Pattern Avoidance in Reverse Double Lists

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Abstract

Pattern avoidance is a branch of combinatorics that arose in 1968 when Donald Knuth began studying stack sorting. One central problem in pattern avoidance is finding the number of permutations of length n that avoid a specific pattern ρ. We expanded this problem to reverse double lists, or lists built by combining a permutation with its reverse. We computed the number of reverse double lists of each length that avoid patterns of up to length four and then conjectured and proved formulas to explain these sequences.

1 Introduction

The term permutation is familiar to many areas of mathematics. For the purpose of this research, a **permutation**, denoted π, of length n is an ordering of the numbers in the set $\{1, 2, \ldots, n\}$ such that no digits are repeated. The set of permutations of length n is denoted $\mathcal{S}_n$. We can use reduction to find smaller permutations within larger ones. The **reduction** of a list of numbers is the list obtained by replacing the i^{th} smallest numbers in the list with i. If we reduce the list 5249, we get $red(5249) = 3124$. A permutation π **contains** ρ as a pattern if π has a subsequence that reduces to ρ. On the other hand, π **avoids** ρ if π has no subsequence that reduces to ρ. We will denote $\mathcal{S}_n(\rho)$ as the set of permutations in $\mathcal{S}_n$ that avoid ρ.

The reverse and complement of a list are significant in pattern avoidance, because they capture relevant structure. The **reverse** of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is simply $\pi^r = \pi_n\pi_{n-1}\cdots\pi_1$. The **complement** of π, denoted π^c, is created by replacing the i^{th} smallest number of π with the i^{th} largest number of π. Furthermore, we use dot diagrams to give a visual representation of permutations. The numbers in a permutation represent relative heights. For example, the dot diagram for 23415 is shown in the figure below:

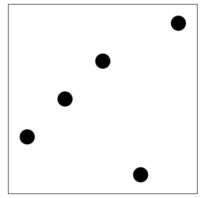

Figure 1: The dot diagram for 23415.

From any dot diagram, the reverse and complement of a permutation can be easily drawn. The reverse is simply flipped over the vertical axis while the complement is flipped over the horizontal axis.

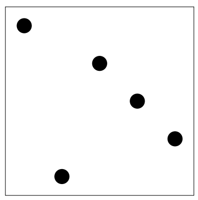

Figure 2: The reverse of 23415.

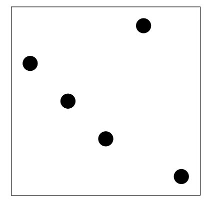

Figure 3: The complement of 23415.

For more background, we will look at other mathematicians who have studied and generalized pattern avoidance. In 1998, Burnstein [1] considered words or lists with repeated digits that avoid given patterns. Others have studied pattern avoidance with special symmetries. For example, Egge [3] looked at pattern avoidance in permutations π such that $\pi^{rc} = \pi$, and Ferrari [4] extended these results to reverse-complement invariant words. Our work considers pattern avoidance in words with a different type of symmetry. Specifically, we call these wodrs reverse double lists. This research was inspired by Cratty, Erickson, Negassi, and Pudwell [2]. We continue their research, but with a different structure of double list. A **reverse double list** is a permutation prepended to its reverse. We write $\mathcal{R}_n$ for the set of reverse double lists, so $\mathcal{R}_n = \{\pi\pi^r | \pi \in \mathcal{S}_n\}$. Below is a quick example of how $\mathcal{S}_3$ builds into $\mathcal{R}_3$.

$$\begin{aligned} \mathcal{S}_3 &= \{123, 132, 213, 231, 312, 321\} \\ \mathcal{R}_3 &= \{123321, 132231, 213312, 231132, 312213, 321123\} \end{aligned}$$

In this paper, we will enumerate and prove how many reverse double lists of each length avoid permutation patterns of up to length four. However, avoiding patterns of length one, two, and three proved to be a trivial task, so we will primarily focus on avoiding patterns of length four.

Since there are 24 patterns of length four, we developed the following theorems, which show that several patterns are guaranteed to have the same enumeration sequence. As a result, instead of proving results for each of the 24 individual patterns, we can simply prove results for the eight equivalence classes of patterns.

Theorem 1.1. *A permutation π contains a pattern ρ iff π^r contains ρ^r.*

Proof. ($\Rightarrow$) Suppose a permutation π contains the pattern ρ. When reading π from right to left, (equivalently, reading π^r from left to right), there is a copy of ρ^r using the same digits that originally reduced to ρ when reading π from left to right.

($\Leftarrow$) Suppose π^r contains ρ^r. When reading π^r from right to left, there is a copy of ρ using the same digits that originally reduced to ρ^r when reading π^r from left to right. $\square$

By the contrapositive, we obtain the following corollary:

Corollary 1.2. *A permutation π avoids a pattern ρ iff π^r avoids ρ^r.*

Theorem 1.3. *A permutation π contains a pattern ρ iff π^c contains ρ^c.*

Proof. ($\Rightarrow$) Suppose a permutation π contains the pattern ρ. Then in π^c, the digits that originally formed ρ will be changed to ρ^c.
($\Leftarrow$) Suppose π^c contains the pattern ρ^c. In $(\pi^c)^c = \pi$, the digits that originally formed ρ^c will be changed to ρ. $\square$

By the contrapositive, we obtain the following corollary:

Corollary 1.4. *A permutation π avoids a pattern ρ iff π^c avoids ρ^c.*

Notice that these arguments also apply to reverse double lists. Let $\mathcal{R}_n(\rho)$ be the set of reverse double lists of semilength n avoiding ρ. We say ρ_1 is Wilf-equivalent to ρ_2 if $|\mathcal{R}_n(\rho_1)| = |\mathcal{R}_n(\rho_2)|$ for $n \geq 1$, and we write $\rho_1 \sim \rho_2$. By the results above, $\rho_1 \sim (\rho_1)^r \sim (\rho_1)^c$ for any pattern ρ. A maximal set of patterns that are Wilf-equivalent are called a Wilf class. Overall, these definitions and theorems will enable us to condense multiple patterns into equivalence classes.

2 Avoiding Patterns of Length 1, 2, and 3

Determining the number of reverse double lists that avoid patterns of length one and two is especially trivial. When avoiding patterns of length one, no reverse double list can avoid the only one length pattern, 1. Since the smallest reverse double list is 11, which contains 1, all longer lists contain 1 as well. Therefore, $|\mathcal{R}_n(1)| = 0$ for $n \geq 1$.

The only two patterns of length two are 12 and 21. By brute force,

$$|\mathcal{R}_n(12)| = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

By Corollary 1.2 and Corollary 1.4, $|\mathcal{R}_n(12)| = |\mathcal{R}_n(21)|$.

Avoiding patterns of length three is less trivial, yet still simple. In this case, there are six permutation patterns (grouped into two equivalence classes) that we will consider.

Theorem 2.1. $|\mathcal{R}_n(123)| = \begin{cases} 1 & \textit{for } n = 1, \\ 2 & \textit{for } n = 2, 3, \\ 0 & \textit{for } n \geq 4. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2 \cdots \pi_n\pi_n \cdots \pi_2\pi_1 \in \mathcal{R}_n(123)$.

The cases when $n \in \{1, 2, 3, 4\}$ can be determined using brute force. Since $|\mathcal{R}_4(123)| = 0$, there can't exist any reverse double lists of longer length that avoid 123. Therefore, $|\mathcal{R}_n(123)| = 0$ for $n \geq 4$. $\square$

Now, by Corollaries 1.2 and 1.4, we obtain Corollary 2.2, which follows from Theorem 2.1.

Corollary 2.2. $|\mathcal{R}_n(321)| = |\mathcal{R}_n(123)|$.

We will be able to generalize Theorem 2.1 and Corollary 2.2 in Section 4.

Theorem 2.3. $|\mathcal{R}_n(132)| = \begin{cases} 1 & \textit{for } n = 1, \\ 2 & \textit{for } n \geq 2. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2 \cdots \pi_n\pi_n \cdots \pi_2\pi_1 \in \mathcal{R}_n(132)$.

The cases when $n \in \{1, 2\}$ can be verified using brute force.

Assume for contradiction that $\pi_1 \neq n$ for $n \geq 3$. This implies that $\pi_1 = 1$ or $2 \leq \pi_1 \leq n-1$. If $\pi_1 = 1$, then there exist 1 and 3 in π, and there exists 2 in π^r such that $red(132) = 132$. If $2 \leq \pi_1 \leq n-1$, then there exists 1 in π, and there exist n and π_1 in π^r such that $red(1n\pi_1) = 132$. Thus, $\pi_1 = n$.

