proves the stability of the system.

Because of the requirement that the global Lyapunov function be infinitely differentiable on all of $\mathbb{R}^2$, the local Lyapunov functions for neighboring regions must be combined smoothly on overlapping regions. This is done with the use of a mollifier function $\phi(t)$, defined as follows:

$$\phi(t) = \frac{1}{m}\int_{-\infty}^{t} \psi(s)ds \text{ with } m = \int_{-\infty}^{\infty} \psi(s)ds$$

and

$$\psi(t) = \begin{cases} \exp\left(\frac{-1}{1-(2t-1)^2}\right) & \text{for } 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

As an example of how this works, consider two overlapping regions $\mathcal{R}_1$ and $\mathcal{R}_2$, with corresponding convex local Lyapunov functions $v_1(x, y)$ and $v_2(x, y)$. We can construct some function $s(x, y)$ such that $s = 0$ on the boundary of $\mathcal{R}_1$ and $s = 1$ on the boundary of $\mathcal{R}_2$. We can then use the mollifier create a new convex combination, $v_{12}(x, y)$, on the overlapping region, where

$$v_{12}(x, y) = \phi(s(x, y))v_1(x, y) + (1 - \phi(s(x, y)))v_2(x, y).$$

This creates a function which is equal to v_2 on the boundary of $\mathcal{R}_1$, equal to v_1 on the boundary of $\mathcal{R}_2$, and infinitely differentiable on the interior of $\mathcal{R}_1 \cap \mathcal{R}_2$. The rest of the proof consists of showing that this newly constructed function satisfies the third Lyapunov condition as well, making it a local Lyapunov function on the overlapping region. Once local Lyapunov functions have have been shown to exist on all regions, and there exist functions on the overlap regions which smoothly connect the different pieces, a global Lyapnuov function can be constructed for the entire plane.

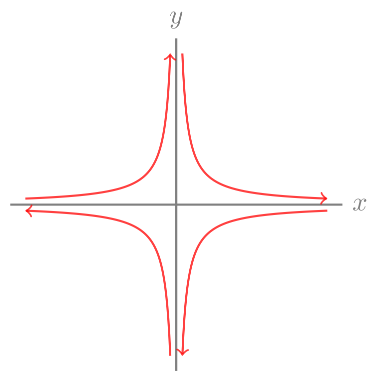

FIGURE 1. Phase portrait of the deterministic Hamiltonian system with $h' > 0$

After proving noise-induced stabilization for the three different Hamiltonian systems, the final section of the paper will address questions that remain to be answered and suggest directions for future research in this area.

2. Hamiltonian System with $H(x,y) = h\left(\frac{(xy)^2}{2}\right)$

We consider the Hamiltonian function

$$H(x,y) = h\left(\frac{(xy)^2}{2}\right) \tag{1}$$

where h is infinitely differentiable on $\mathbb{R}_{\geq 0}$ and $|h'(u)| > 0$ when $u > 0$. The corresponding deterministic Hamiltonian system (x_t, y_t) is the solution to the following two-dimensional system of ODEs:

$$\begin{aligned} \frac{dx_t}{dt} &= \frac{\partial H}{\partial y} &&= h'\left(\frac{(x_t y_t)^2}{2}\right) x_t^2 y_t \\ \frac{dy_t}{dt} &= -\frac{\partial H}{\partial x} &&= -h'\left(\frac{(x_t y_t)^2}{2}\right) x_t y_t^2 \ . \end{aligned} \tag{2}$$

Using the fact that the Hamiltonian function is constant along the solution path, the explicit solution can be found as

$$\begin{aligned} x_t &= x_0 e^{x_0 y_0 h'\left(\frac{(x_0 y_0)^2}{2}\right) t} \\ y_t &= y_0 e^{-x_0 y_0 h'\left(\frac{(x_0 y_0)^2}{2}\right) t} \ . \end{aligned}$$

Hence both the x-axis and y-axis are a continuum of equilibrium points, but for any initial condition off the axes, the solution has the property that

$$y_t = \frac{x_0 y_0}{x_t} \ . \tag{3}$$

You can see in Figure 1 that the system is clearly unstable with solutions going to infinity in each quadrant. However if the addition of noise allows for solutions to cross between quadrants it appears that the solutions will then change directions in each quadrant allowing for a quasi-periodic orbit to occur, so that solutions are bounded and noise induced stabilization can occur.

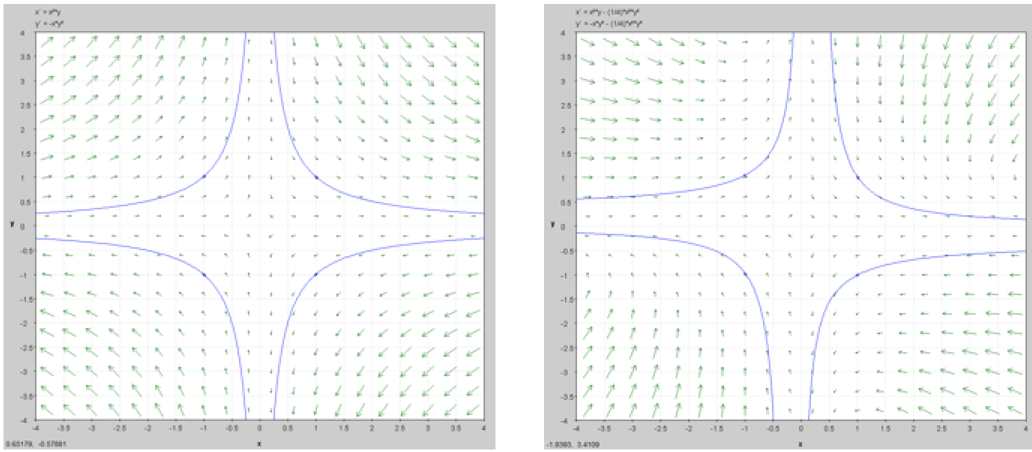

FIGURE 2. Solutions to the original Hamiltonian (left) and modified deterministic system (right) for identical initial conditions.

2.1. **Perturbed Hamiltonian System.** In order to choose the precise form for the additional drift terms, if we let $H_t = H(x_t, y_t)$ we know for Hamiltonian systems that H_t is always a constant, so $\frac{dH_t}{dt} = 0$. To enable noise-induced stabilization to occur, we need to be able to push H_t towards zero. So for this Hamiltonian system we have

$$
\begin{aligned}
\frac{dx_t}{dt} &= \quad (h'\left(\frac{(x_t y_t)^2}{2}\right)x_t y_t - \frac{1}{4}\left|h'\left(\frac{(x_t y_t)^2}{2}\right)\right|(x_t y_t)^2)x_t \\
\frac{dy_t}{dt} &= (-h'\left(\frac{(x_t y_t)^2}{2}\right)x_t y_t - \frac{1}{4}\left|h'\left(\frac{(x_t y_t)^2}{2}\right)\right|(x_t y_t)^2)y_t \ .
\end{aligned} \tag{4}
$$

As with the original Hamiltonian system, the axes consist of a continuum of equilibrium points, while for $x_0 y_0 > 0$, $\lim_{t\to\infty} |x_t| = \infty$ and for $x_0 y_0 < 0$, $\lim_{t\to\infty} |y_t| = \infty$. Thus the additional drift terms in the modified Hamiltonian system given by (4) preserve many of the same qualitative features, including the instability, of the original Hamiltonian system given by (2). The main difference of the modified Hamiltonian system is that its convergence to infinity along the axes is slightly slower than the exponential convergence of the original Hamiltonian system.

We now consider perturbing the modified Hamiltonian system given by (4) with additive white noise to form the following two-dimensional system of stochastic differential equations:

$$
\begin{aligned}
\frac{dX_t}{dt} &= \quad (h'\left(\frac{(X_t Y_t)^2}{2}\right)X_t Y_t - \frac{1}{4}\left|h'\left(\frac{(X_t Y_t)^2}{2}\right)\right|(X_t Y_t)^2)X_t + \epsilon_1 \frac{dB_1(t)}{dt} \\
\frac{dY_t}{dt} &= (-h'\left(\frac{(X_t Y_t)^2}{2}\right)X_t Y_t - \frac{1}{4}\left|h'\left(\frac{(X_t Y_t)^2}{2}\right)\right|(X_t Y_t)^2)Y_t + \epsilon_2 \frac{dB_2(t)}{dt} \ .
\end{aligned} \tag{5}
$$

The phase portraits shown in Figure 2 shows the difference in solutions from the original deterministic system to the perturbed system. Solutions in the perturbed system while unstable in the same way as the original, can be seen to be pushed

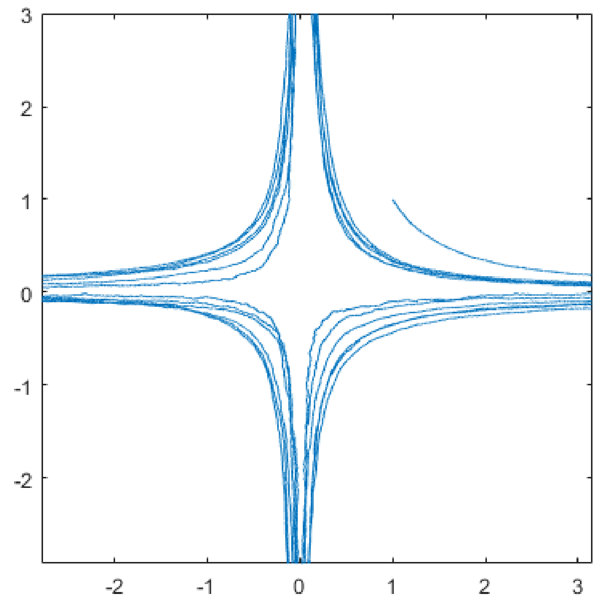

Figure 3. Simulation of the perturbed Hamiltonian system.

towards the axis more quickly giving the desired behavior for noise-induced stabilization to be possible. The simulation in Figure 3 gives a Matlab simulation of a solution for the perturbed system with added noise, using $\epsilon_1 = \epsilon_2 = 0.01$ and initial condition $(1, 1)$. This shows the quasi-periodic behavior intended, but does not give a rigorous proof of stability.

2.2. **Lyapunov Construction.** In order to find local Lyapunov functions, we begin by decomposing the plane into the following regions:

$$\begin{aligned} \mathcal{R}_1 &= \{(x, y) : |xy| \geq c_1\} \\ \mathcal{R}_2 &= \{(x, y) : |x| \geq 1, |xy| \leq c_2\} \\ \mathcal{R}_3 &= \{(x, y) : |y| \geq 1, |xy| \leq c_2\} \end{aligned} \tag{6}$$

where $0 < c_1 < c_2$. The three regions, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, cover the entire plane, minus some ball about the origin, and are depicted in Figure 4. $\mathcal{R}_1$ is the "priming region" where a natural Lyapunov function exists, namely the norm to some power. $\mathcal{R}_2$ and $\mathcal{R}_3$ are "diffusive regions" where the deterministic dynamics are unstable and noise is essential to the existence of a local Lyapunov function.

Due to symmetry in x and y, the existence of a local Lyapunov function $v_2(x, y)$ on $\mathcal{R}_2$ implies that $v_3(x, y) = v_2(y, x)$ is a local Lyapunov function on $\mathcal{R}_3$. Hence, it suffices to just prove the existence of local Lyapunov functions on $\mathcal{R}_1$ and $\mathcal{R}_2$. The priming region was specifically chosen so that a natural Lyapunov function, i.e. the norm to some power, exists in the region. In lemma 2.1 we show that this does indeed hold.

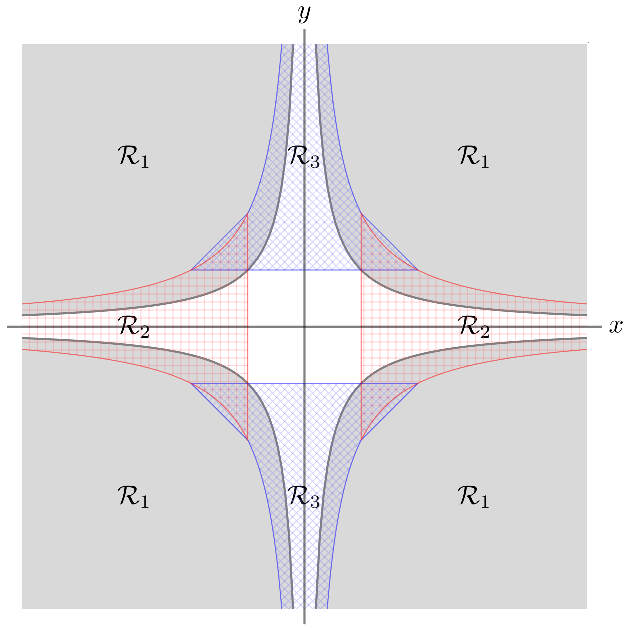

FIGURE 4. Decomposition of plane into priming and diffusive regions.

Theorem 1. *Consider* $H(x,y) = h\left(\frac{(xy)^2}{2}\right)$, *where* h *is infinitely differentiable on* $\mathbb{R}_{\geq 0}$ *and* $|h'(u)| > 0$ *when* $u > 0$. *Then the perturbed Hamiltonian system with*

$$f(x,y) = \frac{1}{4}\left|h'\left(\frac{(xy)^2}{2}\right)\right|x^3y^2 \text{ and}$$
$$g(x,y) = \frac{1}{4}\left|h'\left(\frac{(xy)^2}{2}\right)\right|x^2y^3$$

exhibits noise-induced stabilization.

Proof. In order to prove this theorem we first need to prove the existence of local Lyapunov functions satisfying the properties given above. If h' is negative the direction of solutions reverses, while maintaining the same unstable behavior. For the purposes of proving local and global Lyapunov functions we can assume without loss of generality that h' is positive.

2.2.1. *Priming Region.*

Lemma 2.1. *For any* $c_1 > 4$, $v_1(x,y) = x^2 + y^2$ *is a local Lyapunov function on* $\mathcal{R}_1$.

Proof. v_1 satisfies the first two properties of a local Lyapunov function, so it only remains to show the third property. Applying the generator to v_1 we obtain

$$(\mathcal{L}v_1)(x,y) = (h'\left(\frac{(xy)^2}{2}\right)x^2y - \frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^2)(2x) + (-h'\left(\frac{(xy)^2}{2}\right)xy^2$$
$$- \frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)(2y) + \epsilon_1^2 + \epsilon_2^2$$

$$\begin{aligned} &= (2xy - \frac{1}{2}x^2y^2)h'\left(\frac{(xy)^2}{2}\right)x^2 + (-2xy - \frac{1}{2}x^2y^2)h'\left(\frac{(xy)^2}{2}\right)y^2 + \epsilon_1^2 + \epsilon_2^2 \\ &= h'\left(\frac{(xy)^2}{2}\right)[(2xy - \frac{1}{2}x^2y^2)x^2 + (-2xy - \frac{1}{2}x^2y^2)y^2] + \epsilon_1^2 + \epsilon_2^2 \\ -(\mathcal{L}v_1)(x,y) &= h'\left(\frac{(xy)^2}{2}\right)[(-2xy + \frac{1}{2}x^2y^2)x^2 + (2xy + \frac{1}{2}x^2y^2)y^2] - \epsilon_1^2 - \epsilon_2^2. \end{aligned}$$

Now since the function $f(u) = -2u + \frac{1}{2}u^2$ is positive and strictly increasing for $u > 4$, and since $|xy| \geq c_1 > 4$ on $\mathcal{R}_1$,

$$-2|xy| + \frac{1}{2}|xy|^2 \geq -2c_1 + \frac{1}{2}c_1^2 > 0$$

and

$$\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)} [-(\mathcal{L}v_1)(x,y)] \geq (-2c_1 + \frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)r^2 - \epsilon_1^2 - \epsilon_2^2 \ .$$

Thus,

$$\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)} [-(\mathcal{L}v_1)(x,y)] \to \infty \text{ as } r \to \infty$$

and v_1 is a local Lyapunov function on $\mathcal{R}_1$. □

2.2.2. *Diffusive Regions.* On the diffusive regions $\mathcal{R}_2$ and $\mathcal{R}_3$ the deterministic dynamics are unstable and the noise terms are essential to the existence of a local Lyapunov function. Rather than using an ad hoc method to construct local Lyapunov functions on these regions, we follow the meta-algorithm outlined in [AKM12], which constructs the local Lyapunov function in a diffusive region as the solution to a boundary-value problem of the form

$$\begin{cases} (\tilde{\mathcal{L}}_i v_i)(x,y) = -g_i(x,y) & \text{for } (x,y) \in \mathcal{R}_i \\ v_i(x,y) = v_1(x,y) & \text{for } (x,y) \in \partial\mathcal{R}_i \end{cases} \tag{7}$$

where $\tilde{\mathcal{L}}_i$ consists of the terms in the generator $\mathcal{L}$ that scale dominantly in the region $\mathcal{R}_i$ and g_i is chosen so that $\lim_{r\to\infty} \left[\inf_{(x,y)\in(\mathcal{R}_i\cap B_r^c)} g_i(x,y)\right] = \infty$. This method can be viewed as "propagating" an obvious Lyapunov function to regions of the plane where a Lyapunov function is not obvious.

As mentioned previously, due to the symmetry between $\mathcal{R}_2$ and $\mathcal{R}_3$, it suffices to only consider the construction of v_2 on $\mathcal{R}_2$. In $\mathcal{R}_2$, the dominant term in the generator is $\frac{\epsilon_2^2}{2}\frac{\partial^2}{\partial y^2}$. Hence, $\tilde{\mathcal{L}}_2 = \frac{\epsilon_2^2}{2}\frac{\partial^2}{\partial y^2}$. For simplicity, we choose $g_2(x,y) = k\epsilon_2^2 x^2$, where $k > 0$ will be chosen later to ensure that v_2 is a local Lyapunov function. The function g_2 does converge to infinity on $\mathcal{R}_2$ since $\mathcal{R}_2$ consists of the decaying strip around the x-axis. With this choice for $\tilde{\mathcal{L}}_2$ and g_2, we can find an explicit solution to the boundary-value problem described by (7), which is given in the lemma below.

Lemma 2.2. *For any $c_2 > 0$ and $k > 2\frac{\max_{0\leq u\leq c_2}|h'(u^2/2)|}{\epsilon_2^2}$,*

$$v_2(x,y) = x^2 + \frac{c_2^2}{x^2} + kc_2^2 - kx^2y^2$$

is a local Lyapunov function on $\mathcal{R}_2$.

Proof. Since, $\mathcal{R}_2$ has x bounded away from zero, $v_2 \in C^\infty(\mathcal{R}_2)$. In order to show the second local Lyapunov property, note that on $\mathcal{R}_2$, $|xy| \le c_2$ implies that

$$\begin{aligned}\inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} v_2(x,y) &= \inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} [x^2 + \frac{c_2^2}{x^2} + k(c_2^2 - |xy|^2)x^2] \\ &\ge \inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} [x^2] \to \infty \text{ as } r \to \infty \ .\end{aligned}$$

As for the third local Lyapunov property,

$$\begin{aligned}(\mathcal{L}v_2)(x,y) =& (h'\left(\frac{(xy)^2}{2}\right)x^2y - \frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^2)(2x - \frac{2c_2^2}{x^3} - 2kxy^2) + (-h'\left(\frac{(xy)^2}{2}\right)xy^2 \\ &- \frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)(-2kx^2y) + \epsilon_1^2(1 + \frac{3c_2^2}{x^4} - ky^2) + \epsilon_2^2(-kx^2) \\ &= 2h'\left(\frac{(xy)^2}{2}\right)x^3y - \frac{1}{2}h'\left(\frac{(xy)^2}{2}\right)x^4y^2 - 2kh'\left(\frac{(xy)^2}{2}\right)x^3y^3 - \frac{2c_2^2y}{x}h'\left(\frac{(xy)^2}{2}\right) \\ &- \frac{c_2^2y^2}{2}h'\left(\frac{(xy)^2}{2}\right) + 2kh'\left(\frac{(xy)^2}{2}\right)x^3y^3 + kh'\left(\frac{(xy)^2}{2}\right)x^4y^4 + \epsilon_1^2 + \frac{3c_2^2}{x^4}\epsilon_1^2 - k\epsilon_1^2y^2 - k\epsilon_2^2x^2. \\ &= 2h'\left(\frac{(xy)^2}{2}\right)x^3y - \frac{1}{2}h'\left(\frac{(xy)^2}{2}\right)x^4y^2 - \frac{2c_2^2y}{x}h'\left(\frac{(xy)^2}{2}\right) - \frac{c_2^2y^2}{2}h'\left(\frac{(xy)^2}{2}\right) + kh'\left(\frac{(xy)^2}{2}\right)x^4y^4 \\ &+ \epsilon_1^2(1 + \frac{3c_2^2}{x^4} - ky^2) - k\epsilon_2^2x^2.\end{aligned}$$

Hence, since $|x| \ge 1$ and $|xy| \le c_2$ on $\mathcal{R}_2$,

$$\begin{aligned}\inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} [-(\mathcal{L}v_2)(x,y)] \ge \inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} &[(-2|xy| + \frac{1}{2}|xy|^2)h'\left(\frac{(xy)^2}{2}\right)x^2 - 2c_2^3 \\ &- kc_2^4 - \epsilon_1^2(1 + 3c_2^2) + k\epsilon_2^2x^2].\end{aligned}$$

The smallest value that $-2|xy| + \frac{1}{2}|xy|^2$ can achieve is -2. Hence, as long as we choose $k > 2\frac{\max_{0\le u\le c_2}|h'(u^2/2)|}{\epsilon_2^2}$, then

$$\inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} [-(\mathcal{L}v_2)(x,y)] \to \infty \text{ as } r \to \infty$$

and v_2 is a local Lyapunov function on $\mathcal{R}_2$. □

Therefore, by symmetry,

$$v_3(x,y) = y^2 + \frac{c_2^2}{y^2} + \tilde{k}c_2^2 - \tilde{k}x^2y^2$$

is a local Lyapunov function on $\mathcal{R}_3$, where $\tilde{k} > 2\frac{\max_{0\le u\le c_2}|h'(u^2/2)|}{\epsilon_1^2}$. Note that due to the nature of the deterministic dynamics, it is the noise in the y direction, $\epsilon_2 \neq 0$, that is crucial to the existence of a local Lyapunov function in $\mathcal{R}_2$ and it is the noise in the x direction, $\epsilon_1 \neq 0$, that is crucial to the existence of a local Lyapunov function in $\mathcal{R}_3$. Thus, in order to obtain a global Lyapunov function on the entire plane, noise is needed in both the x and y directions.

2.2.3. *Global Lyapunov Function.* Since we have shown the existence of local Lyapunov functions on regions covering the entire plane, minus some ball about the origin, we now seek to patch them together to form one smooth, global Lyapunov function satisfying 3. In addition, $s(x,y)$ is chosen so that it equals zero on the border of $\mathcal{R}_1$ and one on the border of $\mathcal{R}_2$, i.e.

$$s(x,y) = \begin{cases} \frac{|xy|-c_1}{c_2-c_1} & \text{for } c_1 \leq |xy| \leq c_2 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3. *There exist constants c_1 and c_2 with $4 < c_1 < c_2$, such that for any $k > 2\frac{\max_{0\leq u\leq c_2} |h'(u^2/2)|}{\epsilon_2^2}$,*

$$v_{12}(x,y) = \phi(s(x,y))v_1(x,y) + (1-\phi(s(x,y)))v_2(x,y)$$

is a local Lyapunov function on $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(x,y) : |x| \geq 1, c_1 \leq |xy| \leq c_2\}$.

Proof. Applying the generator to v_{12} we obtain

$$(\mathcal{L}v_{12})(x,y) = \phi(s(x,y))(\mathcal{L}v_1)(x,y) + (1-\phi(s(x,y)))(\mathcal{L}v_2)(x,y) + E(x,y)$$

where

$$\begin{aligned} E(x,y) &= \mathcal{L}[\phi(s(x,y))](v_1(x,y) - v_2(x,y)) \\ &+ \epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)] \\ &+ \epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x,y))]\frac{\partial}{\partial y}[v_1(x,y) - v_2(x,y)]. \end{aligned}$$

From the proofs of the first two lemmas we have, on $\mathcal{R}_1 \cap \mathcal{R}_2$,

$$\begin{aligned} (\mathcal{L}v_1)(x,y) &= (2xy - \frac{1}{2}(xy)^2)h'\left(\frac{(xy)^2}{2}\right)x^2 + (-2xy - \frac{1}{2}(xy)^2)h'\left(\frac{(xy)^2}{2}\right)y^2 + \epsilon_1^2 + \epsilon_2^2 \\ &\leq (2|xy| - \frac{1}{2}|xy|^2)h'\left(\frac{(xy)^2}{2}\right)x^2 + (2|xy| - \frac{1}{2}|xy|^2)h'\left(\frac{(xy)^2}{2}\right)y^2 + \epsilon_1^2 + \epsilon_2^2 \\ &\leq (2c_1 - \frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)x^2 + (2c_1 - \frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)y^2 + \epsilon_1^2 + \epsilon_2^2 \end{aligned}$$

$$\begin{aligned} (\mathcal{L}v_2)(x,y) &= 2h'\left(\frac{(xy)^2}{2}\right)x^3y - \frac{2c_2^2}{x}h'\left(\frac{(xy)^2}{2}\right)y - 2h'\left(\frac{(xy)^2}{2}\right)kx^3y^3 - \frac{1}{2}h'\left(\frac{(xy)^2}{2}\right)x^4y^2 \\ &+ \frac{1}{2}c_2^2h'\left(\frac{(xy)^2}{2}\right)y^2 + \frac{1}{2}kh'\left(\frac{(xy)^2}{2}\right)x^4y^4 + 2h'\left(\frac{(xy)^2}{2}\right)kx^3y^3 + \frac{1}{2}h'\left(\frac{(xy)^2}{2}\right)kx^4y^4 \\ &+ \epsilon_1^2 + \frac{3c_2^2}{x^4}\epsilon_1^2 - k\epsilon_1^2 - k\epsilon_1^2y^2 - k\epsilon_2^2x^2 \\ &\leq (2|xy| - \frac{1}{2}|xy|^2)h'\left(\frac{(xy)^2}{2}\right)x^2 - 2c_2^3 - kc_2^4 - \epsilon_1^2(1+3c_2^2) - k\epsilon_2^2x^2 \\ &\leq (2c_1 - \frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)x^2 - 2c_2^3 - kc_2^4 - \epsilon_1^2(1+3c_2^2) - k\epsilon_2^2x^2 \end{aligned}$$

$$\begin{aligned} &\phi(s(x,y))(\mathcal{L}v_1)(x,y) + (1-\phi(s(x,y)))(\mathcal{L}v_2)(x,y) \\ &\leq \phi(s(x,y))(2c_1 - \frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)x^2 + \phi(s(x,y))(2c_1 - \frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)y^2 \end{aligned}$$

$$
\begin{aligned}
&+\phi(s(x,y))\epsilon_1^2+\phi(s(x,y))\epsilon_2^2+(2c_1-\frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)x^2-2c_2^3-kc_2^4\\
&-\epsilon_1^2(1+3c_2^2)-k\epsilon_2^2x^2-\phi(s(x,y))(2c_1-\frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)x^2+\phi(s(x,y))2c_2^3\\
&+\phi(s(x,y))kc_2^4+\phi(s(x,y))\epsilon_1^2(1+3c_2^2)+\phi(s(x,y))k\epsilon_2^2x^2\\
&=(2c_1-\frac{1}{2}c_1^2)h'\left(\frac{(xy)^2}{2}\right)x^2+k\epsilon_2^2x^2-\phi(s(x,y))k\epsilon_2^2x^2
\end{aligned}
$$

$$
\begin{aligned}
&\phi(s(x,y))(\mathcal{L}v_1)(x,y)+(1-\phi(s(x,y)))(\mathcal{L}v_2)(x,y)\\
&\le(2|xy|-\frac{1}{2}|xy|^2-(1-\phi(s(x,y)))k\epsilon_2^2)x^2+C\\
&\le(2c_1-\frac{1}{2}c_1^2-(1-\phi(s(x,y)))k\epsilon_2^2)x^2+C
\end{aligned}
$$

where C is a constant.

$$
\begin{aligned}
E(x,y)&=(h'\left(\frac{(xy)^2}{2}\right)x^2y^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^3)\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)(y^2-\frac{c_2^2}{x^2}-kc_2^2+kx^2y^2)\\
&+(-h'\left(\frac{(xy)^2}{2}\right)xy^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)(y^2-\frac{c_2^2}{x^2}-kc_2^2+kx^2y^2)\\
&+\frac{1}{2}\epsilon_1^2\phi''(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)^2(y^2-\frac{c_2^2}{x^2}-kc_2^2+kx^2y^2\\
&+\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)(y^2-\frac{c_2^2}{x^2}-kc_2^2+kx^2y^2)\\
&+\epsilon_1^2\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)(\frac{2c_2^2}{x^3}+ky^2)+\epsilon_2^2\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)(2y+2kx^2y)\\
&=(h'\left(\frac{(xy)^2}{2}\right)x^2y^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^3)\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)y^2\\
&-(h'\left(\frac{(xy)^2}{2}\right)x^2y^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^3)\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)\frac{c_2^2}{x^2}\\
&-(h'\left(\frac{(xy)^2}{2}\right)x^2y^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^3)\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)kc_2^2\\
&+(h'\left(\frac{(xy)^2}{2}\right)x^2y^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^3y^3)\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)kx^2y^2\\
&+(-h'\left(\frac{(xy)^2}{2}\right)xy^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)y^2\\
&-(-h'\left(\frac{(xy)^2}{2}\right)xy^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)\frac{c_2^2}{x^2}\\
&-(-h'\left(\frac{(xy)^2}{2}\right)xy^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)kc_2^2\\
&+(-h'\left(\frac{(xy)^2}{2}\right)xy^2-\frac{1}{4}h'\left(\frac{(xy)^2}{2}\right)x^2y^3)\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)kx^2y^2
\end{aligned}
$$

$$+\frac{1}{2}\epsilon_1^2\phi''(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)^2y^2-\frac{1}{2}\epsilon_1^2\phi''(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)^2\frac{c_2^2}{x^2}-\frac{1}{2}\epsilon_1^2\phi''(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)^2kc_2^2$$
$$+\frac{1}{2}\epsilon_1^2\phi''(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)^2kx^2y^2+\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)y^2-\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)\frac{c_2^2}{x^2}$$
$$-\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)kc_2^2+\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)kx^2y^2+\epsilon_1^2\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)\frac{2c_2^2}{x^3}$$
$$+\epsilon_1^2\phi'(s)\left(\frac{\operatorname{sgn}(x)|y|}{c_2-c_1}\right)ky^2+\epsilon_2^2\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)2y+\epsilon_2^2\phi'(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)2kx^2y$$

$$\begin{aligned}E(x,y)&\simeq\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)^2kx^2y^2-\frac{1}{2}\epsilon_2^2\phi''(s)\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)^2kc_2^2+2\epsilon_2^2\phi'\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)kx^2y\\&=\frac{1}{2}k\epsilon_2^2\phi''(s)x^2\left(\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)^2y^2-\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)^2c_2^2\right)+2\epsilon_2^2\phi'\left(\frac{\operatorname{sgn}(y)|x|}{c_2-c_1}\right)kx^2y\\&=k\epsilon_2^2\left(\frac{1}{2}\phi''(s)\left(\frac{|xy|^2-c_2^2}{(c_2-c_1)^2}\right)+2\phi'(s)\left(\frac{|xy|}{c_2-c_1}\right)\right)x^2\end{aligned}$$

From explicit computation, we have that on $\mathcal{R}_1\cap\mathcal{R}_2$ the asymptotic behavior of $E(x,y)$ is

$$E(x,y)\simeq k\epsilon_2^2\left[\frac{2\phi'(s(x,y))|xy|}{c_2-c_1}-\frac{1}{2}\phi''(s(x,y))\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}\right]x^2.$$

Now there exists $M>0$ such that $|\phi'(t)|\leq M$ and $|\phi''(t)|\leq M$ for all t. Then

$$\begin{aligned}&k\epsilon_y^2\left[\frac{2\phi'(h(x,y))|xy|}{c_2-c_1}-\frac{1}{2}\phi''(h(x,y))\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}\right]\\&\leq\left|k\epsilon_y^2\left[\frac{2\phi'(h(x,y))|xy|}{c_2-c_1}-\frac{1}{2}\phi''(h(x,y))\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}\right]\right|\\&=\left|\frac{2\phi'(h(x,y))|xy|}{c_2-c_1}k\epsilon_y^2-\frac{1}{2}\phi''(h(x,y))\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}k\epsilon_y^2\right|\\&\leq\left|\frac{2\phi'(h(x,y))|xy|}{c_2-c_1}k\epsilon_y^2\right|+\left|-\frac{1}{2}\phi''(h(x,y))\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}k\epsilon_y^2\right|\\&=\frac{2|\phi'(h(x,y))||xy|}{c_2-c_1}k\epsilon_y^2+\frac{1}{2}|\phi''(h(x,y))|\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}k\epsilon_y^2\\&\leq\frac{2M|xy|}{c_2-c_1}k\epsilon_y^2+\frac{1}{2}M\frac{c_2^2-|xy|^2}{(c_2-c_1)^2}k\epsilon_y^2\\&=\frac{M}{c_2-c_1}k\epsilon_y^2\left(2|xy|+\frac{1}{2}\frac{c_2^2-|xy|^2}{c_2-c_1}\right)\\&\leq\frac{M}{c_2-c_1}k\epsilon_y^2\left(2c_2+\frac{1}{2}\frac{c_2^2-c_1^2}{c_2-c_1}\right)\\&=\frac{M}{c_2-c_1}k\epsilon_y^2\left(2c_2+\frac{1}{2}(c_2+c_1)\right)\\&\leq\frac{M}{c_2-c_1}k\epsilon_y^2\left(2c_2+\frac{1}{2}(c_2+c_2)\right)\\&=\frac{M}{c_2-c_1}k\epsilon_y^2\left(2c_2+c_2\right)\end{aligned}$$

$$= \frac{3Mc_2}{c_2 - c_1} k\epsilon_y^2$$

for all (x, y) in $\mathcal{R}_1 \cap \mathcal{R}_2$. Therefore, as long as

$$\frac{1}{2}c_1^2 - 2c_1 > k\epsilon_2^2 \frac{3Mc_2}{c_2 - c_1},$$

v_{12} will satisfy the third local Lyapunov property. This can be achieved by setting $c_2 = c_1 + 1$ and choosing c_1 sufficiently large. □