Assume for contradiction that $\pi_2 \neq n-1$ for $n \geq 4$. This implies $\pi_2 = 1$ or $2 \leq \pi_2 \leq n-2$. If $\pi_2 = 1$, then there exist 1 and 3 in π, and there exists 2 in π^r such that $red(132) = 132$. If $2 \leq \pi_2 \leq n-2$, then there exists 1 in π, and there exist $n-1$ and π_2 in π^r such that $red(1(n-1)\pi_2) = 132$. Hence, $\pi_2 = n-1$. By a similar argument, $\pi_i = n-i+1$ for $1 \leq i \leq n-2$. Then, $\pi_{n-1} = 2$ and $\pi_n = 1$ or $\pi_{n-1} = 1$ and $\pi_n = 2$. Hence, $|\mathcal{R}_n(132)| = 2$ when $n \geq 3$. (See Figure 4.)

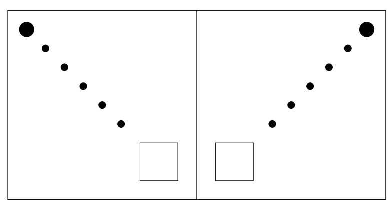

Figure 4: The 2 reverse double lists when $\pi_1 = n$.

□

By Corollaries 1.2 and 1.4, Corollary 2.4 follows from Theorem 2.3.

Corollary 2.4. $|\mathcal{R}_n(132)| = |\mathcal{R}_n(231)| = |\mathcal{R}_n(312)| = |\mathcal{R}_n(213)|$.

Now that we have handled avoiding patterns of lengths less than four, we will move on to avoiding patterns of length four.

3 Avoiding Patterns of Length 4

This section focuses on avoiding patterns of length four. To begin, we enumerate the reverse double lists of length $1 \leq n \leq 9$ that avoid each pattern of length four. Then, we conjecture and prove formulas that explain these enumerations.

3.1 Initial Data

Using Sage, we wrote a computer program to enumerate reverse double lists of length $1 \le n \le 9$ that avoid each pattern of length four. These results have been collected in Table 1, which shows the patterns of length four in the left column and the corresponding enumerations in the right column. By Corollaries 1.2 and 1.4, we know that some patterns are guaranteed to produce the same enumeration. In Table 1, $\rho_1 \sim \rho_2$ denotes that the patterns ρ_1 and ρ_2 are reverses or complements of each other.

Generally, there exist eight Wilf classes of length four. At first glance, it appears that Table 1 gives seven equivalence classes. Upon closer observation, we can see that row five of the table actually groups two Wilf classes together. This is solely based on the enumeration $\{|\mathcal{R}_n(\rho)|\}_{n=1}^{9}$. Observe that $2143 \sim 3412$ are reverses *and* complements of each other, and $1324 \sim 4231$ are reverses *and* complements of each other. However, these two pairs are not otherwise associated by reverses or complements. By coincidence, these two Wilf classes share the same enumeration for $\{|\mathcal{R}_n(\rho)|\}_{n=1}^{9}$.

Pattern ρ	$\{\lvert\mathcal{R}_n(\rho)\rvert\}_{n=1}^{9}$
$1234 \sim 4321$	$1, 2, 6, 16, 32, 32, 0, 0, 0$
$1243 \sim 2134 \sim 3421 \sim 4312$	$1, 2, 6, 16, 34, 62, 102, 156, 226$
$1324 \sim 4231$, $2143 \sim 3412$	$1, 2, 6, 16, 36, 76, 156, 316, 636$
$1423 \sim 2314 \sim 3241 \sim 4132$	$1, 2, 6, 16, 36, 80, 178, 394, 870$
$1432 \sim 2341 \sim 3214 \sim 4123$	$1, 2, 6, 16, 38, 92, 222, 536, 1294$
$1342 \sim 2431 \sim 3124 \sim 4213$	$1, 2, 6, 16, 40, 98, 238, 576, 1392$
$2413 \sim 3142$	$1, 2, 6, 16, 44, 120, 328, 896, 2448$

Table 1: The enumeration of reverse double lists that avoid ρ.

3.2 Proofs

After computing the number of the reverse double lists that avoid each pattern of length four, we conjectured formulas to explain these sequences. In the following section, we will state and prove formulas for each of these patterns.

3.2.1 Avoiding 1234

To begin, we will look at avoiding the pattern 1234.

Theorem 3.1. $|\mathcal{R}_n(1234)| = \begin{cases} n! & \textit{for } n \le 3, \\ 16 & \textit{for } n = 4, \\ 32 & \textit{for } n = 5, 6, \\ 0 & \textit{for } n \ge 7. \end{cases}$

Proof. The cases when $n \le 7$ can be verified using brute force (see Table 1). Since $|\mathcal{R}_7(1234)| = 0$, there are no reverse double lists of longer length that avoid 1234, so $|\mathcal{R}_n(1234)| = 0$ for $n \ge 7$. $\square$

By Corollaries 1.2 and 1.4, we obtain Corollary 3.2, which follows from Theorem 3.1.

Corollary 3.2. $|\mathcal{R}_n(1234)| = |\mathcal{R}_n(4321)|$.

3.2.2 Avoiding 1432

While observing the sequences in the right hand column of Table 1, we noticed that most of them were recursive. We will highlight this recursive structure by avoiding 1432, which has the most easily proven formula.

Theorem 3.3. $|\mathcal{R}_n(1432)| = \begin{cases} n! & \textit{for } n \leq 3, \\ 16 & \textit{for } n = 4, \\ 2|\mathcal{R}_{n-1}(1432)| + |\mathcal{R}_{n-2}(1432)| & \textit{for } n \geq 5. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2\cdots\pi_n\pi_n\cdots\pi_2\pi_1 \in \mathcal{R}_n(1432)$.

The cases when $n \in \{1, 2, 3, 4\}$ can be verified using brute force.

Case 1. Suppose $\pi_1 = n$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(1432)$. Consider $n\sigma\sigma^r n$. The only role n can play in a 1432 pattern is a 4. As a result, it cannot reduce to a 1 to begin the pattern earlier or reduce to a 2 to help finish the pattern. Therefore, $n\sigma\sigma^r n \in \mathcal{R}_n(1432)$. Figure 5 gives a visual representation of n added to the beginning and end of $\sigma\sigma^r$. So, there exist $|\mathcal{R}_{n-1}(1432)|$ reverse double lists with $\pi_1 = n$.

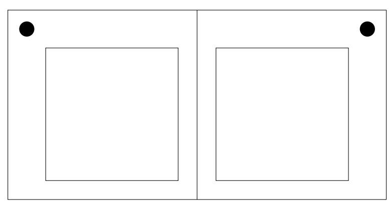

Figure 5: The $|\mathcal{R}_{n-1}(1432)|$ reverse double lists when $\pi_1 = n$.

Case 2. Suppose $\pi_1 = n - 1$.

Since $n-1$ can only play the role of a 3 or a 4 in a 1432 pattern, it cannot play the role of 1 at the beginning or 2 at the end. Also, n and $n-1$ simply switched positions from Case 1, so n just plays the role that $n-1$ did. Hence, there are $|\mathcal{R}_{n-1}(1432)|$ lists where $\pi_1 = n - 1$. (See Figure 6.)

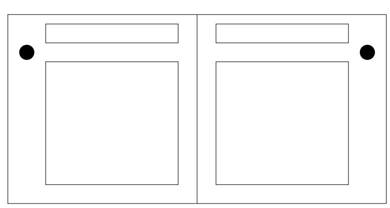

Figure 6: The $|\mathcal{R}_{n-1}(1432)|$ reverse double lists when $\pi_1 = n - 1$.

Case 3. Suppose $\pi_1 = n-2$. Then, $\pi_2 = n$.

Assume for contradiction that $\pi_2 \neq n$. This implies $\pi_2 = n-1$ or $1 \leq \pi_2 \leq n-3$. If $\pi_2 = n-1$, then there exists 1 in π, and there exist $n, n-1$, and $n-2$ in π^r such that $red(1n(n-1)(n-2)) = 1432$. If $1 \leq \pi_2 \leq n-3$, then there exist π_2 and n in π, and there exist $n-1$ and $n-2$ in π^r such that $red(\pi_2 n(n-1)(n-2)) = 1432$. Thus, if $\pi_1 = n-2$, then $\pi_2 = n$.

Together, $(n-2)n$ can only play the role of 24 in a length four pattern. Therefore, $(n-2)n$ cannot play the role of 14 in π or 32 in π^r. Individually, n can only play the role of a 4, so it will neither begin nor end a 1432 pattern. Also, $n-2$ can only play the role of 2, 3, or 4, so it will not play the role of 1 in the beginning of a 1432 pattern. If $n-2$ plays as a 2 in π^r, then $n-1$ in π^r can play as the 3 and n in π can play as the 4. However, there is no number to play the role of 1 that precedes n in π. Thus, the remaining positions can be filled in $|\mathcal{R}_{n-2}(1432)|$ ways. (See Figure 7.)