By symmetry, c_1 and c_2 can also be chosen so that

$$v_{13}(x, y) = \phi(s(x, y))v_1(x, y) + (1 - \phi(s(x, y)))v_3(x, y)$$

is a local Lyapunov function on $\mathcal{R}_1 \cap \mathcal{R}_3 = \{(x, y) : |y| \geq 1, c_1 \leq |xy| \leq c_2\}$. Our global Lyapunov function, $V(x, y) \in C^\infty(\mathbb{R}^2)$, can then be constructed so that

$$V(x, y) = \begin{cases} \tilde{V}(x, y) & \text{for } x^2 + y^2 > \rho^2 \\ \text{arbitrary positive and smooth} & \text{for } x^2 + y^2 \leq \rho^2 \end{cases}$$

where $\rho > 2c_2$ and

$$\tilde{V}(x, y) = \begin{cases} v_1(x, y) & \text{for } (x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3^c \\ v_2(x, y) & \text{for } (x, y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \\ v_3(x, y) & \text{for } (x, y) \in \mathcal{R}_1^c \cap \mathcal{R}_3 \\ v_{12}(x, y) & \text{for } (x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2 \\ v_{13}(x, y) & \text{for } (x, y) \in \mathcal{R}_1 \cap \mathcal{R}_3. \end{cases}$$

Therefore since we have proven the existence of a global Lyapunov function, we know that this perturbed Hamiltonian system exhibits noise induced stabilization. □

3. Hamiltonian System with $H(x, y) = ax^m y^n$

We consider the Hamiltonian function

$$H(x, y) = ax^m y^n \tag{8}$$

where a is some nonzero constant, and m and n are positive integers. The corresponding deterministic Hamiltonian system (x_t, y_t) is the solution to the following two-dimensional system of ODEs:

$$\begin{aligned} \frac{dx_t}{dt} &= \frac{\partial H}{\partial y} = anx_t^m y_t^{n-1} \\ \frac{dy_t}{dt} &= -\frac{\partial H}{\partial x} = -amx_t^{m-1} y_t^n. \end{aligned} \tag{9}$$

For $m, n \geq 2$, the x-axis and the y-axis are a continuum of equilibrium points. For $m = 1$ and $n \geq 2$, only the x-axis is a continuum of equilibrium points. For $m \geq 2$ and $n = 1$, only the y-axis is a continuum of equilibrium points. For $m = n = 1$, only the origin is an equilibrium point. If $x_0^m y_0^n \neq 0$, then the solution of the system has the property that

$$y_t = \pm \sqrt[n]{\left| \frac{x_0^m y_0^n}{x^m} \right|}. \tag{10}$$

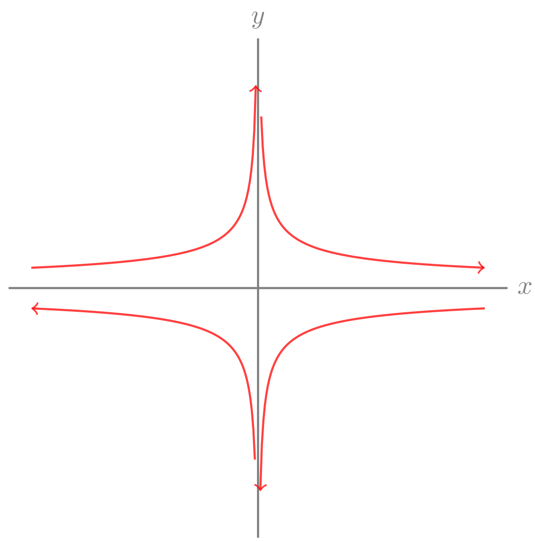

FIGURE 5. Phase portrait of the deterministic Hamiltonian system with $a > 0$ and even m, n with $m < n$

From this, we can see that the deterministic system is unstable, since either $\lim_{t\to\infty} |x_t| = \infty$ or either $\lim_{t\to\infty} |y_t| = \infty$ for any initial condition off of the axes. Because the solution converges to infinity for certain initial conditions, the system is unstable according to the definition of stability given in Section 1.

Although the system is unstable, it possesses characteristics which may lead one to guess that the system will exhibit noise-induced stabilization. For initial conditions off the axes, the solution to the deterministic process remains within the same quadrant for all time, approaching positive infinity or negative infinity along one of the axes as $t \to \infty$. This behavior is visible in Figure 5; different values of the constants m and n in the Hamiltonian function will make the curve change its precise shape. Given this shape, one might guess that the addition of white noise will enable the perturbed process to cross the axis and form a quasi-periodic behavior. For example, for an initial condition in the first quadrant, it would seem that the solution curve will cross the x-axis when the curve is sufficiently close to it, then cross the y-axis, and so on. However, additive white noise is not sufficient for the existence of an invariant probability measure, as explained in Section 1. Hence the system with noise is not stochastically bounded.

We wish to modify the Hamiltonian system defined by (11) in order to create a system which exhibits noise-induced stabilization, and yet retains essentially the same qualitative behavior of the original deterministic dynamics. Due to the fact that any unstable Hamiltonian system remains unstable after the addition of white noise, any modification sufficient to produce noise-induced stabilization must break the Hamiltonian structure. In order to preserve the qualitative behavior of the deterministic dynamics, we add a drift term which points towards the axes but which do not change the limiting behavior of solutions for any initial condition. In the next section we discuss some intuition for the specific drift terms added in order to produce noise-induced stabilization in this particular problem. We then apply the meta-algorithm for the construction of a Lyapunov function in order to prove that the perturbed modified Hamiltonian system is indeed stable.

3.1. **Perturbed Hamiltonian System.** In this problem, we begin with the following deterministic Hamiltonian system:

$$\begin{aligned} \frac{dx_t}{dt} &= anx_t^m y_t^{n-1} \\ \frac{dy_t}{dt} &= -amx_t^{m-1} y_t^n . \end{aligned} \tag{11}$$

As shown in the previous section, this Hamiltonian system is unstable and remains unstable after perturbation by additive white noise. We wish to modify (11) in order to create a new system which exhibits noise-induced stabilization and yet preserves the qualitative features of the original Hamiltonian system, namely the limiting behavior of the solution curves for all initial conditions. Our approach is to add drift terms which point toward the axes, but which scale subdominantly to the original terms from the unperturbed system near the axis. While the original Hamiltonian system remained unstable after the addition of white noise, the additional drift terms in the modified Hamiltonian system should allow for the noise to have a stabilizing effect. The specific $f(x,y)$ and $g(x,y)$ which allow for noise-induced stabilization are given by

$$\begin{aligned} f(x,y) &= a^2n^2x^{2m-1}y^{2n-2} \text{ and} \\ g(x,y) &= a^2m^2x^{2m-2}y^{2n-1}. \end{aligned} \tag{12}$$

This gives us the following S_P:

$$\begin{aligned} \frac{dx_t}{dt} &= (anx_t^{m-1}y_t^{n-1} - (anx_t^{m-1}y_t^{n-1})^2)x_t \\ \frac{dy_t}{dt} &= (-amx_t^{m-1}y_t^{n-1} - (amx_t^{m-1}y_t^{n-1})^2)y_t \ . \end{aligned} \tag{13}$$

This perturbed system has the same set of equilibrium points as the unperturbed system has. Thus the additional drift terms in the modified Hamiltonian system given by (13) preserves many of the qualitative features, including the instability, of the original Hamiltonian system given by (11). Figure 6 shows a side-by-side comparison of phase portraits for the original system and the modified deterministic system.

We now consider further modifying the perturbed system given by (13) by the addition of white noise to form the following two-dimensional system of stochastic differential equations:

$$\begin{aligned} \frac{dX_t}{dt} &= (anX_t^{m-1}Y_t^{n-1} - (anX_t^{m-1}Y_t^{n-1})^2)X_t + \epsilon_1\frac{dB_1(t)}{dt} \\ \frac{dY_t}{dt} &= (-amX_t^{m-1}Y_t^{n-1} - (amX_t^{m-1}Y_t^{n-1})^2)Y_t + \epsilon_2\frac{dB_2(t)}{dt}. \end{aligned} \tag{14}$$

Here $B_1(t)$ and $B_2(t)$ are independent Brownian motions and $\epsilon_1, \epsilon_2 > 0$ represent the strength of the noise in the $x-$ and $y-$directions, respectively. Figure 7 displays a simulation of the solution to (14) with $\epsilon_1 = \epsilon_2 = .01$ and initial condition $(1,1)$, zoomed in about the origin. While the deterministic system was constrained to stay in the same quadrant based on its initial condition and converged to infinity along one of the axes, we observe that the perturbed process exhibits a quasi-periodic

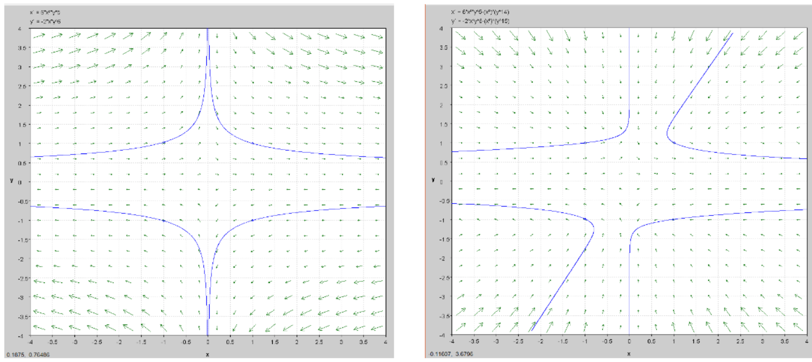

FIGURE 6. Solutions to the original Hamiltonian (left) and modified deterministic system (right) for identical initial conditions

behavior where it travels above and below the axis. While the perturbed process may travel far out along the axes, it does not go off to infinity and remains stochastically bounded. We prove that the modified Hamiltonian system does indeed exhibit noise-induced stabilization in the next section through the construction of a Lyapunov function.

Theorem 2. *Consider* $H(x, y) = ax^m y^n$*, where* a *is an arbitrary constant and* m, n *are positive integers. Then the perturbed Hamiltonian system with*

$$f(x, y) = a^2 n^2 x^{2m-1} y^{2n-2} \text{ and}$$
$$g(x, y) = a^2 m^2 x^{2m-2} y^{2n-1}$$

exhibits noise-induced stabilization.

3.2. **Lyapunov Construction.** We begin by decomposing the plane into the following regions:

$$(15)\qquad \begin{aligned} \mathcal{R}_1 &= \{(x, y) : |x|^{m-1}|y|^{n-1} \geq c_1\} \\ \mathcal{R}_2 &= \{(x, y) : x^2 \geq 1, y^2 \leq \frac{1}{2k}, |x|^{m-1}|y|^{n-1} \leq c_2\} \\ \mathcal{R}_3 &= \{(x, y) : y^2 \geq 1, x^2 \leq \frac{1}{2\tilde{k}}, |x|^{m-1}|y|^{n-1} \leq c_2\} \end{aligned}$$

where $0 < c_1 < c_2$ and $c_1 < c_3$. The precise values of the constants will be specified later to facilitate local Lyapunov function constructions and patching. The three regions, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, cover the entire plane, minus some ball about the origin, similarly to the decomposition of the plane depicted in Figure 4 from Section 2. $\mathcal{R}_1$ is the "priming region" where a natural Lyapunov function exists, namely the norm to some power. $\mathcal{R}_2$ and $\mathcal{R}_3$ are "diffusive regions" where the deterministic dynamics are unstable and noise is essential to the existence of a local Lyapunov function.

We seek to show the existence of local Lyapunov functions, v_1, v_2, and v_3 on $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{R}_3$, respectively.

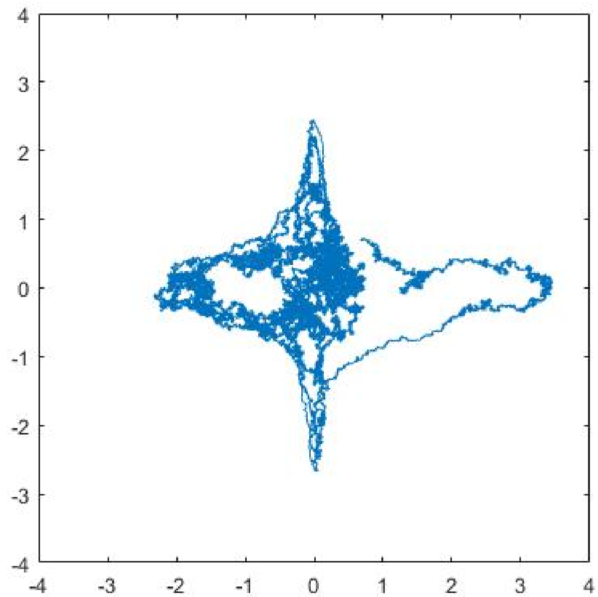

FIGURE 7. A simulation of the modified Hamiltonian system with noise

3.2.1. *Priming Region.* The priming region was specifically chosen so that a natural Lyapunov function, i.e. the norm to some power, exists in the region. In Lemma 3.1 we show that this does indeed hold. In the following five lemmas, we assume without loss of generality that $m < n$.

Lemma 3.1. *For any* $c_1 > \frac{n}{|a|m^2}$, $v_1(x, y) = x^2 + y^2$ *is a local Lyapunov function on* $\mathcal{R}_1$.

Proof. v_1 is clearly infinitely differentiable on $\mathcal{R}_1$. So, $v_1 \in C^\infty(\mathcal{R}_1)$ and the first property of a Lyapunov functions holds for v_1. Also,

$$\lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)} v_1(x,y)\right] = \lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)} x^2+y^2\right] = \lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)} r^2\right] = \infty,$$

and so v_1 satisfies the second property of a Lyapunov function. So, it only remains to show the third property. Applying the generator to v_1 we obtain

$$\begin{aligned}
(\mathcal{L}v_1)(x,y) &= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]x(2x) \\
&\quad + [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]y(2y) + \frac{1}{2}\epsilon_1^2(2) + \frac{1}{2}\epsilon_2^2(2) \\
&= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]2x^2 \\
&\quad + [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]2y^2 + \epsilon_1^2 + \epsilon_2^2 \\
&\le [(|an||x|^{m-1}|y|^{n-1}) - (|an||x|^{m-1}|y|^{n-1})^2]2x^2 \\
&\quad + [(|am||x|^{m-1}|y|^{n-1}) - (|am||x|^{m-1}|y|^{n-1})^2]2y^2 + \epsilon_1^2 + \epsilon_2^2 \\
&\le [(|an||x|^{m-1}|y|^{n-1}) - (|am||x|^{m-1}|y|^{n-1})^2]2x^2
\end{aligned}$$

$$\begin{aligned}
&+ [(|an||x|^{m-1}|y|^{n-1}) - (|am||x|^{m-1}|y|^{n-1})^2]2y^2 + \epsilon_1^2 + \epsilon_2^2. \\
&= [(|an||x|^{m-1}|y|^{n-1}) - (|am||x|^{m-1}|y|^{n-1})^2]2(x^2+y^2) + \epsilon_1^2 + \epsilon_2^2 \\
&= [(|an||x|^{m-1}|y|^{n-1}) - (|am||x|^{m-1}|y|^{n-1})^2]2r^2 + \epsilon_1^2 + \epsilon_2^2.
\end{aligned}$$

That is,

$$-(\mathcal{L}v_1)(x,y) \geq [-(|an||x|^{m-1}|y|^{n-1}) + (|am||x|^{m-1}|y|^{n-1})^2]2r^2 - \epsilon_1^2 - \epsilon_2^2.$$

The function $p(u) := -|an|u + (|am|u)^2$ is positive and strictly increasing for $u > \frac{n}{|a|m^2}$. Thus, since $|x|^{m-1}|y|^{n-1} \geq c_1 > \frac{n}{|a|m^2} > 0$ on $\mathcal{R}_1$, we have that $p(|x|^{m-1}|y|^{n-1}) \geq p(c_1) > p(0)$, and so

$$-(|an||x|^{m-1}|y|^{n-1}) + (|am||x|^{m-1}|y|^{n-1})^2 \geq -(|an|c_1) + (|am|c_1)^2 > 0.$$

Therefore,

$$-(\mathcal{L}v_1)(x,y) \geq [-(|an|c_1) + (|am|c_1)^2]2r^2 - \epsilon_1^2 - \epsilon_2^2$$

and so

$$\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)}[-(\mathcal{L}v_1)(x,y)] \geq \inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)}[[-(|an|c_1) + (|am|c_1)^2]2r^2 - \epsilon_1^2 - \epsilon_2^2]$$

which implies

$$\begin{aligned}
&\lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)}[-(\mathcal{L}v_1)(x,y)]\right] \\
&\quad \geq \lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap B_r^c)}[[-(|an|c_1) + (|am|c_1)^2]2r^2 - \epsilon_1^2 - \epsilon_2^2]\right] \\
&\quad = \infty
\end{aligned}$$

thus proving the third Lyapunov property. Therefore, v_1 is a local Lyapunov function on $\mathcal{R}_1$. □

3.2.2. *Diffusive Regions.* We follow a similar procedure as described in Section 2.2.2 for constructing local Lyapunov functions on the diffusive regions. However, for this Hamiltonian system, we use the asymptotic behavior of v_1 as the boundary condition instead of the exact v_1 for simplicity.