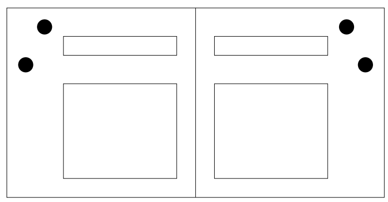

Figure 7: The $|\mathcal{R}_{n-2}(1432)|$ reverse double lists when $\pi_1 = n-2$.

Case 4. Assume for contradiction $1 \leq \pi_1 \leq n-3$. This implies $\pi_1 = 1$ or $2 \leq \pi_1 \leq n-3$.

If $\pi_1 = 1$, then $2 \leq \pi_2 \leq n-2$ or $\pi_2 \in \{n-1, n\}$. If $2 \leq \pi_2 \leq n-2$, then there exist 1 and n in π, and there exist $n-1$ and π_2 in π^r such that $red(1n(n-1)\pi_2) = 1432$. If $\pi_2 = n-1$ or $\pi_2 = n$, then there exist $1, \pi_2$, and $n-2$ in π, and there exists $n-3$ in π^r such that $red(1\pi_2(n-2)(n-3)) = 1432$. Thus, $\pi_1 \neq 1$.

If $2 \leq \pi_1 \leq n-3$, then $\pi_2 = n$ or $1 \leq \pi_2 \leq n-1$. If $\pi_2 = n$, then there exist π_1, n, and $n-1$ in π, and there exists $n-2$ in π^r, such that $red(\pi_1 n(n-1)(n-2)) = 1432$. If $1 \leq \pi_2 \leq n-1$, then two cases must be considered. If $\pi_1 < \pi_2$, then there exists 1 in π, and there exist n, π_2, and π_1 in π^r, such that $red(1n\pi_2\pi_1) = 1432$. If $\pi_1 > \pi_2$, then there exist π_2 and n in π, and there exist $n-1$ and π_1 in π^r, such that $red(\pi_2 n(n-1)\pi_1) = 1432$. Thus, $\pi_1 \geq n-2$.

Therefore, $|\mathcal{R}_n(1432)| = 2|\mathcal{R}_{n-1}(1432)| + |\mathcal{R}_{n-2}(1432)|$ for $n \geq 5$. $\square$

Now, by Corollaries 1.2 and 1.4, we obtain Corollary 3.4, which follows from Theorem 3.3.

Corollary 3.4. $|\mathcal{R}_n(1432)| = |\mathcal{R}_n(2341)| = |\mathcal{R}_n(3214)| = |\mathcal{R}_n(4123)|$.

3.2.3 Avoiding 1342

When avoiding 1342, $\mathcal{R}_n(1342)$ appears to be primarily built from members of $\mathcal{R}_{n-1}(1342)$ and $\mathcal{R}_{n-2}(1342)$, which is similar to $\mathcal{R}_n(1432)$. In Theorem 3.5, we observe that the formulas for avoiding 1432 and 1342 are very similar except $\mathcal{R}_n(1342)$ contains an extra two reverse double lists compared to $\mathcal{R}_n(1432)$.

Theorem 3.5. $|\mathcal{R}_n(1342)| = \begin{cases} n! & \textit{for } n \leq 3, \\ 2|\mathcal{R}_{n-1}(1342)| + |\mathcal{R}_{n-2}(1342)| + 2 & \textit{for } n \geq 4. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2 \cdots \pi_n\pi_n \cdots \pi_2\pi_1 \in \mathcal{R}_n(1342)$.

The cases when $n \in \{1, 2, 3\}$ can be verified using brute force.

Case 1. Suppose $\pi_1 = n$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(1342)$. Consider $n\sigma\sigma^r n$. Since n can only play the role of a 4 in a 1342 pattern, $n\sigma\sigma^r n \in \mathcal{R}_n(1342)$. So, there are $|\mathcal{R}_{n-1}(1342)|$ reverse double lists with $\pi_1 = n$. (See Figure 5.)

Case 2. Suppose $\pi_1 = n - 1$.

Since $n - 1$ can only play the role of a 3 or a 4 in a 1342 pattern, it will not aid in the generation of a 1342 pattern. Now, n plays the role that $n - 1$ played in the reverse double lists from Case 1. Hence, there are $|\mathcal{R}_{n-1}(1342)|$ lists where $\pi_1 = n - 1$. (See Figure 6.)

Case 3. Suppose $\pi_1 = n - 2$. Then, $\pi_2 = n - 1$.

Assume for contradiction that $\pi_2 \neq n - 1$. This implies that $\pi_2 = n$ or $1 \leq \pi_2 \leq n - 3$. If $\pi_2 = n$, then there exists 1 in π, and there exist $n - 1, n$, and $n - 2$ in π^r such that $red(1(n-1)n(n-2)) = 1342$. If $1 \leq \pi_2 \leq n - 3$, then there exist π_2 and $n - 1$ in π, and there exist n and $n - 2$ in π^r such that $red(\pi_2(n-1)n(n-2)) = 1342$. Thus, if $\pi_1 = n - 2$, then $\pi_2 = n - 1$.

Together $(n-2)(n-1)$ can only play the role of 23 or 34 in π, so it will not aid in the generation of a 1342 pattern. Individually, $n - 1$ can only play the role of a 3 or 4, so it will not play as a 1 in π or 2 in π^r. Also, $n - 2$ cannot play the role of a 1. If $n - 2$ plays the role of 2, there exists n in π^r to play the role of 4. Now, the only number to play 3 is $n - 1$ in π. However, numbers that could play the role of a 1 do not precede $n - 1$ in π. Thus, the remaining positions can be filled in $|\mathcal{R}_{n-2}(1342)|$ ways. (See Figure 8.)

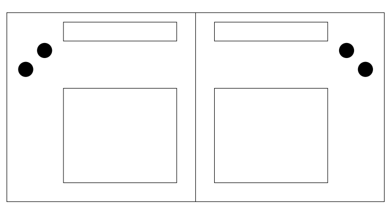

Figure 8: The $|\mathcal{R}_{n-2}(1342)|$ reverse double lists when $\pi_1 = n - 2$.

Case 4. Suppose $\pi_1 = 1$. Then $\pi_2 = n$.

Assume for contradiction that $\pi_2 \neq n$. This implies that $\pi_2 = 2$ or $3 \leq \pi_2 \leq n - 1$. If $\pi_2 = 2$, then there exist 1 and 3 in π, and there exist 4 and 2 in π^r such that $red(1342) = 1342$. If $3 \leq \pi_2 \leq n - 1$, then there exist $1, \pi_2$ and n in π, and there exists 2 in π^r such that $red(1\pi_2 n2) = 1342$. Hence, when $\pi_1 = 1$, $\pi_2 = n$. By a similar argument, $\pi_i = n - i + 2$ for $2 \leq i \leq n - 2$. Finally, either $\pi_{n-1} = 2$ and $\pi_n = 3$ or $\pi_{n-1} = 3$ and $\pi_n = 2$. Thus, there are two ways to avoid 1342 when $\pi_1 = 1$. (See Figure 9.)

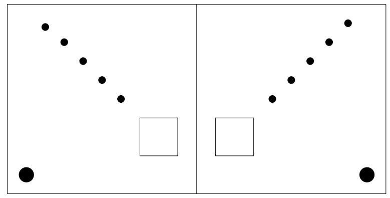

Figure 9: The 2 reverse double lists when $\pi_1 = 1$.

Case 5. Assume for contradiction $2 \leq \pi_1 \leq n-3$.

Then $\pi_2 = 1$, $2 \leq \pi_2 \leq n-2$, or $\pi_2 \in \{n-1, n\}$. If $\pi_2 = 1$, then there exist 1 and $n-1$ in π, and there exist n and π_1 in π^r such that $red(1(n-1)n\pi_1) = 1342$. If $2 \leq \pi_2 \leq n-2$, then two cases must be considered. If $\pi_1 < \pi_2$, then there exist π_1 and $n-1$ in π, and there exist n and π_2 in π^r such that $red(\pi_1(n-1)n\pi_2) = 1342$. If $\pi_1 > \pi_2$, then there exist π_2 and $n-1$ in π, and there exist n and π_1 in π^r such that $red(\pi_2(n-1)n\pi_1) = 1342$. If $\pi_2 \in \{n-1, n\}$, then there exists 1 in π, and there exist $n-2, \pi_2$, and π_1 and π^r such that $red(1(n-2)\pi_2\pi_1) = 1342$. Hence, $\pi_1 \in \{1, n-2, n-1, n\}$.