Lemma 3.2. *For any $c_2 > 0$ and $k > \frac{1}{2\epsilon_2^2}$,*

$$v_2(x,y) = x^2(1-ky^2)$$

is a local Lyapunov function on $\mathcal{R}_2$.

Proof. v_2 is clearly infinitely differentiable on $\mathcal{R}_2$. So, $v_2 \in C^\infty(\mathcal{R}_2)$ and the first property of a Lyapunov functions holds for v_2.

In order to show the second local Lyapunov property, note that on $\mathcal{R}_2$,

$$y^2 \leq \frac{1}{2k} \implies ky^2 \leq \frac{1}{2} \implies 0 \leq \frac{1}{2} - ky^2 \implies \frac{1}{2} \leq 1 - ky^2$$

and so

$$\inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} v_2(x,y) = \inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} [x^2(1-ky^2)]$$
$$\geq \inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)} [\frac{1}{2}x^2] \to \infty \text{ as } r\to\infty \ .$$

As for the third local Lyapunov property,

$$\begin{aligned}
(\mathcal{L}v_2)(x,y) &= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]x(2x-2kxy^2) \\
&+ [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]y(-2kx^2y) \\
&+ \frac{1}{2}\epsilon_1^2(2-2ky^2) + \frac{1}{2}\epsilon_2^2(-2kx^2) \\
&= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2](2x^2-2kx^2y^2) \\
&+ [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2](-2kx^2y^2) \\
&+ \epsilon_1^2(1-ky^2) + \epsilon_2^2(-kx^2) \\
&\simeq [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]2x^2 - \epsilon_2^2kx^2 \\
&\leq [(|an||x|^{m-1}|y|^{n-1}) - (|an||x|^{m-1}|y|^{n-1})^2]2x^2 - \epsilon_2^2kx^2
\end{aligned}$$

That is,

$$(-\mathcal{L}v_2)(x,y) \gtrsim \{[-(|an||x|^{m-1}|y|^{n-1}) + (|an||x|^{m-1}|y|^{n-1})^2]2 + \epsilon_2^2k\}x^2.$$

The minimum of $q(u) := -|an|u + (|an|u)^2$ occurs when $u = \frac{1}{2|an|}$. In particular, the minimum value is equal to $-\frac{1}{4}$. Thus, letting $u = |x|^{m-1}|y|^{n-1}$, we see that

$$-(|an||x|^{m-1}|y|^{n-1}) + (|an||x|^{m-1}|y|^{n-1})^2 \geq -\frac{1}{4}.$$

So,

$$(-\mathcal{L}v_2)(x,y) \gtrsim [-\frac{1}{4}2 + \epsilon_2^2k]x^2 = (-\frac{1}{2} + \epsilon_2^2k)x^2.$$

Since $k > \frac{1}{2\epsilon_2^2}$, it follows that $-\frac{1}{2} + \epsilon_2^2k > 0$. Thus,

$$\lim_{r\to\infty}\Big[\inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)}[-(\mathcal{L}v_2)(x,y)]\Big] \geq \lim_{r\to\infty}\Big[\inf_{(x,y)\in(\mathcal{R}_2\cap B_r^c)}[(-\frac{1}{2}+\epsilon_2^2k)x^2]\Big] = \infty$$

and so v_2 satisfies the third Lyapunov property. Thus, v_2 is a local Lyapunov function on $\mathcal{R}_2$. ☐

Lemma 3.3. *For any $c_3 > 0$ and $\tilde{k} > \frac{1}{2\epsilon_1^2}$,*

$$v_3(x,y) = y^2(1-\tilde{k}x^2)$$

is a local Lyapunov function on $\mathcal{R}_3$.

Proof. v_3 is clearly infinitely differentiable on $\mathcal{R}_3$. So, $v_3 \in C^\infty(\mathcal{R}_3)$ and the first property of a Lyapunov functions holds for v_3.

In order to show the second local Lyapunov property, note that on $\mathcal{R}_3$,

$$x^2 \leq \frac{1}{2\tilde{k}} \implies \tilde{k}x^2 \leq \frac{1}{2} \implies 0 \leq \frac{1}{2} - \tilde{k}x^2 \implies \frac{1}{2} \leq 1 - \tilde{k}x^2$$

and so

$$\begin{aligned}\inf_{(x,y)\in(\mathcal{R}_3\cap B_r^c)} v_3(x,y) &= \inf_{(x,y)\in(\mathcal{R}_3\cap B_r^c)} [y^2(1-\tilde{k}x^2)] \\ &\geq \inf_{(x,y)\in(\mathcal{R}_3\cap B_r^c)} [\frac{1}{2}y^2] \to \infty \text{ as } r\to\infty.\end{aligned}$$

As for the third local Lyapunov property,

$$\begin{aligned}(\mathcal{L}v_3)(x,y) &= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]x(-2\tilde{k}y^2x) \\ &+ [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]y(2y-2\tilde{k}yx^2) \\ &+ \frac{1}{2}\epsilon_1^2(-2\tilde{k}y^2) + \frac{1}{2}\epsilon_2^2(2-2\tilde{k}x^2) \\ &= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2](-2\tilde{k}y^2x^2) \\ &+ [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2](2y^2-2\tilde{k}y^2x^2) \\ &+ \epsilon_1^2(-\tilde{k}y^2) + \epsilon_2^2(1-\tilde{k}x^2) \\ &\simeq [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]2y^2 - \epsilon_1^2\tilde{k}y^2 \\ &\leq [(|am||x|^{m-1}|y|^{n-1}) - (|am||x|^{m-1}|y|^{n-1})^2]2y^2 - \epsilon_1^2\tilde{k}y^2.\end{aligned}$$

That is,

$$(-\mathcal{L}v_3)(x,y) \gtrsim \{[-(|am||x|^{m-1}|y|^{n-1}) + (|am||x|^{m-1}|y|^{n-1})^2]2 + \epsilon_1^2\tilde{k}\}y^2.$$

The minimum of $r(u) := -|am|u + (|am|u)^2$ occurs when $u = \frac{1}{2|am|}$. In particular, the minimum value is equal to $-\frac{1}{4}$. Thus, letting $u = |x|^{m-1}|y|^{n-1}$, we see that

$$-(|am||x|^{m-1}|y|^{n-1}) + (|am||x|^{m-1}|y|^{n-1})^2 \geq -\frac{1}{4}.$$

So,

$$(-\mathcal{L}v_3)(x,y) \gtrsim [-\frac{1}{4}2 + \epsilon_1^2\tilde{k}]y^2 = (-\frac{1}{2} + \epsilon_1^2\tilde{k})y^2.$$

Since $\tilde{k} > \frac{1}{2\epsilon_1^2}$, it follows that $-\frac{1}{2} + \epsilon_1^2\tilde{k} > 0$. Thus,

$$\lim_{r\to\infty}\Big[\inf_{(x,y)\in(\mathcal{R}_3\cap B_r^c)}[-(\mathcal{L}v_3)(x,y)]\Big] \geq \lim_{r\to\infty}\Big[\inf_{(x,y)\in(\mathcal{R}_3\cap B_r^c)}[(-\frac{1}{2}+\epsilon_1^2k)y^2]\Big] = \infty$$

and so v_3 satisfies the third Lyapunov property. Thus, v_3 is a local Lyapunov function on $\mathcal{R}_3$. □

Note that due to the nature of the deterministic dynamics, it is the noise in the y direction, $\epsilon_2 \neq 0$, that is crucial to the existence of a local Lyapunov function in $\mathcal{R}_2$ and it is the noise in the x direction, $\epsilon_1 \neq 0$, that is crucial to the existence of a local Lyapunov function in $\mathcal{R}_3$. Thus, in order to obtain a global Lyapunov function on the entire plane, noise is needed in both the x and y directions.

3.2.3. *Global Lyapunov Function.* Since we have shown the existence of local Lyapunov functions on regions covering the entire plane, minus some ball about the origin, we now seek to patch them together to form one smooth, global Lyapunov function satisfying 3. Since the local regions overlap, the straightforward approach is to construct convex combinations of the form

$$v_{ij}(x,y) = \phi(s(x,y))v_i(x,y) + (1-\phi(s(x,y)))v_j(x,y)$$

on the overlap regions such that $\phi(x,y)$ is a smooth function with $\phi(x,y) = 0$ on one border and $\phi(x,y) = 1$ on the other (similarly for the overlapping region between v_1 and v_3 as well). These convex combinations clearly satisfy the first two local Lyapunov properties (i.e. the smoothness and growth conditions) on the appropriate overlap regions. However, it is not guaranteed that the convex combinations will satisfy the third local Lyapunov property since additional terms result after the application of the generator.

On $\mathcal{R}_1 \cap \mathcal{R}_2$ we define

$$v_{12}(x,y) = \phi(s(x,y))v_1(x,y) + (1-\phi(s(x,y)))v_2(x,y)$$

and on $\mathcal{R}_1 \cap \mathcal{R}_3$ we define

$$v_{13}(x,y) = \phi(s(x,y))v_1(x,y) + (1-\phi(s(x,y)))v_3(x,y)$$

where $\phi(t)$ is the standard mollifier that was defined in Section 1.2. In addition, $s(x,y)$ is chosen so that it equals zero on the border of $\mathcal{R}_1$ and one on the border of $\mathcal{R}_2$, i.e.

$$s(x,y) = \frac{|x|^{m-1}|y|^{n-1} - c_1}{c_2 - c_1}.$$

Lemma 3.4. *There exist constants c_1 and c_2 with $\frac{n}{|a|m^2} < c_1 < c_2$, such that for any $k > \frac{1}{2\epsilon_2^2}$*

$$v_{12}(x,y) = \phi(s(x,y))v_1(x,y) + (1-\phi(s(x,y)))v_2(x,y)$$

is a local Lyapunov function on $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(x,y) : x^2 \geq 1, y^2 \leq \frac{1}{2k}, c_1 \leq |x|^{m-1}|y|^{n-1} \leq c_2\}$.

Proof. v_{12} clearly satisfies the first two properties of a local Lyapunov function, so it only remains to show the third property. Applying the generator to v_{12} we obtain

$$(\mathcal{L}v_{12})(x,y) = D(x,y) + E(x,y)$$

where

$$D(x,y) = \phi(s(x,y))(\mathcal{L}v_1)(x,y) + (1-\phi(s(x,y)))(\mathcal{L}v_2)(x,y)$$

and

$$\begin{aligned} E(x,y) &= \mathcal{L}[\phi(s(x,y))](v_1(x,y) - v_2(x,y)) \\ &+ \epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x,y))] \frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)] \\ &+ \epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x,y))] \frac{\partial}{\partial y}[v_1(x,y) - v_2(x,y)], \end{aligned}$$

which can be verified from explicit computation.

To show that the third Lyapunov property holds, we need to show that

$$\lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap\mathcal{R}_2\cap B_r^c)} -[D(x,y)+E(x,y)]\right] = \infty.$$

We will do this in three steps involving an examination of the behavior of $-D(x,y)$ or $-E(x,y)$ as $r\to\infty$.

(1) First, we will find some A for which $Ax^2 \lesssim -D(x,y)$.
(2) Second, we will find some B for which $Bx^2 \lesssim -E(x,y)$.
(3) Third, we will show that we can always choose c_1 and c_2 so that $A+B>0$, which would imply that the limit inferior above holds, since

$$(A+B)x^2 \lesssim -[D(x,y)+E(x,y)] = -(\mathcal{L}v_{12})(x,y).$$

Note that $\phi'(s(x,y))$, and $\phi''(s(x,y))$ exist and are continuous on the interval $[0,1]$, it follows from the boundedness theorem that there exists an $M>0$ such that

$$|\phi'(s(x,y))| < M \text{ and } |\phi''(s(x,y))| < M.$$

We proceed with Step 1 of the proof. From the expressions for $(\mathcal{L}v_1)(x,y)$ and $(\mathcal{L}v_2)(x,y)$ that were calculated in Lemmas 3.1 and 3.3 respectively, on $\mathcal{R}_1\cap\mathcal{R}_2$ we have that

$$\begin{aligned}(\mathcal{L}v_1)(x,y) &\le [(|an|c_1)-(|am|c_1)^2]2(x^2+y^2)+\epsilon_1^2+\epsilon_2^2\\ &\simeq [(|an|c_1)-(|am|c_1)^2]2x^2\end{aligned}$$

and

$$(\mathcal{L}v_2)(x,y) \lesssim (\frac{1}{2}-\epsilon_2^2k)x^2.$$

Using these inequalities, we find that

$$\begin{aligned}D(x,y) &= \phi(s(x,y))(\mathcal{L}v_1)(x,y)+(1-\phi(s(x,y)))(\mathcal{L}v_2)(x,y)\\ &\lesssim \phi(s(x,y))\Big[[(|an|c_1)-(|am|c_1)^2]2x^2\Big]\\ &\quad+(1-\phi(s(x,y)))\Big[[\frac{1}{2}-\epsilon_2^2k]x^2\Big]\\ &\le \phi(s(x,y))\Big[2[(|an|c_1)-(|am|c_1)^2]x^2\Big]\\ &\le 2[(|an|c_1)-(|am|c_1)^2]x^2\end{aligned}$$

Hence, $-D(x,y) \gtrsim 2[-(|an|c_1)+(|am|c_1)^2]x^2$. By letting $A = 2[-(|an|c_1)+(|am|c_1)^2]$, we have found some constant A for which $Ax^2 \lesssim -D(x,y)$.

Now we will move on to Step 2 of the proof. For convenience, we reproduce $E(x,y)$ below.

$$E(x,y) = \mathcal{L}[\phi(s(x,y))](v_1(x,y)-v_2(x,y))$$

$$+\epsilon_1^2\frac{\partial}{\partial x}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y)-v_2(x,y)]$$
$$+\epsilon_2^2\frac{\partial}{\partial y}[\phi(s(x,y))]\frac{\partial}{\partial y}[v_1(x,y)-v_2(x,y)].$$

Our aim is to find some constant B for which $Bx^2 \lesssim -E(x,y)$. We will do this by calculating the terms in the expression for $E(x,y)$ above and determining which terms of $E(x,y)$ are dominant.

Note that

$$\frac{\partial}{\partial x}[\phi(s(x,y))] = \phi'(s(x,y))\frac{(m-1)|x|^{m-2}|y|^{n-1}\mathrm{sgn}(x)}{c_2-c_1}$$
$$\frac{\partial}{\partial y}[\phi(s(x,y))] = \phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-2}\mathrm{sgn}(y)}{c_2-c_1}$$
$$\frac{\partial^2}{\partial x^2}[\phi(s(x,y))] = \frac{(m-1)|y|^{n-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_2-c_1}+\phi'(s(x,y))(m-2)|x|^{m-3}\Big]$$
$$\frac{\partial^2}{\partial y^2}[\phi(s(x,y))] = \frac{(n-1)|x|^{m-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_2-c_1}+\phi'(s(x,y))(n-2)|y|^{n-3}\Big].$$

We begin by calculating $\mathcal{L}[\phi(s(x,y))]$:

$$\begin{aligned}\mathcal{L}[\phi(s(x,y))] &= [(anx^{m-1}y^{n-1})-(anx^{m-1}y^{n-1})^2]x\frac{\partial}{\partial x}[\phi(s(x,y))]\\
&\quad+[-(amx^{m-1}y^{n-1})-(amx^{m-1}y^{n-1})^2]y\frac{\partial}{\partial y}[\phi(s(x,y))]\\
&\quad+\frac{1}{2}\epsilon_1^2\frac{\partial^2}{\partial x^2}[\phi(s(x,y))]+\frac{1}{2}\epsilon_2^2\frac{\partial^2}{\partial y^2}[\phi(s(x,y))]\\
&=[(anx^{m-1}y^{n-1})-(anx^{m-1}y^{n-1})^2]x\phi'(s(x,y))\frac{(m-1)|x|^{m-2}|y|^{n-1}\mathrm{sgn}(x)}{c_2-c_1}\\
&\quad+[-(amx^{m-1}y^{n-1})-(amx^{m-1}y^{n-1})^2]y\phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-2}\mathrm{sgn}(y)}{c_2-c_1}\\
&\quad+\frac{1}{2}\epsilon_1^2\frac{(m-1)|y|^{n-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_2-c_1}+\phi'(s(x,y))(m-2)|x|^{m-3}\Big]\\
&\quad+\frac{1}{2}\epsilon_2^2\frac{(n-1)|x|^{m-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_2-c_1}+\phi'(s(x,y))(n-2)|y|^{n-3}\Big]\\
&=[(anx^{m-1}y^{n-1})-(anx^{m-1}y^{n-1})^2]\phi'(s(x,y))\frac{(m-1)|x|^{m-1}|y|^{n-1}}{c_2-c_1}\\
&\quad+[-(amx^{m-1}y^{n-1})-(amx^{m-1}y^{n-1})^2]\phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_2-c_1}\\
&\quad+\frac{1}{2}\epsilon_1^2\frac{(m-1)|y|^{n-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_2-c_1}+\phi'(s(x,y))(m-2)|x|^{m-3}\Big]\\
&\quad+\frac{1}{2}\epsilon_2^2\frac{(n-1)|x|^{m-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_2-c_1}+\phi'(s(x,y))(n-2)|y|^{n-3}\Big]\end{aligned}$$

By determining which terms are dominant, we find that

$$\mathcal{L}[\phi(s(x,y))] \simeq \frac{1}{2}\epsilon_2^2 \frac{(n-1)|x|^{m-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_2-c_1} + \phi'(s(x,y))(n-2)|y|^{n-3}\Big].$$

We now calculate $v_1(x,y) - v_2(x,y)$:

$$\begin{aligned} v_1(x,y) - v_2(x,y) &= x^2 + y^2 - [x^2(1-ky^2)] \\ &= x^2 + y^2 - x^2 + kx^2y^2) \\ &= y^2 + kx^2y^2. \end{aligned}$$

We find that

$$\begin{aligned} &\mathcal{L}[\phi(s(x,y))](v_1(x,y) - v_2(x,y)) \\ &\simeq \frac{1}{2}\epsilon_2^2 \frac{(n-1)|x|^{m-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_2-c_1} + \phi'(s(x,y))(n-2)|y|^{n-3}\Big](y^2 + kx^2y^2) \\ &\simeq \frac{1}{2}\epsilon_2^2 \frac{(n-1)|x|^{m-1}}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_2-c_1} + \phi'(s(x,y))(n-2)|y|^{n-3}\Big]kx^2y^2 \\ &= \frac{1}{2}\epsilon_2^2 \frac{(n-1)}{c_2-c_1}\Big[\frac{\phi''(s(x,y))(n-1)(|x|^{m-1}|y|^{(n-1)})^2}{c_2-c_1} + \phi'(s(x,y))(n-2)|x|^{m-1}|y|^{n-1}\Big]kx^2 \end{aligned}$$

We now calculate $\frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)]$:

$$\begin{aligned} \frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)] &= \frac{\partial}{\partial x}\Big[y^2 + kx^2y^2\Big] \\ &= 2kxy^2. \end{aligned}$$

We find that

$$\begin{aligned} &\epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)] \\ &= \epsilon_1^2 \phi'(s(x,y))\frac{(m-1)|x|^{m-2}|y|^{n-1}\operatorname{sgn}(x)}{c_2-c_1}2kxy^2 \\ &= \epsilon_1^2 \phi'(s(x,y))\frac{(m-1)|x|^{m-1}|y|^{n-1}}{c_2-c_1}2ky^2 \end{aligned}$$

which is bounded.

We now calculate $\frac{\partial}{\partial y}[v_1(x,y) - v_2(x,y)]$:

$$\begin{aligned} \frac{\partial}{\partial y}[v_1(x,y) - v_2(x,y)] &= \frac{\partial}{\partial y}\Big[y^2 + kx^2y^2\Big] \\ &= 2y + 2kx^2y. \end{aligned}$$

We find that

$$
\begin{aligned}
&\epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x,y))] \frac{\partial}{\partial x}[v_1(x,y) - v_2(x,y)] \\
&\quad = \epsilon_2^2 \phi'(s(x,y)) \frac{(n-1)|x|^{m-1}|y|^{n-2}\mathrm{sgn}(y)}{c_2 - c_1}(2y + 2kx^2 y) \\
&\quad = \epsilon_2^2 \phi'(s(x,y)) \frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_2 - c_1} 2 + \epsilon_2^2 \phi'(s(x,y)) \frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_2 - c_1} 2kx^2 \\
&\quad \simeq \epsilon_2^2 \phi'(s(x,y)) \frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_2 - c_1} 2kx^2
\end{aligned}
$$

These calculations yield

$$
\begin{aligned}
E(x,y) \simeq & \frac{1}{2}\epsilon_2^2 \frac{(n-1)}{c_2 - c_1}\Big[\frac{\phi''(s(x,y))(n-1)(|x|^{m-1}|y|^{(n-1)})^2}{c_2 - c_1} + \phi'(s(x,y))(n-2)|x|^{m-1}|y|^{n-1}\Big] kx^2 \\
& + \epsilon_2^2 \phi'(s(x,y)) \frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_2 - c_1} 2kx^2 \\
& \simeq k\epsilon_2^2 \Big[\frac{\phi''(s(x,y))(n-1)^2(|x|^{m-1}|y|^{n-1})^2}{2(c_2 - c_1)^2} + \frac{\phi'(s(x,y))(n-1)(n-2)(|x|^{m-1}|y|^{n-1})}{2(c_2 - c_1)} \\
& + \frac{2\phi'(s(x,y))(n-1)(|x|^{m-1}|y|^{n-1})}{(c_2 - c_1)}\Big] x^2 \\
& \leq k\epsilon_2^2 \Big[\frac{M(n-1)^2 c_2^2}{2(c_2 - c_1)^2} + \frac{M(n-1)(n-2)c_2}{2(c_2 - c_1)} + \frac{2M(n-1)c_2}{(c_2 - c_1)}\Big] x^2.
\end{aligned}
$$

If we let $B = -k\epsilon_2^2 \Big[\frac{M(n-1)^2 c_2^2}{2(c_2-c_1)^2} + \frac{M(n-1)(n-2)c_2}{2(c_2-c_1)} + \frac{2M(n-1)c_2}{(c_2-c_1)}\Big]$, then we have found an expression of the form $Bx^2 \lesssim -E(x,y)$ which is what we wanted to show.

Now we move on to Step 3.

We have

$$
\begin{aligned}
(\mathcal{L}v_{12})(x,y) &= D(x,y) + E(x,y) \\
&\lesssim [2|an|c_1 - 2(|am|c_1)^2]x^2 + k\epsilon_2^2 \Big[\frac{M(n-1)^2 c_2^2}{2(c_2 - c_1)^2} + \frac{M(n-1)(n-2)c_2}{2(c_2 - c_1)} + \frac{2M(n-1)c_2}{(c_2 - c_1)}\Big] x^2 \\
&= \left\{2|an|c_1 - 2(|am|c_1)^2 + k\epsilon_2^2 \Big[\frac{M(n-1)^2 c_2^2}{2(c_2 - c_1)^2} + \frac{M(n-1)(n-2)c_2}{2(c_2 - c_1)} + \frac{2M(n-1)c_2}{(c_2 - c_1)}\right\} x^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(-\mathcal{L}v_{12})(x,y) &= -[D(x,y) + E(x,y)] \\
&\gtrsim -\left\{2|an|c_1 - 2(|am|c_1)^2 + k\epsilon_2^2 \Big[\frac{M(n-1)^2 c_2^2}{2(c_2 - c_1)^2} + \frac{M(n-1)(n-2)c_2}{2(c_2 - c_1)} + \frac{2M(n-1)c_2}{(c_2 - c_1)}\Big]\right\} x^2.
\end{aligned}
$$

Let $c_2 = 2c_1$. Then, $c_2 - c_1 = \frac{c_2}{2}$ and the equation above becomes

$$
(-\mathcal{L}v_{12})(x,y) = -[D(x,y) + E(x,y)]
$$

$$\gtrsim -\left\{|an|c_2 - 2(|am|\frac{c_2}{2})^2 + k\epsilon_2^2\Big[2M(n-1)^2 + M(n-1)(n-2) + 4M(n-1)\Big]\right\}x^2$$
$$= \left\{\frac{1}{2}a^2m^2c_2^2 - |an|c_2 - k\epsilon_2^2\Big[2M(n-1)^2 + M(n-1)(n-2) + 4M(n-1)\Big]\right\}x^2$$

The coefficient in brackets in the last line (i.e. $A+B$) is a second-degree polynomial in c_2, taking the form $d_1c_2^2 + d_2c_2 + d_3$ for some constants $d_1, d_2,$ and d_3. Since $d_1 = \frac{1}{2}a^2m^2$, we know that $d_1 > 0$. Thus, as is true for any even-degree polynomial in c_2 with positive leading coefficient, we can choose c_2 sufficiently large so that we make the polynomial positive, implying that $A + B > 0$. This proves the lemma. □

Lemma 3.5. *There exist constants c_1 and c_3 with $\frac{n}{|a|m^2} < c_1 < c_3$, such that for any $\tilde{k} > \frac{1}{2\epsilon_1^2}$*

$$v_{13}(x,y) = \phi(s(x,y))v_1(x,y) + (1-\phi(s(x,y)))v_3(x,y)$$

is a local Lyapunov function on $\mathcal{R}_1 \cap \mathcal{R}_3 = \{(x,y) : y^2 \geq 1, x^2 \leq \frac{1}{2\tilde{k}}, c_1 \leq |x|^{m-1}|y|^{n-1} \leq c_3\}$.

Proof. v_{13} clearly satisfies the first two properties of a local Lyapunov function, so it only remains to show the third property. Applying the generator to v_{13} we obtain

$$(\mathcal{L}v_{13})(x,y) = D(x,y) + E(x,y)$$

where

$$D(x,y) = \phi(s(x,y))(\mathcal{L}v_1)(x,y) + (1-\phi(s(x,y)))(\mathcal{L}v_3)(x,y)$$

and

$$E(x,y) = \mathcal{L}[\phi(s(x,y))](v_1(x,y) - v_3(x,y))$$
$$+ \epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y) - v_3(x,y)]$$
$$+ \epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x,y))]\frac{\partial}{\partial y}[v_1(x,y) - v_3(x,y)],$$

which can be verified from explicit computation.

To show that the third Lyapunov property holds, we need to show that

$$\lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_1\cap\mathcal{R}_3\cap B_r^c)} -[D(x,y)+E(x,y)]\right] = \infty.$$

We will do this in three steps involving an examination of the behavior of $-D(x,y)$ or $-E(x,y)$ as $r \to \infty$.

(1) First, we will find some A for which $Ay^2 \lesssim -D(x,y)$.
(2) Second, we will find some B for which $By^2 \lesssim -E(x,y)$.
(3) Third, we will show that we can always choose c_1 and c_3 so that $A + B > 0$, which would imply that the limit inferior above holds, since

$$(A+B)y^2 \lesssim -[D(x,y)+E(x,y)] = -(\mathcal{L}v_{13})(x,y).$$

Note that $\phi'(s(x,y))$, and $\phi''(s(x,y))$ exist and are continuous on the interval $[0,1]$, it follows from the boundedness theorem that there exists an $M>0$ such that

$$|\phi'(s(x,y))| < M \text{ and } |\phi''(s(x,y))| < M.$$

We proceed with Step 1 of the proof. From the expressions for $(\mathcal{L}v_1)(x,y)$ and $(\mathcal{L}v_3)(x,y)$ that were calculated in Lemmas 3.1 and 3.3 respectively, on $\mathcal{R}_1 \cap \mathcal{R}_3$ we have that

$$\begin{aligned}(\mathcal{L}v_1)(x,y) &\leq [(|an|c_1) - (|am|c_1)^2]2(x^2+y^2) + \epsilon_1^2 + \epsilon_2^2 \\ &\simeq [(|an|c_1) - (|am|c_1)^2]2y^2\end{aligned}$$

and

$$(\mathcal{L}v_3)(x,y) \lesssim (\frac{1}{2} - \epsilon_1^2\tilde{k})y^2.$$

Using these inequalities, we find that

$$\begin{aligned}D(x,y) &= \phi(s(x,y))(\mathcal{L}v_1)(x,y) + (1-\phi(s(x,y)))(\mathcal{L}v_3)(x,y) \\ &\lesssim \phi(s(x,y))\Big[[(|an|c_1) - (|am|c_1)^2]2y^2\Big] \\ &\quad + (1-\phi(s(x,y)))\Big[[\frac{1}{2} - \epsilon_1^2\tilde{k}]y^2\Big] \\ &\leq \phi(s(x,y))\Big[2[(|an|c_1) - (|am|c_1)^2]y^2\Big] \\ &\leq 2[(|an|c_1) - (|am|c_1)^2]y^2\end{aligned}$$

Hence, $-D(x,y) \gtrsim 2[-(|an|c_1) + (|am|c_1)^2]y^2$. By letting $A = 2[-(|an|c_1) + (|am|c_1)^2]$, we have found some constant A for which $Ay^2 \lesssim -D(x,y)$.

Now we will move on to Step 2 of the proof. For convenience, we reproduce $E(x,y)$ below.

$$\begin{aligned}E(x,y) &= \mathcal{L}[\phi(s(x,y))](v_1(x,y) - v_3(x,y)) \\ &\quad + \epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y) - v_3(x,y)] \\ &\quad + \epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x,y))]\frac{\partial}{\partial y}[v_1(x,y) - v_3(x,y)].\end{aligned}$$

Our aim is to find some constant B for which $By^2 \lesssim -E(x,y)$. We will do this by calculating the terms in the expression for $E(x,y)$ above and determining which terms of $E(x,y)$ are dominant.

Note that

$$\frac{\partial}{\partial x}[\phi(s(x,y))] = \phi'(s(x,y))\frac{(m-1)|x|^{m-2}|y|^{n-1}\text{sgn}(x)}{c_3 - c_1}$$

$$\frac{\partial}{\partial y}[\phi(s(x,y))] = \phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-2}\text{sgn}(y)}{c_3-c_1}$$
$$\frac{\partial^2}{\partial x^2}[\phi(s(x,y))] = \frac{(m-1)|y|^{n-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_3-c_1} + \phi'(s(x,y))(m-2)|x|^{m-3}\Big]$$
$$\frac{\partial^2}{\partial y^2}[\phi(s(x,y))] = \frac{(n-1)|x|^{m-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_3-c_1} + \phi'(s(x,y))(n-2)|y|^{n-3}\Big].$$

We begin by calculating $\mathcal{L}[\phi(s(x,y))]$:

$$\begin{aligned}
\mathcal{L}[\phi(s(x,y))] &= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]x\frac{\partial}{\partial x}[\phi(s(x,y))] \\
&\quad + [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]y\frac{\partial}{\partial y}[\phi(s(x,y))] \\
&\quad + \frac{1}{2}\epsilon_1^2\frac{\partial^2}{\partial x^2}[\phi(s(x,y))] + \frac{1}{2}\epsilon_2^2\frac{\partial^2}{\partial y^2}[\phi(s(x,y))] \\
&= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]x\phi'(s(x,y))\frac{(m-1)|x|^{m-2}|y|^{n-1}\text{sgn}(x)}{c_3-c_1} \\
&\quad + [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]y\phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-2}\text{sgn}(y)}{c_3-c_1} \\
&\quad + \frac{1}{2}\epsilon_1^2\frac{(m-1)|y|^{n-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_3-c_1} + \phi'(s(x,y))(m-2)|x|^{m-3}\Big] \\
&\quad + \frac{1}{2}\epsilon_2^2\frac{(n-1)|x|^{m-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_3-c_1} + \phi'(s(x,y))(n-2)|y|^{n-3}\Big] \\
&= [(anx^{m-1}y^{n-1}) - (anx^{m-1}y^{n-1})^2]\phi'(s(x,y))\frac{(m-1)|x|^{m-1}|y|^{n-1}}{c_3-c_1} \\
&\quad + [-(amx^{m-1}y^{n-1}) - (amx^{m-1}y^{n-1})^2]\phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_3-c_1} \\
&\quad + \frac{1}{2}\epsilon_1^2\frac{(m-1)|y|^{n-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_3-c_1} + \phi'(s(x,y))(m-2)|x|^{m-3}\Big] \\
&\quad + \frac{1}{2}\epsilon_2^2\frac{(n-1)|x|^{m-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(n-1)|x|^{m-1}|y|^{2(n-2)}}{c_3-c_1} + \phi'(s(x,y))(n-2)|y|^{n-3}\Big]
\end{aligned}$$