Ultimately, $|\mathcal{R}_n(1342)| = 2|\mathcal{R}_{n-1}(1342)| + |\mathcal{R}_{n-2}(1342)| + 2$ for $n \geq 4$. □

With this result, we obtain the following corollary when considering the statements of Corollaries 1.2 and 1.4.

Corollary 3.6. $|\mathcal{R}_n(1342)| = |\mathcal{R}_n(2431)| = |\mathcal{R}_n(3124)| = |\mathcal{R}_n(4213)|$.

3.2.4 Avoiding 2413

Now, we look at avoiding the pattern 2413. In Theorem 3.7, we observe that the reverse double lists of length n are built out of the reverse double lists of semilength $n-1$ and $n-2$.

Theorem 3.7. $|\mathcal{R}_n(2413)| = \begin{cases} n! & \textit{for } n \leq 2, \\ 2|\mathcal{R}_{n-1}(2413)| + 2|\mathcal{R}_{n-2}(2413)| & \textit{for } n \geq 3. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2 \cdots \pi_n\pi_n \cdots \pi_2\pi_1$ where $\pi\pi^r \in \mathcal{R}_n(2413)$.

The cases when $n \in \{1, 2\}$ can be verified using brute force methods.

Case 1. Suppose $\pi_1 = n$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(2413)$. Consider $n\sigma\sigma^r n$. Since n can only play the role of a 4 in a 2413 pattern, $n\sigma\sigma^r n \in \mathcal{R}_n(2413)$. So, there are $|\mathcal{R}_{n-1}(2413)|$ reverse double lists with $\pi_1 = n$. (See Figure 5.)

Case 2. Suppose $\pi_1 = n-1$. Then, $\pi_2 = n$.

Assume for contradiction that $\pi_2 \neq n$. This implies $\pi_2 = 1$ or $2 \leq \pi_2 \leq n-2$. If $\pi_2 = 1$, then there exists 2 in π, and there exist $n, 1$, and $n-1$ in π^r such that $red(2n1(n-1)) = 2413$. If $2 \leq \pi_2 \leq n-2$, then there exist π_2 and n in π, and there exist 1 and $n-1$ in π^r such that $red(\pi_2 n1(n-1)) = 2413$. Hence, if $\pi_1 = n-1$, then $\pi_2 = n$.

Together, $(n-1)n$ can only play the role of 34 in π, so they will not play the role of 24 in π or 13 in π^r to help generate a 2413 pattern. When considered individually, $n-1$ can only play the role of a 3 or 4, so it cannot play the role of 2 at the beginning of a 2413 pattern. If $n-1$ plays the role of 3, the numbers 1 in π^r and n in π can play the roles of

1 and 4 respectively. However, there does not exist a number before n in π to play the role of 2. Also, n can only play the role of a 4, so it will neither begin nor end a 2413 pattern. Now, the remaining positions can be filled in $|\mathcal{R}_{n-2}(2413)|$ ways to avoid a 2413 pattern. (See Figure 10.)

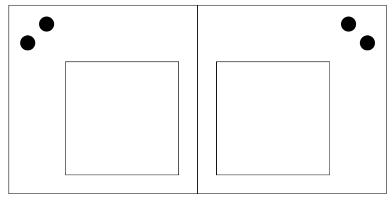

Figure 10: The $|\mathcal{R}_{n-2}(2413)|$ reverse double lists when $\pi_1 = n-1$.

Case 3. Suppose $\pi_1 = 2$. Then, $\pi_2 = 1$.

Assume for contradiction that $\pi_2 \neq 1$. This implies that $\pi_2 = n$ or $3 \leq \pi_2 \leq n-1$. If $\pi_2 = n$, then there exist $2, n$, and 1 in π, and there exists 3 in π^r such that $red(2n13) = 2413$. If $3 \leq \pi_2 \leq n-1$, then there exist 2 and n in π, and there exist 1 and π_2 in π^r such that $red(2n1\pi_2) = 2413$. Hence, if $\pi_1 = 2$, then $\pi_2 = 1$.

Together, 21 can only play the role of 21 in π, so they will neither begin nor end a 2413 pattern. Individually, 2 can only play the role of 1 or 2. If it plays the role of 2 in π, then there exist 2 and 4 in π, but a number to play the role of a 3 after 1 in π^r cannot be found. Also, 1 can only play the role of 1, so it will neither begin nor end a 2413 pattern. Now, the remaining positions can be filled in $|\mathcal{R}_{n-2}(2413)|$ ways to avoid a 2413 pattern. (See Figure 11.)

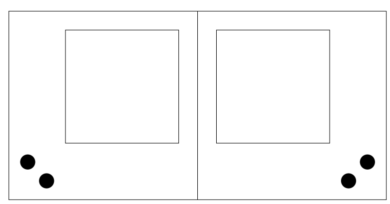

Figure 11: The $|\mathcal{R}_{n-2}(2413)|$ reverse double lists when $\pi_1 = 2$.

Case 4. Suppose $\pi_1 = 1$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(2413)$. Consider $(1 \oplus \sigma)(\sigma^r \ominus 1)$. Since 1 can only play the role of a 1 in a 2413 pattern, adding 1 to the beginning and end of $\sigma\sigma^r$ will not aid in the creation of a 2413 pattern. Thus, there are $|\mathcal{R}_{n-1}(2413)|$ ways to avoid a 2413 pattern. (See Figure 12.)

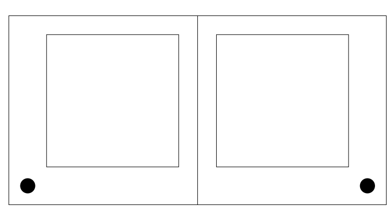

Figure 12: The $|\mathcal{R}_{n-1}(2413)|$ reverse double lists when $\pi_1 = 1$.

Case 5. Assume for contradiction that $3 \leq \pi_1 \leq n-2$.

Now, $\pi_2 = n$, $2 \leq \pi_2 \leq n-1$, or $\pi_2 = 1$. If $\pi_2 = n$, there exist π_1, n, and 1 in π, and there exists $n-1$ in π^r such that $red(\pi_1 n 1 (n-1)) = 2413$. If $2 \leq \pi_2 \leq n-1$, two cases need to be considered. If $\pi_1 > \pi_2$, then there exist π_2 and n in π, and there exist 1 and π_1 in π^r such that $red(\pi_2 n 1 \pi_1) = 2413$. If $\pi_1 < \pi_2$, then there exist π_1 and n in π, and there exist 1 and π_2 in π^r such that $red(\pi_1 n 1 \pi_2) = 2413$. If $\pi_2 = 1$, there exists 2 in π, and there exist n, π_2, and π_1 in π^r such that $red(2n\pi_2\pi_1) = 2413$. Hence, $\pi_1 \in \{1, 2, n-1, n\}$.

Ultimately, $|\mathcal{R}_n(2413)| = 2|\mathcal{R}_{n-1}(2413)| + 2|\mathcal{R}_{n-2}(2413)|$ for $n \geq 3$. □

Using Corollaries 1.2 and 1.4, Corollary 3.8 follows from Theorem 3.7.

Corollary 3.8. $|\mathcal{R}_n(2413)| = |\mathcal{R}_n(3142)|$.

3.2.5 Avoiding 2143

Now, we focus on avoiding the pattern 2143. Theorem 3.9, shows that $\mathcal{R}_n(2143)$ is mainly built out of members of $\mathcal{R}_{n-1}(2143)$.

Theorem 3.9. $|\mathcal{R}_n(2143)| = \begin{cases} n! & \textit{for } n \leq 3, \\ 2|\mathcal{R}_{n-1}(2143)| + 4 & \textit{for } n \geq 4. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2 \cdots \pi_n\pi_n \cdots \pi_2\pi_1 \in \mathcal{R}_n(2143)$.

The cases when $n \in \{1, 2, 3\}$ can be verified using brute force.

Case 1. Suppose $\pi_1 = n$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(2143)$. Consider $n\sigma\sigma^r n$. Since n can only play the role of a 4 in a 2143 pattern, $n\sigma\sigma^r n \in \mathcal{R}_n(2143)$. So, there exist $|\mathcal{R}_{n-1}(2143)|$ reverse double lists with $\pi_1 = n$. (See Figure 5.)

Case 2. Suppose $\pi_1 = n-1$. Then, $\pi_2 = 1$.