By determining which terms are dominant, we find that

$$\mathcal{L}[\phi(s(x,y))] \simeq \frac{1}{2}\epsilon_1^2\frac{(m-1)|y|^{n-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_3-c_1} + \phi'(s(x,y))(m-2)|x|^{m-3}\Big].$$

We now calculate $v_1(x,y) - v_3(x,y)$:

$$\begin{aligned}
v_1(x,y) - v_3(x,y) &= x^2 + y^2 - [y^2(1-\tilde{k}x^2)] \\
&= x^2 + y^2 - y^2 + \tilde{k}x^2y^2) \\
&= x^2 + \tilde{k}x^2y^2.
\end{aligned}$$

We find that

$$
\begin{aligned}
&\mathcal{L}[\phi(s(x,y))](v_1(x,y)-v_3(x,y)) \\
&\simeq \frac{1}{2}\epsilon_1^2 \frac{(m-1)|y|^{n-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_3-c_1} + \phi'(s(x,y))(m-2)|x|^{m-3}\Big](x^2+\tilde{k}x^2y^2) \\
&\simeq \frac{1}{2}\epsilon_1^2 \frac{(m-1)|y|^{n-1}}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)|x|^{2(m-2)}|y|^{n-1}}{c_3-c_1} + \phi'(s(x,y))(m-2)|x|^{m-3}\Big]\tilde{k}x^2y^2 \\
&= \frac{1}{2}\epsilon_1^2 \frac{(m-1)}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)(|x|^{m-1}|y|^{n-1})^2}{c_3-c_1} + \phi'(s(x,y))(n-2)|x|^{m-1}|y|^{n-1}\Big]\tilde{k}y^2
\end{aligned}
$$

We now calculate $\frac{\partial}{\partial x}[v_1(x,y)-v_3(x,y)]$:

$$
\begin{aligned}
\frac{\partial}{\partial x}[v_1(x,y)-v_3(x,y)] &= \frac{\partial}{\partial x}\Big[x^2+\tilde{k}x^2y^2\Big] \\
&= 2x + 2\tilde{k}xy^2.
\end{aligned}
$$

We find that

$$
\begin{aligned}
&\epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y)-v_3(x,y)] \\
&= \epsilon_1^2 \phi'(s(x,y))\frac{(m-1)|x|^{m-2}|y|^{n-1}\mathrm{sgn}(x)}{c_3-c_1}(2x+2\tilde{k}xy^2) \\
&\simeq \epsilon_1^2 \phi'(s(x,y))\frac{(m-1)|x|^{m-1}|y|^{n-1}}{c_3-c_1}2\tilde{k}y^2.
\end{aligned}
$$

We now calculate $\frac{\partial}{\partial y}[v_1(x,y)-v_3(x,y)]$:

$$
\begin{aligned}
\frac{\partial}{\partial y}[v_1(x,y)-v_3(x,y)] &= \frac{\partial}{\partial y}\Big[x^2+\tilde{k}x^2y^2\Big] \\
&= 2\tilde{k}x^2y.
\end{aligned}
$$

We find that

$$
\begin{aligned}
&\epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x,y))]\frac{\partial}{\partial x}[v_1(x,y)-v_3(x,y)] \\
&= \epsilon_2^2 \phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-2}\mathrm{sgn}(y)}{c_3-c_1}2\tilde{k}x^2y \\
&= \epsilon_2^2 \phi'(s(x,y))\frac{(n-1)|x|^{m-1}|y|^{n-1}}{c_3-c_1}2\tilde{k}x^2
\end{aligned}
$$

which is bounded.

These calculations yield

$$
E(x,y) \simeq = \frac{1}{2}\epsilon_1^2 \frac{(m-1)}{c_3-c_1}\Big[\frac{\phi''(s(x,y))(m-1)(|x|^{m-1}|y|^{n-1})^2}{c_3-c_1} + \phi'(s(x,y))(n-2)|x|^{m-1}|y|^{n-1}\Big]\tilde{k}y^2
$$

$$+ \epsilon_1^2 \phi'(s(x,y)) \frac{(m-1)|x|^{m-1}|y|^{n-1}}{c_3 - c_1} 2\tilde{k} y^2$$
$$\simeq \tilde{k}\epsilon_1^2 \Big[\frac{\phi''(s(x,y))(m-1)^2(|x|^{m-1}|y|^{n-1})^2}{2(c_3-c_1)^2} + \frac{\phi'(s(x,y))(m-1)(m-2)(|x|^{m-1}|y|^{n-1})}{2(c_3-c_1)}$$
$$+ \frac{2\phi'(s(x,y))(m-1)(|x|^{m-1}|y|^{n-1})}{(c_3-c_1)} \Big] y^2$$
$$\leq \tilde{k}\epsilon_1^2 \Big[\frac{M(m-1)^2 c_3^2}{2(c_3-c_1)^2} + \frac{M(m-1)(m-2)c_3}{2(c_3-c_1)} + \frac{2M(m-1)c_3}{(c_3-c_1)} \Big] y^2.$$

If we let $B = -\tilde{k}\epsilon_1^2 \Big[\frac{M(m-1)^2 c_3^2}{2(c_3-c_1)^2} + \frac{M(m-1)(m-2)c_3}{2(c_3-c_1)} + \frac{2M(m-1)c_3}{(c_3-c_1)} \Big]$, then we have found an expression of the form $By^2 \lesssim -E(x,y)$ which is what we wanted to show.

Now we move on to Step 3.

We have

$$(\mathcal{L}v_{13})(x,y) = D(x,y) + E(x,y)$$
$$\lesssim [2|an|c_1 - 2(|am|c_1)^2] y^2 + \tilde{k}\epsilon_1^2 \Big[\frac{M(m-1)^2 c_3^2}{2(c_3-c_1)^2} + \frac{M(m-1)(m-2)c_3}{2(c_3-c_1)} + \frac{2M(m-1)c_3}{(c_3-c_1)} \Big] y^2$$
$$= \left\{ 2|an|c_1 - 2(|am|c_1)^2 + \tilde{k}\epsilon_1^2 \Big[\frac{M(m-1)^2 c_3^2}{2(c_3-c_1)^2} + \frac{M(m-1)(m-2)c_3}{2(c_3-c_1)} + \frac{2M(m-1)c_3}{(c_3-c_1)} \Big] \right\} y^2.$$

Thus,

$$(-\mathcal{L}v_{13})(x,y) = -[D(x,y) + E(x,y)]$$
$$\gtrsim -\left\{ 2|an|c_1 - 2(|am|c_1)^2 + \tilde{k}\epsilon_1^2 \Big[\frac{M(m-1)^2 c_3^2}{2(c_3-c_1)^2} + \frac{M(m-1)(m-2)c_3}{2(c_3-c_1)} + \frac{2M(m-1)c_3}{(c_3-c_1)} \Big] \right\} y^2.$$

Let $c_3 = 2c_1$. Then, $c_3 - c_1 = \frac{c_3}{2}$ and the equation above becomes

$$(-\mathcal{L}v_{13})(x,y) = -[D(x,y) + E(x,y)]$$
$$\gtrsim -\left\{ |an|c_3 - 2(|am|\frac{c_3}{2})^2 + \tilde{k}\epsilon_1^2 \Big[2M(m-1)^2 + M(m-1)(m-2) + 4M(m-1) \Big] \right\} y^2$$
$$= \left\{ \frac{1}{2}a^2 m^2 c_3^2 - |an|c_3 - \tilde{k}\epsilon_1^2 \Big[2M(m-1)^2 + M(m-1)(m-2) + 4M(m-1) \Big] \right\} y^2$$

The coefficient in brackets in the last line (i.e. $A+B$) is a second-degree polynomial in c_3, taking the form $d_1 c_3^2 + d_2 c_3 + d_3$ for some constants $d_1, d_2,$ and d_3. Since $d_1 = \frac{1}{2}a^2 m^2$, we know that $d_1 > 0$. Thus, as is true for any even-degree polynomial in c_3 with positive leading coefficient, we can choose c_3 sufficiently large so that we make the polynomial positive, implying that $A + B > 0$. This proves the lemma.

□

Our global Lyapunov function, $V(x,y) \in C^\infty(\mathbb{R}^2)$, can be constructed so that

$$V(x,y) = \begin{cases} \tilde{V}(x,y) & \text{for } x^2 + y^2 > \rho^2 \\ \text{arbitrary positive and smooth} & \text{for } x^2 + y^2 \leq \rho^2 \end{cases}$$

where ρ is sufficiently large and

$$\tilde{V}(x,y) = \begin{cases} v_1(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3^c \\ v_2(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \\ v_3(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_3 \\ v_{12}(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2 \\ v_{13}(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_3. \end{cases}$$

4. Hamiltonian System with $H(x,y) = \frac{e^{\alpha x^2} y^2}{2}$

We consider the Hamiltonian function

$$H(x,y) = \frac{e^{\alpha x^2} y^2}{2} \tag{16}$$

where α is some positive constant. The corresponding deterministic Hamiltonian system (x_t, y_t) is the solution to the following two-dimensional system of ODEs:

$$\begin{aligned} \frac{dx_t}{dt} &= \frac{\partial H}{\partial y} = y_t e^{\alpha x_t^2} \\ \frac{dy_t}{dt} &= -\frac{\partial H}{\partial x} = -\alpha x_t y_t^2 e^{\alpha x_t^2} \ . \end{aligned} \tag{17}$$

Hence the x-axis is a continuum of equilibrium points, but for any initial condition with $y_0 \neq 0$, the solution has the property that

$$y_t = y_0 e^{\alpha(x_0^2 - x_t^2)/2} \ . \tag{18}$$

Now that we have solved for y_t in terms of x_0 and y_0, we can substitute into the expression for dx_t and solve for x_t.

$$\begin{aligned} \frac{dx_t}{dt} &= y e^{\alpha x_t^2} \\ &= (y_0 e^{\alpha(x_0^2 - x_t^2)/2}) e^{\alpha x_t^2} \\ \int e^{\alpha x_t^2/2} dx_t &= \int y_0 e^{\alpha x_0^2/2} dt \\ \int \frac{e^{\frac{-x_t^2}{2/\alpha}}}{\sqrt{2\pi/\alpha}} dx_t &= \frac{y_0 e^{\alpha x_0^2/2} t}{\sqrt{2\pi/\alpha}} + C \end{aligned}$$

The left-hand side is now the probability distribution function of a normally distributed random variable x_t with mean zero and variance $\frac{1}{\alpha}$. This is equivalent to saying that $\sqrt{\alpha} x_t$ is normally distributed with mean zero and variance 1. We use this fact, and the cumulative distribution function Φ, to re-write the left-hand side of the equation above and continue.

$$\begin{aligned} \Phi(\sqrt{\alpha} x_t) &= \frac{y_0 e^{\alpha x_0^2/2} t}{\sqrt{2\pi/\alpha}} + C \\ \sqrt{\alpha} x_t &= \Phi^{-1}\left(\frac{y_0 e^{\alpha x_0^2/2} t}{\sqrt{2\pi/\alpha}} + C \right) \end{aligned}$$

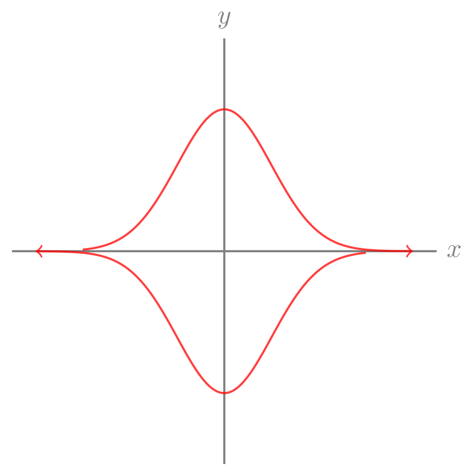

FIGURE 8. Phase portrait of the deterministic Hamiltonian system for $H(x,y) = \frac{e^{x^2} y^2}{2}$

$$x_t = \frac{1}{\sqrt{\alpha}} \Phi^{-1} \left(\frac{y_0 e^{\alpha x_0^2/2} t}{\sqrt{2\pi/\alpha}} + C \right)$$

If we set $t = 0$, we can see that $C = \Phi(\sqrt{\alpha} x_0)$. Therefore we have an explicit solution for x_t in this Hamiltonian system:

$$x_t = \frac{1}{\sqrt{\alpha}} \Phi^{-1} \left(\frac{y_0 e^{\alpha x_0^2/2} t}{\sqrt{2\pi/\alpha}} + \Phi(\sqrt{\alpha} x_0) \right)$$

Since $\Phi^{-1}(z)$ is the inverse of the cumulative distribution function of the standard normal distribution, it blows up to infinity when $z = 0$ or 1. Thus, x_t blows up when $\frac{y_0 e^{\alpha x_0^2/2} t}{\sqrt{2\pi/\alpha}} + \Phi(\sqrt{\alpha} x_0) = 0$ or 1. If $y_0 > 0$, the blow-up occurs when the expression is equal to 1. If $y_0 < 0$, it occurs when the expression is equal to zero. Therefore we have two different blow-up times, for $y_0 > 0$ and $y_0 < 0$, respectively. They are:

$$t_{y_0 > 0} = \frac{\sqrt{2\pi/\alpha}(1 - \Phi(\sqrt{\alpha} x_0))}{y_0 e^{\alpha x_0^2/2}}$$

$$t_{y_0 < 0} = \frac{\sqrt{2\pi/\alpha}\Phi(\sqrt{\alpha} x_0)}{|y_0| e^{\alpha x_0^2/2}}$$

In both cases, x_t blows up to infinity in finite time. Because the solution converges to infinity for certain initial conditions, the system is unstable according to the definition of stability given in Section 1.

Although the system is unstable, it possesses characteristics which may lead one to guess that the system will exhibit noise-induced stabilization. For initial conditions off the x-axis, the solution to the deterministic process remains above or below the axis for all time, approaching positive (for $y_0 > 0$) or negative (for $y_0 < 0$) infinity along the axis as y_t tends toward (but does not reach) zero. This behavior is visible in Figure 8; different values of the constant α in the Hamiltonian function will make the curve steeper or flatter. Given this shape, one might guess that the addition of white noise will enable the perturbed process to cross the axis and form a quasi-periodic behavior, moving to the right above the x-axis and to the left below the x-axis. However, additive white noise is not sufficient for the existence of an invariant probability measure, as explained in Section 1. Hence the system with noise is not stochastically bounded.

We wish to modify the Hamiltonian system defined by (17) in order to create a system which exhibits noise-induced stabilization, and yet retains essentially the same qualitative behavior of the original deterministic dynamics. Due to the fact that any unstable Hamiltonian system remains unstable after the addition of white noise, any modification sufficient to produce noise-induced stabilization must break the Hamiltonian structure. In order to preserve the qualitative behavior of the deterministic dynamics, we add a drift term which points towards the x-axis but which does not change the limiting behavior of solutions for any initial condition. In the next section we discuss some intuition for the specific drift terms added in order to produce noise-induced stabilization in this particular problem. We then apply the meta-algorithm for the construction of a Lyapunov function in order to prove that the perturbed modified Hamiltonian system is indeed stable.

4.1. **Perturbed Hamiltonian System.** In this problem, we begin with the following deterministic Hamiltonian system:

$$\begin{aligned} \frac{dx_t}{dt} &= \quad y_t e^{\alpha x_t^2} \\ \frac{dy_t}{dt} &= -(\alpha x_t y_t^2) e^{\alpha x_t^2} \ . \end{aligned} \tag{19}$$

As shown in the previous section, this Hamiltonian system is unstable and remains unstable after perturbation by additive white noise. We wish to modify (19) in order to create a new system which exhibits noise-induced stabilization and yet preserves the qualitative features of the original Hamiltonian system, namely the limiting behavior of the solution curves for all initial conditions. Our approach is to add a drift term which points toward the x-axis, but which scales subdominantly to the original drift terms near the axis. While the original Hamiltonian system remained unstable after the addition of white noise, the additional drift terms in the modified Hamiltonian system should allow for the noise to have a stabilizing effect.