Assume for contradiction that $\pi_2 \neq 1$. This implies $\pi_2 = n$ or $2 \leq \pi_2 \leq n-2$. If $\pi_2 = n$, then there exists 2 in π, and there exist $1, n$, and $n-1$ in π^r such that $red(21n(n-1)) = 2143$. If $2 \leq \pi_2 \leq n-2$, then there exist π_2 and 1 in π, and there exist n and $n-1$ in π^r such that $red(\pi_2 1 n(n-1)) = 2143$. Hence, if $\pi_1 = n-1$, then $\pi_2 = 1$. By a similar argument, $\pi_i = i-1$ for $2 \leq i \leq n-2$. Finally, $\pi_{n-1} = n-2$ and $\pi_n = n$ or $\pi_{n-1} = n$ and $\pi_n = n-2$. Thus, there are 2 ways to avoid a 2143 when $\pi_1 = n-1$. (See Figure 13.)

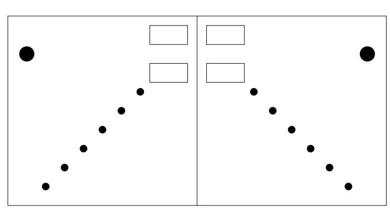

Figure 13: The 2 reverse double lists when $\pi_1 = n-1$.

Case 3. Suppose $\pi_1 = 2$. Then, $\pi_2 = n$.

Assume for contradiction that $\pi_2 \neq n$. This implies $\pi_2 = 1$ or $3 \leq \pi_2 \leq n-1$. If $\pi_2 = 1$, then there exist $2, 1$, and n in π, and there exist $n-1$ in π^r such that $red(21n(n-1)) = 2143$. If $3 \leq \pi_2 \leq n-1$, then there exist 2 and 1 in π, and there exist n and π_2 in π^r such that $red(21n\pi_2) = 2143$. Thus, if $\pi_1 = 2$, $\pi_2 = n$. By a similar argument, $\pi_i = n - i + 2$ for $2 \leq i \leq n-2$. Finally, $\pi_{n-1} = 1$ and $\pi_n = 3$ or $\pi_{n-1} = 3$ and $\pi_n = 1$. Thus, there are 2 ways to avoid a 2143 pattern when $\pi_1 = 2$. (See Figure 14.)

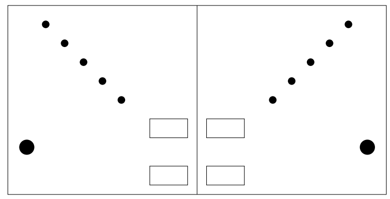

Figure 14: The 2 reverse double lists when $\pi_1 = 2$.

Case 4. Suppose $\pi_1 = 1$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(2143)$. Consider $(1 \oplus \sigma)(\sigma^r \ominus 1)$. Since 1 can only play the role of a 1 in a 2143 pattern, adding 1 to the beginning and end of $\sigma\sigma^r$ will not aid in the creation of a 2143 pattern. Thus, there are $|\mathcal{R}_{n-1}(2143)|$ ways to avoid a 2143 pattern. (See Figure 12.)

Case 5. Assume for contradiction that $3 \leq \pi_1 \leq n-2$.

Then, $\pi_2 = 1$, $2 \leq \pi_2 \leq n-1$, or $\pi_2 = n$. If $\pi_2 = 1$, there exist $\pi_1, 1$, and n in π, and there exists $n-1$ in π^r such that $red(\pi_1 1n(n-1)) = 2143$. If $2 \leq \pi_2 \leq n-1$, then two cases must be considered. If $\pi_1 < \pi_2$, then there exist π_1 and 1 in π, and there exist n and π_2 in π^r such that $red(\pi_1 1n\pi_2) = 2143$. If $\pi_1 > \pi_2$, then there exist π_2 and 1 in π, and there exist n and π_1 in π^r such that $red(\pi_2 1n\pi_1) = 2143$. If $\pi_2 = n$, then there exists 2 in π, and there exist $1, n$, and π_1 in π^r such that $red(21n\pi_1) = 2143$. Thus, $\pi_1 \in \{1, 2, n, n-1\}$.

Ultimately, $|\mathcal{R}_n(2143)| = 2|\mathcal{R}_{n-1}(2143)| + 4$ for $n \geq 4$. $\square$

Now, using Corollaries 1.2 and 1.4, Corollary 3.10 follows from Theorem 3.9.

Corollary 3.10. $|\mathcal{R}_n(2143)| = |\mathcal{R}_n(3412)|$.

3.2.6 Avoiding 1324

Recall from Table 1 that $2143 \sim 3412$ and $1324 \sim 4231$ shared the same enumeration for $\{|\mathcal{R}_n(\rho)|\}_{n=1}^{9}$. When comparing Theorems 3.9 and 3.11, we observe that these two Wilf classes actually do share the same enumeration for all $n \geq 1$. However, going through the proof of Theorem 3.11, we will see that the reverse double lists that avoid these patterns are formed in very different ways.

Theorem 3.11. $|\mathcal{R}_n(1324)| = \begin{cases} 1 & \textit{for } n = 1, \\ 2 & \textit{for } n = 2, \\ 6 & \textit{for } n = 3, \\ 2|\mathcal{R}_{n-1}(1324)| + 4 & \textit{for } n \geq 4. \end{cases}$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2\cdots\pi_n\pi_n\cdots\pi_2\pi_1 \in \mathcal{R}_n(1324)$.

The cases when $n \in \{1, 2, 3\}$ can be verified using brute force.

Case 1. Suppose $\pi_1 = 1$. Then $\pi_2 = 2$.

Assume for contradiction $\pi_2 \neq 2$. This implies $3 \leq \pi_2 \leq n-1$ or $\pi_2 = n$. If $3 \leq \pi_2 \leq n-1$, there exist $1, \pi_2$, and 2 in π, and there exists n in π^r such that $red(1\pi_2 2n) = 1324$. If $\pi_2 = n$, there exist 1 and 3 in π, and there exist 2 and n in π^r such that $red(132n) = 1324$. Therefore, when $\pi_1 = 1$, $\pi_2 = 2$. By a similar argument, $\pi_i = i$ where $1 \leq i \leq n-2$. Then $\pi_{n-1} = n-1$ and $\pi_n = n$ or $\pi_{n-1} = n$ and $\pi_n = n-1$. Thus, there exist two reverse double lists that avoid 1324 when $\pi_1 = 1$. (See Figure 15.)

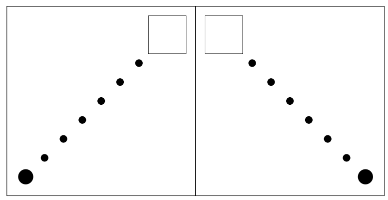

Figure 15: The 2 reverse double lists when $\pi_1 = 1$.

Case 2. Suppose $\pi_1 = n$. Then $\pi_2 = n-1$.

Assume for contradiction $\pi_2 \neq n-1$. This implies that $\pi_2 = 1$ or $2 \leq \pi_2 \leq n-2$. If $\pi_2 = 1$, there exist 1 and 3 in π, and there exist 2 and n in π^r such that $red(132n) = 1324$. If $2 \leq \pi_2 \leq n-2$, then there exists 1 in π, and there exist $n-1$, π_2, and n in π^r such that $red(1(n-1)\pi_2 n) = 1324$. Hence, if $\pi_1 = n$, then $\pi_2 = n-1$. By a similar argument, $\pi_i = n-i+1$ for $1 \leq i \leq n-2$. Then, $\pi_{n-1} = 2$ and $\pi_n = 1$ or $\pi_{n-1} = 1$ and $\pi_n = 2$. Thus, there exist two reverse double lists that avoid 1324 when $\pi_1 = n$. (See Figure 4.)

Case 3. Suppose $\pi_1 = a$ where $2 \leq a \leq n-1$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(1324)$. In general, $\pi\pi^r \in \mathcal{R}_n(1324)$ can be built from lists $\sigma\sigma^r$ where $\sigma_1 = a$ or $\sigma_1 = a-1$. To create $\pi\pi^r$, every $\sigma_j \geq a$ is increased by 1 and a is prepended to σ, where $1 \leq j \leq n-1$. This gives $\pi_1\pi_2 = a(a-1)$ or $\pi_1\pi_2 = a(a+1)$.