In order to choose the precise form for the additional drift terms $f(x, y)$ and $g(x, y)$, we analyze the process $H_t = y_t^2 e^{\alpha x_t^2}$, which is always constant due to the Hamiltonian structure. As in the previous problems, our initial approach is to modify the system so that H_t tends toward zero. In this case, we can modify the system so that $dH_t = -\tilde{a} H_t^q$, where q is some positive integer which can be chosen to control the strength of the deterministic perturbation. Since $dH_t = (d(e^{\alpha x_t^2}))(y_t^2) + (e^{\alpha x_t^2})(dy_t^2)$, we can deduce that the modified system will have the desired dH_t if we set

$$\begin{aligned} &f(x, y) = 0 \text{ and} \\ &g(x, y) = a y^{(2q-1)} e^{(q-1)\alpha x^2} \ . \end{aligned} \tag{20}$$

Intuitively, the reasoning behind only modifying the system in the y direction is because the desired result is to induce the system with noise to cross over the

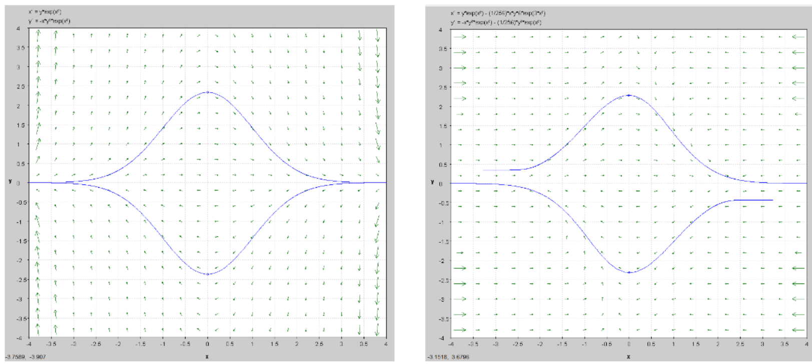

FIGURE 9. Solutions to the original Hamiltonian (left) and modified deterministic system (right) for identical initial conditions

x-axis. In addition, if $f(x, y) = 0$, then the sign of dx_t in the modified system is the same as in the original system for all locations in the plane. This is "good" in the sense that it preserves the behavior of the original system to a large extent.

Using only the modification given above, local Lyapunov functions were found for all regions of the plane (subject to constraints on choices of constants). These Lyapunov functions are identical to the ones used in the proof that follows. However, no method was found to combine these functions into a global Lyapunov function which is infinitely differentiable at all points. Therefore a deterministic modification in the x direction was also made to the system. It was chosen specifically to provide terms which become crucial in the section of the proof dealing with the combined Lyapunov function $v_{12}(x, y)$. This new modification is given by $f(x, y) = xy^6 e^{3\alpha x^2}$.

This gives us the following S_P:

$$
\begin{aligned}
\frac{dx_t}{dt} &= \quad (y_t e^{\alpha x_t^2} - x_t y_t^6 e^{3\alpha x_t^2}) \\
\frac{dy_t}{dt} &= (-(\alpha x_t y_t^2) e^{\alpha x_t^2} - a y_t^{(2q-1)} e^{(q-1)\alpha x_t^2}).
\end{aligned} \tag{21}
$$

As in the original Hamiltonian system, the x-axis is a continuum of equilibrium points, while for $y_0 > 0$, $\lim_{t\to\infty} x_t = \infty$ and for $y_0 < 0$, $\lim_{t\to\infty} x_t = -\infty$. Thus the additional drift terms in the modified Hamiltonian system given by (21) preserves many of the qualitative features, including the instability, of the original Hamiltonian system given by (19). Figure 9 shows a side-by-side comparison of phase portraits for the original system and the modified deterministic system.

We now consider perturbing the modified system given by (21) with additive white noise to form the following two-dimensional system of stochastic differential equations:

$$
\begin{aligned}
\frac{dX_t}{dt} &= \quad (Y_t e^{\alpha X_t^2} - X_t Y_t^6 e^{3\alpha X_t^2}) + \epsilon_1 \frac{dB_1(t)}{dt} \\
\frac{dY_t}{dt} &= (-(\alpha X_t Y_t^2) e^{\alpha X_t^2} - a Y_t^3 e^{\alpha X_t^2}) dt + \epsilon_2 \frac{dB_2(t)}{dt} \quad .
\end{aligned} \tag{22}
$$

Here $B_1(t)$ and $B_2(t)$ are independent Brownian motions and $\epsilon_1, \epsilon_2 > 0$ represent the strength of the noise in the x and y directions, respectively. Figure 10 displays

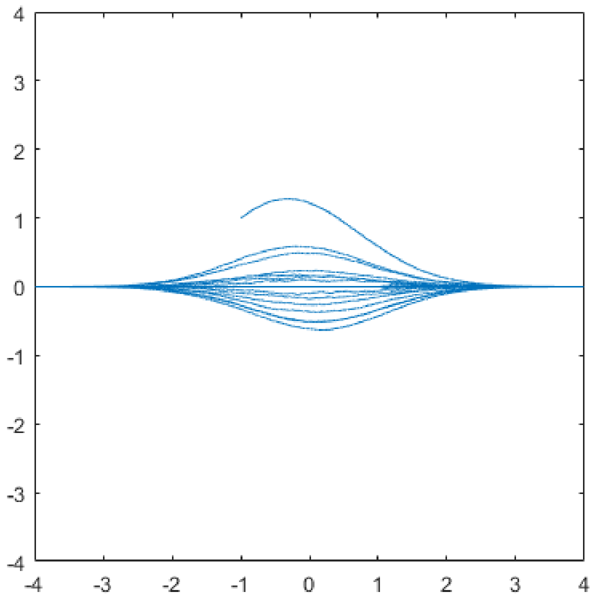

FIGURE 10. A simulation of the modified Hamiltonian system with noise

a simulation of the solution to (22) with $\epsilon_1 = \epsilon_2 = .01$ and initial condition $(-1, 1)$, zoomed in about the origin. While the deterministic system was constrained to be above or below the x-axis based on its initial condition and converged to infinity along the x-axis, we observe that the perturbed process exhibits a quasi-periodic behavior where it travels above and below the axis. While the perturbed process does travel far out along the x-axis in either direction, it does not go off to infinity and remains stochastically bounded. We prove that the modified Hamiltonian system does indeed exhibit noise-induced stabilization in the next section through the construction of a Lyapunov function.

Theorem 3. *Consider* $H(x, y) = \frac{y^2 e^{\alpha x^2}}{2}$, *where* α *is any positive real number. Then the perturbed Hamiltonian system with*

$$f(x, y) = xy^6 e^{3\alpha x^2} \text{ and}$$
$$g(x, y) = ay^3 e^{\alpha x^2}$$

where a *is any positive real number, exhibits noise-induced stabilization*

4.2. **Lyapunov Construction.** If we can show the existence of a Lyapunov function which satisfies the conditions given in Section 1, then we have shown that (X_t, Y_t) is positive recurrent, which in turn implies that the system is stochastically bounded and, hence, stable. Generally, showing the existence of a Lyapunov function can be quite ad hoc and tedious. However, we apply the systematic method developed in [AKM12] in order to construct local Lyapunov functions on various

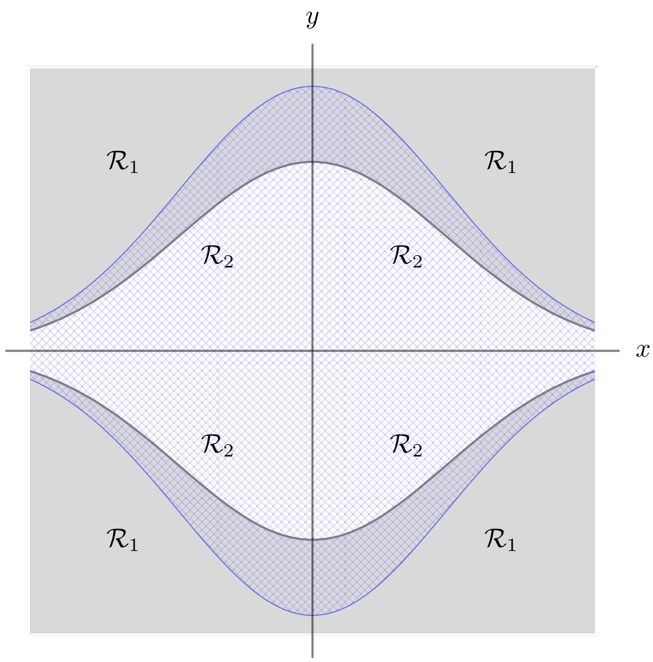

FIGURE 11. Decomposition of plane into priming and diffusive regions.

regions of the plane and then patch them together to form one global, smooth Lyapunov function.

We begin by decomposing the plane into the following regions:

$$\begin{aligned} \mathcal{R}_1 &= \{(x,y) : y^2 e^{p\alpha x^2} \geq c_1\} \\ \mathcal{R}_2 &= \{(x,y) : y^2 e^{p\alpha x^2} \leq c_2\} \end{aligned} \tag{23}$$

where $0 < c_1 < c_2$. The precise values of these constants, and of the constant p, can be specified later to facilitate local Lyapunov function constructions and patching. The two regions, $\mathcal{R}_1$ and $\mathcal{R}_2$, cover the entire plane, and are depicted in Figure 11. $\mathcal{R}_1$ is the "priming region" where a natural Lyapunov function exists, here the natural logarithm of the Hamiltonian function. $\mathcal{R}_2$ is the "diffusive region" where the deterministic dynamics are unstable and noise is essential to the existence of a local Lyapunov function.

We seek to show the existence of local Lyapunov functions, v_1 and v_2, on $\mathcal{R}_1$ and $\mathcal{R}_2$,, respectively. As before, we define v_i to be a *local Lyapunov function* on $\mathcal{R}_i$ if it satisfies the three conditions stated in Section 1 everywhere on $\mathcal{R}_i$. After showing the existence of these local Lyapunov functions, we will patch them together to form one smooth, global Lyapunov function satisfying the stability conditions for the whole plane.

4.2.1. *Priming Region.* The priming region was chosen so that a "natural" Lyapunov function (the natural logarithm of the Hamiltonian function) exists in the region (as shown in Lemma 4.1). Note that the boundary is defined by a function closely related to the Hamiltonian function; any solution curve in our perturbed deterministic system will eventually move from the priming region to the diffusive region because we have made $\frac{dH_t}{dt}$ negative.

Lemma 4.1. $v_1(x,y) = \alpha x^2 + 2\ln|y|$ *is a local Lyapunov function on* $\mathcal{R}_1$*, for* $p < 1$ *and* $q > 1$*.*

Since $y \neq 0$ on $\mathcal{R}_1$, it is clear that v_1 is infinitely differentiable. This satisfies the first Lyapunov condition.

First we need to show that $\lim_{|x,y|\to\infty} v_1(x,y) = \infty$. We will use two cases to show this. In the first case, when $|y| \geq |x|$, $v_1(x,y)$ clearly goes to infinity because $|y|$ goes to infinity. In the second case, when $|y| < |x|$ in $\mathcal{R}_1$, we have:

$$\begin{aligned} v_1(x,y) = \alpha x^2 + 2\text{ln}|y| &\geq \alpha x^2 + 2\text{ln}(\sqrt{c_1}e^{-\frac{p}{2}\alpha x^2}) \\ &= \alpha x^2 + \text{ln}(c_1) + \text{ln}(e^{-p\alpha x^2}) \\ &= (1-p)\alpha x^2 + \text{ln}(c_1) \end{aligned}$$

Since $|x| \to \infty$ in this case, v_1 must also go to infinity, as long as $p < 1$. Thus v_1 satisfies the second Lyapunov condition everywhere on $\mathcal{R}_1$.

Now we need to show that $\lim_{|x,y|\to\infty} \mathcal{L}v_1(x,y) = -\infty$.

$$\begin{aligned} \mathcal{L}v_1 &= (ye^{\alpha x^2} - xy^6e^{3\alpha x^2})(2\alpha x) - (\alpha xy^2e^{\alpha x^2} + ay^{(2q-1)}e^{(q-1)\alpha x^2})(\frac{2}{y}) + \frac{\epsilon_1^2}{2}(2\alpha) + \frac{\epsilon_2^2}{2}(\frac{-2}{y^2}) \\ &= 2\alpha xye^{\alpha x^2} - 2\alpha x^2y^6e^{3\alpha x^2} - 2\alpha xye^{\alpha x^2} - 2ay^{(2q-2)}e^{(q-1)\alpha x^2} + \alpha\epsilon_1^2 - \frac{\epsilon_2^2}{y^2} \\ &= -2\alpha x^2y^6e^{3\alpha x^2} - 2ay^{2(q-1)}e^{(q-1)\alpha x^2} + \alpha\epsilon_1^2 - \frac{\epsilon_2^2}{y^2} \end{aligned}$$

If we are in the case where $|y| \geq |x|$, then we know that the above expression goes to negative infinity due to the second term, as long as $q > 1$. If $|y| < |x|$, then we need to continue:

$$\begin{aligned} -2\alpha x^2y^6e^{3\alpha x^2} - 2ay^{2(q-1)}e^{(q-1)\alpha x^2} + \alpha\epsilon_1^2 - \frac{\epsilon_2^2}{y^2} &\leq -2\alpha x^2[c_2^3e^{-3p\alpha x^2}]e^{3\alpha x^2} \\ &\quad - 2a[c_1^{(q-1)}e^{-p\alpha x^2(q-1)}]e^{(q-1)\alpha x^2} + \alpha\epsilon_1^2 \\ &= -2\alpha c_2^3x^2e^{(3-3p)\alpha x^2} - 2ac_1^{(q-1)}e^{(q-1)(1-p)\alpha x^2} + \alpha\epsilon_1^2 \end{aligned}$$

This expression clearly goes to negative infinity as $|x| \to \infty$ for $p < 1$. Thus v_1 satisfies the third Lyapunov property everywhere on $\mathcal{R}_1$.

4.2.2. *Diffusive Regions.* In the diffusive region $\mathcal{R}_2$, the deterministic dynamics are unstable and the noise terms are essential to the existence of a local Lyapunov function. This local function, v_2, is constructed using the algorithm described in [AKM12]. It is a solution to a boundary-value problem of the form

$$\begin{cases} (\tilde{\mathcal{L}}_2v_2)(x,y) = -g_2(x,y) & \text{for } (x,y) \in \mathcal{R}_2 \\ v_2(x,y) \simeq v_1(x,y) & \text{for } (x,y) \in \partial\mathcal{R}_2 \end{cases} \tag{24}$$

where $\tilde{\mathcal{L}}_2$ consists of the terms in the generator $\mathcal{L}$ that scale dominantly in the region $\mathcal{R}_2$ and g_2 is chosen so that $\lim_{r\to\infty}\left[\inf_{(x,y)\in(\mathcal{R}_i\cap B_r^c)} g_i(x,y)\right] = \infty$. This method ensures that the construction satisfies the Lyapunov properties.

In $\mathcal{R}_2$, the dominant term in the generator is $\frac{\epsilon_2^2}{2}\frac{\partial^2}{\partial y^2}$. Hence, $\tilde{\mathcal{L}}_2 = \frac{\epsilon_2^2}{2}\frac{\partial^2}{\partial y^2}$. For simplicity (so that the chosen term is bounded in v_2), we choose $g_2(x,y) = k\epsilon_2^2 e^{p\alpha x^2}$, where $k > 0$ will be chosen later to ensure that v_2 is a local Lyapunov function. The function g_2 does converge to infinity on $\mathcal{R}_2$ since $\mathcal{R}_2$ consists of the decaying strip around the x-axis. With this choice for $\tilde{\mathcal{L}}_2$ and g_2, we can find an explicit solution to the boundary-value problem described by (24), which is given in the theorem below.

Lemma 4.2. *$v_2(x,y) = (1-p)\alpha x^2 - ky^2 e^{p\alpha x^2}$ is a local Lyapunov function on $\mathcal{R}_2$, for $\frac{3}{4} \geq p$ and $q > 1$.*

It is clear that v_2 is infinitely differentiable, so it satisfies the first Lyapunov condition. To see that it also satisfies the second condition, note that on $\mathcal{R}_2$, $y^2 e^{p\alpha x^2} \leq c_2$, so $v_2(x,y) \geq (1-p)\alpha x^2 - kc_2$. On $\mathcal{R}_2$, the only way for $|(x,y)|$ to approach infinity is with $|x| \to \infty$, so $v_2(x,y)$ must also tend toward infinity for $p < 1$.