Consider the case when $a \notin \{3, n-2\}$. Assume for contradiction that $\pi_2 \notin \{a-1, a+1\}$. This implies $\pi_2 = 1$, $2 \leq \pi_2 \leq a-2$, $a+2 \leq \pi_2 \leq n-1$, or $\pi_2 = n$. If $\pi_2 = 1$, then $\pi_1 \neq 2$. Therefore, there exist 1 and 3 in π, and there exist 2 and a in π^r such that $red(132a) = 1324$. If $2 \leq \pi_2 \leq a-2$, then there exists 1 in π, and there exist $a-1, \pi_2$ and a in π^r such that $red(1(a-1)\pi_2 a) = 1324$. If $a+2 \leq \pi_2 \leq n-1$, then there exist a, π_2 and $a+1$ in π, and there exists n in π^r such that $red(a\pi_2(a+1)n) = 1324$. If $\pi_2 = n$, then $\pi_1 \neq n-1$. Therefore, there exist π_1 and $n-1$ in π, and there exist $n-2$ and n in π^r such that $red(\pi_1(n-1)(n-2)n) = 1324$. Hence, $\pi_2 \in \{a+1, a-1\}$ when $a \notin \{3, n-2\}$.

Consider when $a = 3$ or $a = n-2$. If $a = 3$, then it is possible to have $\pi_1\pi_2 = 31$ in addition to $\pi_2 = 2$ or $\pi_2 = 4$ as discussed previously. This is because the number 1 can only play the role of 1, so the subsequence 13 in π^r cannot play the role of 24 in a 1324 pattern. If 3 is looked at individually, then it cannot play the 4 in a 1324 pattern. If 3 plays the role of 1, a 1324 pattern is still avoided because $\sigma\sigma^r$ avoids 324 after σ_1 in π. Similarly, if $a = n-2$, then it is possible to have $\pi_1\pi_2 = (n-2)n$ in addition to $\pi_2 = n-1$ or $\pi_2 = n-3$. This is because the number n can only play the role of a 4, so the subsequence $(n-2)n$ in π cannot play the role of 13 in a 1324 pattern. If $n-2$ is looked at individually, then it cannot play the role of 1 in a 1324 pattern. If $n-2$ plays the role of 4, a 1342 pattern is still avoided because $\sigma\sigma^r$ avoids 132 before σ_1 in π^r. For all other values of π_2, a 1324 pattern cannot be avoided by previous arguments.

In general, when $\sigma_1 = i$ where $1 \leq i \leq n-1$, it is possible to build reverse double lists where $\pi_1 = i$ or $\pi_1 = i+1$. So, for each $\sigma\sigma^r \in \mathcal{R}_{n-1}(1324)$, we can build two lists that are in $\mathcal{R}_n(1324)$. Hence, there exist $2|\mathcal{R}_{n-1}(1324)|$ reverse double lists when $\pi_1 = a$ for $2 \leq a \leq n-1$.

Thus, $|\mathcal{R}_n(1324)| = 2|\mathcal{R}_{n-1}(1324)| + 4$ for $n \geq 4$. □

From Theorem 3.11, we can obtain Corollary 3.12. This is due to Corollaries 1.2 and 1.4.

Corollary 3.12. $|\mathcal{R}_n(1324)| = |\mathcal{R}_n(4231)|$.

3.2.7 Avoiding 1243

The formula for $|\mathcal{R}_n(1243)|$ is a little more complex than for previous patterns. Notice in Theorem 3.13 that the formula for $|\mathcal{R}_n(1243)|$ involves a summation. For this reason, the proof for avoiding 1243 will use a few new techniques.

Theorem 3.13. $$|\mathcal{R}_n(1243)| = \begin{cases} n! & \textit{for } n \leq 2, \\ |\mathcal{R}_{n-1}(1243)| + \displaystyle\sum_{i=2}^{n-1} 2i & \textit{for } n \geq 3. \end{cases}$$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2\cdots\pi_n\pi_n\cdots\pi_2\pi_1 \in \mathcal{R}_n(1243)$.

The cases when $n \in \{1, 2\}$ can be verified using brute force.

Case 1. Suppose $\pi_1 = n$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(1243)$. Consider $n\sigma\sigma^r n$. Since n can only play the role of a 4 in a 1243 pattern, $n\sigma\sigma^r n \in \mathcal{R}_n(1243)$. So, there exist $|\mathcal{R}_{n-1}(1243)|$ reverse double lists with $\pi_1 = n$. (See Figure 5.)

Case 2. Suppose $\pi_1 = 1$. Then, $\pi_2 = n$.

Assume for contradiction that $\pi_2 \neq n$. This implies $2 \leq \pi_2 \leq n-2$ or $\pi_2 = n-1$. If $2 \leq \pi_2 \leq n-2$, then there exist $1, \pi_2$ and n in π, and there exists $n-1$ in π^r such that $red(1\pi_2 n(n-1)) = 1243$. If $\pi_2 = n-1$, then there exist 1 and 2 in π, and there exist n and $n-1$ in π^r such that $red(12n(n-1)) = 1243$. Hence, if $\pi_1 = 1$, then $\pi_2 = n$. By a similar argument, $\pi_i = n-i+2$ for $2 \leq i \leq n-2$. Finally, either $\pi_{n-1} = 2$ and $\pi_n = 3$ or $\pi_{n-1} = 3$ and $\pi_n = 2$. Thus, there are 2 ways to avoid a 1243 pattern when $\pi_1 = 1$. (See Figure 9.)

Case 3. Suppose $\pi_1 = n-1$. Then, $\pi_2 = n-2$.

Assume for contradiction that $\pi_2 \neq n-2$. This implies $\pi_2 = n$ or $1 \leq \pi_2 \leq n-3$. If $\pi_2 = n$, then there exists 1 in π, and there exist $n-2$, n, and $n-1$ in π^r such that $red(1(n-2)n(n-1)) = 1243$. If $1 \leq \pi_2 \leq n-3$, then there exist π_2 and $n-2$ in π, and there exist n and $n-1$ in π^r such that $red(\pi_2(n-2)n(n-1) = 1243$. Thus, if $\pi_1 = n-1$, $\pi_2 = n-2$. By a similar argument, $\pi_i = n-i$ for $1 \leq i \leq n-2$. Finally, $\pi_{n-1} = 1$ and

$\pi_n = n$ or $\pi_{n-1} = n$ and $\pi_n = 1$. Thus there exist 2 ways to avoid a 1243 pattern when $\pi_1 = n-1$. (See Figure 16.)

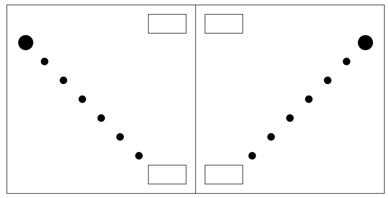

Figure 16: The 2 reverse double lists when $\pi_1 = n-1$.

Case 4. Suppose $\pi_1 = a$ where $2 \leq a \leq n-2$.

To avoid a 1243 pattern, $\pi_1\pi_2\cdots\pi_{a-1} = a(a-1)\cdots 2$. The number 1 can take position π_h where $a \leq h \leq n$. The remaining positions must be filled with either $n(n-1)\cdots(a+3)(a+2)(a+1)$ or $n(n-1)\cdots(a+3)(a+1)(a+2)$.

Suppose $a \neq 2$. Assume for contradiction, $\pi_2 \neq a-1$. This implies $1 \leq \pi_2 \leq a-2$ or $a+1 \leq \pi_2 \leq n$. If $1 \leq \pi_2 \leq a-2$, there exist π_2 and $a-1$ in π, and there exist n and a in π^r such that $red(\pi_2(a-1)na) = 1243$. On the other hand, consider $a+1 \leq \pi_2 \leq n$. Then, there exists 1 in π, and there exist 2, π_2, and a in π^r such that $red(12\pi_2 a) = 1243$. Hence, $\pi_2 = a-1$. By a similar argument, $\pi_j = a-j+1$ where $2 \leq j \leq a-1$.

Suppose $\pi_{a-1} = 2$. Assume for contradiction, $\pi_a \notin \{n, 1\}$. This implies $a+1 \leq \pi_a \leq n-2$ or $\pi_a = n-1$. If $a+1 \leq \pi_a \leq n-2$, there exist 2, π_a, and n in π, and there exists $n-1$ in π^r such that $red(2\pi_2 n(n-1)) = 1243$. If $\pi_a = n-1$, then there exist 2 and $a+1$ in π, and there exist n and $n-1$ in π^r such that $red(2(a+1)n(n-1)) = 1243$. Hence, $\pi_a = 1$ or $\pi_a = n$. By a similar argument, there are a finite number of possible values for π_k where $a+1 \leq k \leq n-2$. If $\pi_{k-1} = 1$, then $\pi_k = \pi_{k-2} - 1$. If $\pi_{k-1} \neq 1$, then $\pi_k = \pi_{k-1} - 1$ or $\pi_k = 1$.

Finally, $a+2$ does not need to precede $a+1$ in π. The number 1 can take position π_h where $a \leq h \leq n$, and the remaining positions must be filled with either $n(n-1)\cdots(a+3)(a+2)(a+1)$ or $n(n-1)\cdots(a+3)(a+1)(a+2)$.