Now we need to show that v_2 satisfies the third Lyapunov condition on $\mathcal{R}_2$:

$$
\begin{aligned}
\mathcal{L}v_2 &= (ye^{\alpha x^2} - xy^6 e^{3\alpha x^2})(2(1-p)\alpha x - 2pk\alpha xy^2 e^{p\alpha x^2}) - (\alpha xy^2 e^{\alpha x^2} + ay^{(2q-1)}e^{(q-1)\alpha x^2})(-2kye^{p\alpha x^2}) \\
&\quad + \frac{\epsilon_1^2}{2}[2(1-p)\alpha - 2pk\alpha y^2 e^{p\alpha x^2}(2p\alpha x^2+1)] + \frac{\epsilon_2^2}{2}(-2ke^{p\alpha x^2}) \\
&= 2(1-p)\alpha xye^{\alpha x^2} - 2pk\alpha xy^3 e^{(1+p)\alpha x^2} - 2(1-p)\alpha x^2 y^6 e^{3\alpha x^2} + 2pk\alpha x^2 y^8 e^{(p+3)\alpha x^2} \\
&\quad + 2k\alpha xy^3 e^{(1+p)\alpha x^2} + 2aky^{2q}e^{(p+q-1)\alpha x^2} + (1-p)\epsilon_1^2\alpha - \epsilon_1^2 pk\alpha y^2 e^{p\alpha x^2}(2p\alpha x^2+1) - \epsilon_2^2 ke^{p\alpha x^2} \\
&= 2(1-p)\alpha xye^{\alpha x^2} + 2(1-p)k\alpha xy^3 e^{(1+p)\alpha x^2} + 2aky^{2q}e^{(p+q-1)\alpha x^2} + (1-p)\epsilon_1^2\alpha \\
&\quad - \epsilon_1^2 pk\alpha y^2 e^{p\alpha x^2}(2p\alpha x^2+1) - \epsilon_2^2 ke^{p\alpha x^2} - 2(1-p)\alpha x^2 y^6 e^{3\alpha x^2} + 2pk\alpha x^2 y^8 e^{(p+3)\alpha x^2} \\
&\leq 2(1-p)\alpha|x||y|e^{\alpha x^2} + 2(1-p)k\alpha|x||y|c_2 e^{\alpha x^2} + 2akc_2^q e^{(p+q-1-pq)\alpha x^2} + (1-p)\epsilon_1^2\alpha \\
&\quad - \epsilon_1^2 pk\alpha c_2(2p\alpha x^2+1) - \epsilon_2^2 ke^{p\alpha x^2} - 2(1-p)\alpha x^2 c_2^3 e^{(3-3p)\alpha x^2} + 2pk\alpha x^2 c_2^4 e^{(3-3p)\alpha x^2} \\
&\leq 2(1-p)\alpha|x|(\sqrt{c_2}e^{\frac{-p\alpha x^2}{2}})e^{\alpha x^2} + 2(1-p)k\alpha|x|(\sqrt{c_2}e^{\frac{-p\alpha x^2}{2}})c_2 e^{\alpha x^2} + 2akc_2^q e^{(p+q-1-pq)\alpha x^2} \\
&\quad + (1-p)\epsilon_1^2\alpha - \epsilon_2^2 ke^{p\alpha x^2} + c_2^3\alpha x^2(-2(1-p) + 2pkc_2)e^{(3-3p)\alpha x^2} \\
&= \frac{\alpha}{2}|x|\sqrt{c_2}e^{(1-\frac{p}{2})\alpha x^2} + 2(1-p)k\alpha|x|c_2^{3/2}e^{(1-\frac{p}{2})\alpha x^2} + 2akc_2^q e^{(p+q-1-pq)\alpha x^2} + (1-p)\epsilon_1^2\alpha - \epsilon_2^2 ke^{p\alpha x^2} \\
&\quad + c_2^3\alpha x^2(-2(1-p) + 2pkc_2)e^{(3-3p)\alpha x^2}
\end{aligned}
$$

If $3/4 \geq p$ and $q < \frac{1}{1-p}$, then the last term is dominant. Thus we need the following inequality to hold:

$$
\begin{aligned}
-2(1-p) + 2pkc_2 &< 0 \\
pkc_2 &< 1-p \\
k &< \frac{1-p}{pc_2}
\end{aligned}
$$

If this is true, then the coefficient on the dominant term is negative. Thus $\mathcal{L}v_2$ goes to negative infinity on $\mathcal{R}_2$, and the third Lyapunov condition is satisfied.

4.2.3. *Global Lyapunov Function.* We now have local Lyapunov functions covering the entire plane; in order to show the stability of the system, we need to combine these local Lyapunov functions into one global Lyapunov function. This will be done in the same way as for the previous problems, using the mollifier function $\phi(t)$ to create a convex combination of v_1 and v_2 on the region where $\mathcal{R}_1$ and $\mathcal{R}_2$ overlap, with the result being differentiable at all points. More formally, we define $v_{12}(x, y)$ in the overlap region (denoted $\mathcal{R}_{12}$) as follows:

$$v_{12}(x, y) = \phi(s(x, y))v_1(x, y) + (1 - \phi(s(x, y)))v_2(x, y).$$

In this problem, we define $s(x, y) = \frac{y^2 e^{p\alpha x^2} - \ln(c_1)}{\ln(c_2) - \ln(c_1)}$ so that $s(x, y) = 0$ on the boundary of $\mathcal{R}_1$ and $s(x, y) = 1$ on the boundary of $\mathcal{R}_2$. We also set $c_1 = 1$ so that $s(x, y)$ reduces to $\frac{y^2 e^{p\alpha x^2}}{\ln(c_2)}$. Now we can show that v_{12} is a local Lyapunov function on the region where $\mathcal{R}_1$ and $\mathcal{R}_2$ overlap.

Lemma 4.3. *For constants $3/4 \geq p$, $c_1 = 1$, $c_2 > c_1$, and $k < \frac{1-p}{pc_2}$,*

$$v_{12}(x, y) = \phi(s(x, y))v_1(x, y) + (1 - \phi(s(x, y)))v_2(x, y)$$

is a local Lyapunov function on $\mathcal{R}_1 \cap \mathcal{R}_2$.

Since v_1, v_2, and $\phi(s)$ are all differentiable on this region, we know that v_{12} is differentiable everywhere it is defined. In this region, $|x| \to \infty$ whenever $|(x, y)| \to \infty$. We have already shown that both v_1 and v_2 tend to infinity as $|x| \to \infty$; therefore, v_{12} satisfies the second Lyapunov condition on $\mathcal{R}_{12}$. Now we need to show that v_{12} satisfies the third Lyapunov condition.

Applying the generator to v_{12}, we get:

$$\begin{aligned}
\mathcal{L}v_{12}(x, y) = {} & \phi(s(x, y))\mathcal{L}v_1(x, y) + (1 - \phi(s(x, y)))\mathcal{L}v_2(x, y) \\
& + \mathcal{L}[\phi(s(x, y))](v_1(x, y) - v_2(x, y)) \\
& + \epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x, y))]\frac{\partial}{\partial x}[v_1(x, y) - v_2(x, y)] \\
& + \epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x, y))]\frac{\partial}{\partial y}[v_1(x, y) - v_2(x, y)]
\end{aligned}$$

Now we will break apart this expression to analyze its terms:

$$\begin{aligned}
D &= \phi(s(x, y))\mathcal{L}v_1(x, y) + (1 - \phi(s(x, y)))\mathcal{L}v_2(x, y) \\
E &= \mathcal{L}[\phi(s(x, y))](v_1(x, y) - v_2(x, y)) \\
&\quad + \epsilon_1^2 \frac{\partial}{\partial x}[\phi(s(x, y))]\frac{\partial}{\partial x}[v_1(x, y) - v_2(x, y)] \\
&\quad + \epsilon_2^2 \frac{\partial}{\partial y}[\phi(s(x, y))]\frac{\partial}{\partial y}[v_1(x, y) - v_2(x, y)]
\end{aligned}$$

From the proofs for v_1 and v_2, we know that both of the terms of D approach negative infinity as $|x| \to \infty$, with dominant terms of order $x^2 e^{(3-3p)\alpha x^2}$. Now we will look at the first line of E. First, consider $(v_1 - v_2)$ on $\mathcal{R}_{12}$:

$$v_1 - v_2 = p\alpha x^2 + 2\ln|y| + ky^2 e^{p\alpha x^2}$$

$$\begin{aligned}
&\geq p\alpha x^2 + 2\ln(\sqrt{c_1}e^{-p\alpha x^2/2}) + kc_1 \\
&= p\alpha x^2 + \ln(c_1) + \ln(e^{-p\alpha x^2}) + kc_1 \\
&= \ln(c_1) + kc_1
\end{aligned}$$

By choosing $c_1 = 1$, we ensure that $v_1 - v_2$ is always positive on this region. Now we turn to $\mathcal{L}[\phi(s)]$. Since we are trying to show that $D+E$ goes to negative infinity, and we already know that D goes to negative infinity, we only need to consider the case where E is positive.

$$\begin{aligned}
\mathcal{L}[\phi(s)] &= (ye^{p\alpha x^2} - xy^6e^{3\alpha x^2})\phi'(s)\frac{\partial s}{\partial x} + (-xy^2e^{p\alpha x^2} - ay^{(2q-1)}e^{(q-1)\alpha x^2})\phi'(s)\frac{\partial s}{\partial y} \\
&+ \frac{\epsilon_1^2}{2}[\phi'(s)\frac{\partial^2 s}{\partial x^2} + \phi''(s)(\frac{\partial s}{\partial x})^2] + \frac{\epsilon_2^2}{2}[\phi'(s)\frac{\partial^2 s}{\partial y^2} + \phi''(s)(\frac{\partial s}{\partial y})^2] \\
&= (ye^{p\alpha x^2} - xy^6e^{3\alpha x^2})\phi'(s)(\frac{2p\alpha xy^2e^{p\alpha x^2}}{\ln(c_2)}) + (-xy^2e^{p\alpha x^2} - ay^{(2q-1)}e^{(q-1)\alpha x^2})\phi'(s)(\frac{2ye^{p\alpha x^2}}{\ln(c_2)}) \\
&+ \frac{\epsilon_1^2}{2}[\phi'(s)(\frac{2p\alpha y^2e^{p\alpha x^2}}{\ln(c_2})(2p\alpha x^2+1) + \phi''(s)(\frac{4p^2\alpha^2x^2y^4e^{2p\alpha x^2}}{\ln(c_2)^2})] \\
&+ \frac{\epsilon_2^2}{2}[\phi'(s)\frac{2e^{p\alpha x^2}}{\ln(c_2)} + \phi''(s)(\frac{4y^2e^{2p\alpha x^2}}{\ln(c_2)^2})] \\
&= \frac{\phi'(s)}{\ln(c_2)}(2p\alpha xy^3e^{2p\alpha x^2} - 2p\alpha x^2y^8e^{(3+p)\alpha x^2}) + \frac{\phi'(s)}{\ln(c_2)}(-2xy^3e^{2p\alpha x^2} - 2ay^{2q}e^{(p+q-1)\alpha x^2}) \\
&+ \frac{\epsilon_1^2}{2}[\phi'(s)(\frac{2p\alpha y^2e^{p\alpha x^2}}{\ln(c_2})(2p\alpha x^2+1) + \phi''(s)(\frac{4p^2\alpha^2x^2y^4e^{2p\alpha x^2}}{\ln(c_2)^2})] \\
&+ \frac{\epsilon_2^2}{2}[\phi'(s)\frac{2e^{p\alpha x^2}}{\ln(c_2)} + \phi''(s)(\frac{4y^2e^{2p\alpha x^2}}{\ln(c_2)^2})] \\
&\leq \frac{\phi'(s)}{\ln(c_2)}(2p\alpha c_2|x||y|e^{p\alpha x^2} - 2p\alpha x^2c_2^4e^{(3-3p)\alpha x^2}) + \frac{\phi'(s)}{\ln(c_2)}(2c_2|x||y|e^{p\alpha x^2} - 2ac_2^qe^{(p+q-1-pq)\alpha x^2}) \\
&+ \frac{\epsilon_1^2}{2}[\phi'(s)(\frac{2p\alpha c_2}{\ln(c_2})(2p\alpha x^2+1) + \phi''(s)(\frac{4p^2\alpha^2x^2c_2^2}{\ln(c_2)^2})] \\
&+ \frac{\epsilon_2^2}{2}[\phi'(s)\frac{2e^{p\alpha x^2}}{\ln(c_2)} + \phi''(s)(\frac{4c_2e^{p\alpha x^2}}{\ln(c_2)^2})]
\end{aligned}$$

The dominant term in this part of E is of order $x^2e^{(3-3p)\alpha x^2}$ and is always negative. Now we move on to the second part of E:

$$\begin{aligned}
\epsilon_1^2\frac{\partial}{\partial x}[\phi(s)]\frac{\partial}{\partial x}[v_1-v_2] &= \epsilon_1^2\phi'(s)\frac{\partial s}{\partial x}\frac{\partial}{\partial x}[v_1-v_2] \\
&= \epsilon_1^2\phi'(s)(\frac{2p\alpha xy^2e^{p\alpha x^2}}{\ln(c_2)})(2p\alpha x + 2pk\alpha xy^2e^{p\alpha x^2}) \\
&\leq \frac{\epsilon_1^2|\phi'(s)|}{\ln(c_2)}(4p^2\alpha^2x^2y^2e^{p\alpha x^2} + 4p^2\alpha^2x^2y^4e^{2p\alpha x^2})
\end{aligned}$$

$$\leq \frac{\epsilon_1^2|\phi'(s)|}{\ln(c_2)}(4p^2\alpha^2x^2c_2 + 4p^2\alpha^2x^2c_2^2)$$

The dominant terms in this part are order x^2, so they are not big enough to affect the asymptotic behavior of $D + E$. There is one more part of E that must be examined. It is:

$$\begin{aligned}
\epsilon_2^2\frac{\partial}{\partial y}[\phi(s)]\frac{\partial}{\partial y}[v_1 - v_2] &= \epsilon_2^2\phi'(s)\frac{\partial s}{\partial y}\frac{\partial}{\partial y}[v_1 - v_2] \\
&= \epsilon_2^2\phi'(s)(\frac{2ye^{p\alpha x^2}}{\ln(c_2)})(\frac{2}{y} + 2kye^{p\alpha x^2}) \\
&= \frac{\epsilon_2^2\phi'(s)}{\ln(c_2)}(4e^{p\alpha x^2} + 4ky^2e^{2p\alpha x^2}) \\
&\leq \frac{4\epsilon_2^2\phi'(s)}{\ln(c_2)}(e^{p\alpha x^2} + kc_2e^{p\alpha x^2})
\end{aligned}$$

The dominant terms here are order $e^{p\alpha x^2}$: smaller than the dominant terms in D as long as $p \leq 3/4$. Thus we know that, as $|x| \to \infty$, $\mathcal{L}v_{12}(x, y) \to -\infty$. This completes the proof that $v_{12}(x, y)$ is a Lyapunov function on $\mathcal{R}_{12}$.

Now we can define a single global Lyapunov function which is needed to show that the perturbed Hamiltonian system with noise is stable. We call this function $V(x, y)$, and define it as follows:

$$V(x,y) = \begin{cases} v_1(x,y) & \text{for} \quad y^2e^{p\alpha x^2} > c_2 \\ v_2(x,y) & \text{for} \quad y^2e^{p\alpha x^2} < c_1 \\ v_{12}(x,y) & \text{for} \quad c_1 \leq y^2e^{p\alpha x^2} \leq c_2 \end{cases} \tag{25}$$

By the lemmas above, we know that $V(x, y)$ satisfies the conditions for stability given in Section 1. Therefore the stochastic system

$$\begin{aligned}
\frac{dX_t}{dt} &= \quad (Y_te^{\alpha X_t^2} - X_tY_t^6e^{3\alpha X_t^2}) + \epsilon_1\frac{dB_1(t)}{dt} \\
\frac{dY_t}{dt} &= (-(\alpha X_tY_t^2)e^{\alpha X_t^2} - aY_t^3e^{\alpha X_t^2}) + \epsilon_2\frac{dB_2(t)}{dt}
\end{aligned}$$

is stable. This completes the proof of Theorem 3.

5. Conclusion

This paper demonstrated three cases in which perturbed Hamiltonian systems were stabilized by noise. It is likely that there are many other unstable Hamiltonian systems which could be modified and then stabilized by noise in a similar manner. One of the purposes of this paper is to provide a systematic framework for deterministically perturbing such systems to allow for noise-induced stabilization. However, we provide no systematic means of identifying Hamiltonian systems which are candidates for this type of phenomenon. One feature that is common to all of the systems studied here is a line that forms a continuum of equilibrium points; the change of sign of dX_t and dY_t on either side of this line is one of the fundamental features that allows for noise-induced stabilization to occur.

Another question that remains open is how to identify the smallest possible perturbation which allows noise-induced stabilization to occur. Modifications of larger magnitude can make proof of stability substantially easier, but also entail greater deterministic deviation from the original system which we sought to stabilize. Our modified deterministic systems exhibit the same limiting behavior as the original Hamiltonian systems for all initial conditions; we do not know if a stricter definition of "qualitatively similar behavior" allows for modifications which enable noise-induced stabilization. Much space remains for further research into these questions and others related to noise-induced stabilization.

References

[AKM12] Avanti Athreya, Tiffany Kolba, and Jonathan C Mattingly. Propagating lyapunov functions to prove noise-induced stabilization. *Electron. J. Probab*, 17(96):1–38, 2012.

[CHM+11] Ben Cooke, David P Herzog, Jonathan C Mattingly, Scott A McKinley, and Scott C Schmidler. Geometric ergodicity of two–dimensional hamiltonian systems with a lennard–jones–like repulsive potential. *arXiv preprint arXiv:1104.3842*, 2011.

[HM+15] David Herzog, Jonathan Mattingly, et al. Noise-induced stabilization of planar flows i. *Electronic Journal of Probability*, 20, 2015.

[HSD12] Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos.* Academic press, 2012.

[Kha11] Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.

[Law06] Gregory F Lawler. *Introduction to stochastic processes.* CRC Press, 2006.

[LS17] Matti Leimbach and Michael Scheutzow. Blow-up of a stable stochastic differential equation. *Journal of Dynamics and Differential Equations*, 29(2):345–353, 2017.