Since $\pi_{a-1} = 2$ and $\pi_h = 1$ where $a \leq h \leq n$, there are $n-a+1$ choices for the location of 1 where $2 \leq a \leq n-2$. Furthermore, since $a+1$ and $a+2$ can be interchanged, there are two choices for how to complete the rest of the list. Let $i = n-a+1$. Since $2 \leq a \leq n-2$, then $2 \leq n-a \leq n-2$. This implies $3 \leq i \leq n-1$. Since $2(n-a+1) = 2i$, there are $\displaystyle\sum_{i=3}^{n-1} 2i$ reverse double lists that avoid 1243 when $\pi_1 = a$ and $2 \leq a \leq n-2$. (See Figure 17.)

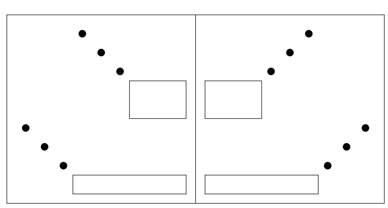

Figure 17: The $\displaystyle\sum_{i=3}^{n-1} 2i$ reverse double lists when $2 \leq \pi_1 \leq n-2$.

Thus,

$$\begin{aligned}|\mathcal{R}_n(1243)| &= |\mathcal{R}_{n-1}(1243)| + 2 + 2 + \sum_{i=3}^{n-1} 2i \\ &= |\mathcal{R}_{n-1}(1243)| + \sum_{i=2}^{n-1} 2i\end{aligned}$$

for $n \geq 3$. $\square$

When we search for $\{|\mathcal{R}_n(1243)|\}_{n=1}^{9}$ in the Online Encyclopedia of Integer Sequences (OEIS) [5], we obtain an interesting result. According to OEIS, $\{|\mathcal{R}_n(1243)|\}_{n=1}^{9}$ yields the following equation

$$|\mathcal{R}_n(1243)| = \frac{1}{3}(n-3)^3 + 3(n-3)^2 + \frac{20}{3}(n-3) + 6,$$

which can be further simplified to

$$|\mathcal{R}_n(1243)| = \frac{n^3}{3} - \frac{7n}{3} + 4.$$

Using the principle of mathematical induction, we can prove

$$\begin{aligned}|\mathcal{R}_n(1243)| &= |\mathcal{R}_{n-1}(1243)| + \sum_{i=2}^{n-1} 2i \\ &= \frac{n^3}{3} - \frac{7n}{3} + 4.\end{aligned}$$

Theorem 3.14. $|\mathcal{R}_n(1243)| = \dfrac{n^3}{3} - \dfrac{7n}{3} + 4.$

Proof.
Base Case. Let $n = 3$. Then, $|\mathcal{R}_3(1243)| = 6$, and $\dfrac{3^3}{3} - \dfrac{7(3)}{3} + 4 = 6$. Thus, the claim holds for $n = 3$.

Inductive Step. Suppose $|\mathcal{R}_k(1243)| = \dfrac{k^3}{3} - \dfrac{7k}{3} + 4$ for some $k \geq 4$. By Theorem 3.13,

$$|\mathcal{R}_{k+1}(1243)| = |\mathcal{R}_k(1243)| + \sum_{i=2}^{k} 2i.$$

This means

$$\begin{aligned}
|\mathcal{R}_{k+1}(1243)| &= \frac{k^3}{3} - \frac{7k}{3} + 4 + \sum_{i=2}^{k} 2i \\
&= \frac{k^3}{3} - \frac{7k}{3} + 4 + (k+2)(k-1) \\
&= \frac{k^3}{3} - \frac{7k}{3} + 4 + k^2 + k - 2 \\
&= \frac{k^3}{3} + k^2 + k + \frac{1}{3} - \frac{7k}{3} - \frac{7}{3} + 4 \\
&= \frac{1}{3}(k+1)^3 - \frac{7}{3}(k+1) + 4.
\end{aligned}$$

Hence, by the principle of mathematical induction, $|\mathcal{R}_n(1243)| = \dfrac{n^3}{3} - \dfrac{7n}{3} + 4$ for $n \geq 3$. □

Now, by Corollaries 1.2 and 1.4, we obtain Corollary 3.15 from Theorem 3.13.

Corollary 3.15. $|\mathcal{R}_n(1243)| = |\mathcal{R}_n(2134)| = |\mathcal{R}_n(3421)| = |\mathcal{R}_n(4312)|$.

3.2.8 Avoiding 1423

When we consider avoiding 1423, we can see that the enumeration is much more complex than the previous formulas. This formula is stated in Theorem 3.16.

Theorem 3.16.

$$|\mathcal{R}_n(1423)| = \begin{cases} n! & \textit{for } n \leq 3, \\ 16 & \textit{for } n = 4, \\ |\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-2}(1423)| + 2n + \displaystyle\sum_{i=2}^{n-3} 2|\mathcal{R}_i(1423)| & \textit{for } n \geq 5. \end{cases}$$

Proof. Suppose $\pi\pi^r = \pi_1\pi_2 \cdots \pi_n\pi_n \cdots \pi_2\pi_1 \in \mathcal{R}_n(1423)$.

The cases when $n \in \{1, 2, 3, 4\}$ can be verified using brute force.

Case 1. Suppose $\pi_1 = n$.

Let $\sigma\sigma^r \in \mathcal{R}_{n-1}(1423)$. Consider $n\sigma\sigma^r n$. Since n can only play the role of a 4 in a 1423 pattern, $n\sigma\sigma^r n \in \mathcal{R}_n(1423)$. So, there exist $|\mathcal{R}_{n-1}(1423)|$ reverse double lists with $\pi_1 = n$. (See Figure 5.)

Case 2. Suppose $\pi_1 = n - 1$. Then, $\pi_2 = n$.

Assume for contradiction $\pi_2 \neq n$. This implies $\pi_2 = 1$ or $2 \leq \pi_2 \leq n-2$. If $\pi_2 = 1$, then there exist 1 and n in π, and there exist 2 and $n-1$ in π^r such that $red(1n2(n-1)) = 1423$. If $2 \leq \pi_2 \leq n-2$, then there exists 1 in π and n, π_2, and $n-1$ in π^r such that $red(1n\pi_2(n-1)) = 1423$. Thus, when $\pi_1 = n-1$, $\pi_2 = n$.

Together, the numbers $(n-1)n$ can only play the role of 34 at the beginning of $\pi\pi^r$ and the role of a 43 at the end of $\pi\pi^r$. Therefore, $(n-1)n$ will not aid in the creation of a 1423 pattern. When considered individually, n can only play the role of 4. Thus, it will not aid in the creation of a 1423 pattern. On the other hand, $n-1$ can play a 3 or 4. If it plays the role of a 3 in π^r, then $n(n-2)$ can be found in π to play the role of 42, but there is no number to play the role of 1 that precedes n. This gives $|\mathcal{R}_{n-2}(1423)|$ reverse double lists where $\pi_1 = n-1$ and $\pi_2 = n$. (See Figure 10.)

Case 3. Suppose $\pi_1 = i+1$ where $2 \leq i \leq n-3$. Then, $\pi_2 = i+2$.

Assume for contradiction $\pi_2 \neq i+2$. This implies $\pi_2 = 1$, $2 \leq \pi_2 \leq i$, $i+3 \leq \pi_2 \leq n-1$, or $\pi_2 = n$. If $\pi_2 = 1$, then there exist 1 and n in π, and there exist 2 and $i+1$ in π^r such that $red(1n2(i+1)) = 1423$. If $2 \leq \pi_2 \leq i$, then there exists 1 in π, and there exist n, π_2, and $i+1$ in π^r such that $red(1n\pi_2(i+1)) = 1423$. If $i+3 \leq \pi_2 \leq n-1$, then there exist $i+1$ and n in π, and there exist $i+2$ and π_2 in π^r such that $red((i+1)n(i+2)\pi_2) = 1423$. If $\pi_2 = n$, then there exist $i+1, n$ and $i+2$ in π, and there exists $n-1$ in π^r such that $red((i+1)n(i+2)(n-1)) = 1423$. Thus, when $\pi_1 = i+1$, then $\pi_2 = i+2$. By a similar argument, $\pi_j = i+j$ where $1 \leq j \leq n-i-3$.

Suppose $\pi_k = n-2$, where $1 \leq k \leq n-i-3$. Assume for contradiction $\pi_{k+1} \neq n-1$ and $\pi_{k+1} \neq n$. This implies $\pi_{k+1} \leq \pi_k$, but it was previously proven that $\pi_{k+1} > \pi_k$. Thus, when $\pi_k = n-2$, $\pi_{k+1} \in \{n-1, n\}$. Ultimately, $\pi_{n-i-2} = n-1$ and $\pi_{n-i-1} = n$ or $\pi_{n-i-2} = n$ and $\pi_{n-i-1} = n-1$.

Now, the remaining i positions of π can be filled with σ, where $\sigma\sigma^r \in \mathcal{R}_i(1423)$. Since n and $n-1$ can be interchanged and there exist $|\mathcal{R}_i(1423)|$ reverse double lists for $2 \leq i \leq n-3$, there are $\displaystyle\sum_{i=2}^{n-3} 2|\mathcal{R}_i(1423)|$ reverse double lists where $\pi_1 = i+1$. (See Figure 18.)

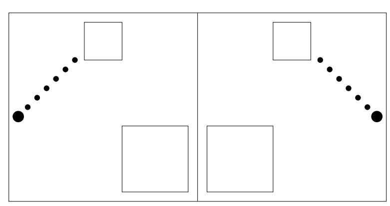

Figure 18: The $\displaystyle\sum_{i=2}^{n-3} 2|\mathcal{R}_i(1423)|$ reverse double lists when $\pi_1 = i+1$.

Case 4. Suppose $\pi_1 = 2$. Then, $\pi_2 = 1$ or $\pi_2 = 3$.

Assume for contradiction that $\pi_2 \notin \{1, 3\}$. This implies $\pi_2 = n$ or $4 \leq \pi_2 \leq n-1$. If $\pi_2 = n$, then there exist 2, n, and 3 in π, and there exists $n-1$ in π^r such that $red(2n3(n-1)) = 1423$. If $4 \leq \pi_2 \leq n-1$, then there exist 2 and n in π, and there exist 3 and π_2 in π^r such that $red(2n3\pi_2) = 1423$. Therefore, if $\pi_1 = 2$, then $\pi_2 = 1$ or $\pi_2 = 3$.

Subcase 1. Suppose $\pi_2 = 3$. Then, $\pi_3 \in \{1, 4\}$.

Assume for contradiction $\pi_3 \notin \{1, 4\}$ when $n \geq 6$. This implies $\pi_3 = n$ or $5 \leq \pi_3 \leq n-1$. If $\pi_3 = n$, then there exist 2, n, and $n-2$ in π, and there exists $n-1$ in π^r such that $red(2n(n-2)(n-1)) = 1423$. If $5 \leq \pi_3 \leq n-1$, then there exist 2 and n in π, and there exist 4 and π_3 in π^r such that $red(2n4\pi_3) = 1423$. Thus, if $\pi_2 = 3$, then $\pi_3 = 1$ or $\pi_3 = 4$. By a similar argument, there are a finitely many possible values where π_i for $3 \leq i \leq n-3$. If $\pi_{i-1} = 1$, then $\pi_i = \pi_{i-2} + 1$. If $\pi_{i-1} \neq 1$, then $\pi_i = 1$ or $\pi_i = \pi_{i-1} + 1$. Finally, numbers $n-1$ and n can be interchanged while still avoiding a 1423 pattern.

Since there are $n-2$ possible positions 1 can take and $n-1$ and n can be interchanged, this gives $2(n-2)$ reverse double lists when $\pi_1 = 2$ and $\pi_2 = 3$. (See Figure 19.)

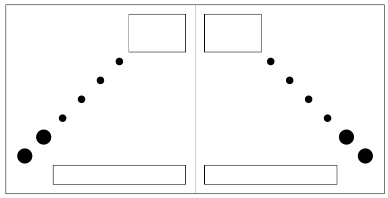

Figure 19: The $2(n-2)$ reverse double lists when $\pi_1 = 2$ and $\pi_2 = 3$.

Subcase 2. Suppose $\pi_2 = 1$. Then, $\pi_3 = 3$.

Assume for contradiction $\pi_3 \neq 3$. This implies $\pi_3 = n$ or $4 \leq \pi_3 \leq n-1$. If $\pi_3 = n$, then there exist 1, n, and 3 in π, and there exists 4 in π^r such that $red(1n34) = 1423$. If $4 \leq \pi_3 \leq n-1$, then there exist 1 and n in π, and there exist 3 and π_3 in π^r such that $red(1n3\pi_3) = 1423$. Thus, if $\pi_2 = 1$, then $\pi_3 = 3$. By a similar argument, $\pi_i = i$ for $3 \leq i \leq n-2$. Finally, $\pi_{n-1} = n-1$ and $\pi_n = n$ or $\pi_{n-1} = n$ and $\pi_n = n-1$, giving two reverse double lists when $\pi_1 = 2$ and $\pi_2 = 1$. (See Figure 20.)

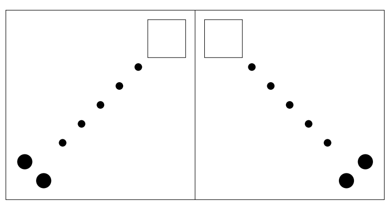

Figure 20: The 2 reverse double lists when $\pi_1 = 2$ and $\pi_2 = 1$.

Case 5. Suppose $\pi_1 = 1$. Then, $\pi_2 = 2$.

Assume for contradiction $\pi_2 \neq 2$. However, following a similar argument as in Case 4, Subcase 2, $\pi_2 = 2$. In general, $\pi_i = i$ for $1 \leq i \leq n-2$. Finally, $\pi_{n-1} = n-1$ and $\pi_n = n$ or $\pi_{n-1} = n$ and $\pi_n = n-1$, giving two reverse double lists when $\pi_1 = 1$ and $\pi_2 = 2$. (See Figure 15.)

Hence,

$$\begin{aligned}|\mathcal{R}_n(1423)| &= |\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-2}(1423)| + 2 + 2 + 2(n-2) + \sum_{i=2}^{n-3} 2|\mathcal{R}_i(1423)| \\ &= |\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-2}(1423)| + 2n + \sum_{i=2}^{n-3} 2|\mathcal{R}_i(1423)|.\end{aligned}$$

for $n \geq 5$. □

This formula, though complex, is the most straightforward to prove when observing the actual reverse double lists that avoid 1423 and how they are recursively built from smaller lists. However, upon closer inspection of $\{|\mathcal{R}_n(1423)|\}_{i=1}^{9}$, we conjecture the formula

$$|\mathcal{R}_n(1423)| = 2|\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-3}(1423)| + 2.$$

Now, using the principle of mathematical induction, we can prove

$$\begin{aligned}|\mathcal{R}_n(1423)| &= |\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-2}(1423)| + 2n + \sum_{i=2}^{n-3} 2|\mathcal{R}_i(1423)| \\ &= 2|\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-3}(1423)| + 2.\end{aligned}$$

Theorem 3.17. $|\mathcal{R}_n(1423)| = 2|\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-3}(1423)| + 2.$

Proof.
Base Case. Let $n = 5$. Then, $|\mathcal{R}_5(1423)| = 36$, and $2|\mathcal{R}_4(1423)| + |\mathcal{R}_2(1423)| + 2 = 2(16) + 2 + 2 = 36$. Thus, the claim holds for $n = 5$.

Inductive Step. Suppose $|\mathcal{R}_k(1423)| = 2|\mathcal{R}_{k-1}(1423)| + |\mathcal{R}_{k-3}(1423)| + 2$ for some $k \geq 6$. When $n = k + 1$, Theorem 3.16 gives

$$\begin{aligned}|\mathcal{R}_{k+1}(1423)| &= |\mathcal{R}_k(1423)| + |\mathcal{R}_{k-1}(1423)| + 2(k+1) + \sum_{i=2}^{k-2} 2|\mathcal{R}_i(1423)| \\ &= |\mathcal{R}_k(1423)| + |\mathcal{R}_{k-1}(1423)| + 2(k+1) + \sum_{i=2}^{k-3} 2|\mathcal{R}_i(1423)| + 2|\mathcal{R}_{k-2}(1423)| \\ &= |\mathcal{R}_k(1423)| + 2|\mathcal{R}_{k-2}(1423)| + |\mathcal{R}_{k-1}(1423)| + 2k + 2 + \sum_{i=2}^{k-3} 2|\mathcal{R}_i(1423)|. \qquad (1)\end{aligned}$$

When $n = k$, Theorem 3.16 implies

$$\sum_{i=2}^{k-3} 2|\mathcal{R}_i(1423)| = |\mathcal{R}_k(1423)| - |\mathcal{R}_{k-1}(1423)| - |\mathcal{R}_{k-2}(1423)| - 2k. \qquad (2)$$

After substituting Equation 2 into Equation 1, this gives

$$\begin{aligned}|\mathcal{R}_{k+1}(1423)| = |\mathcal{R}_k(1423)| + 2|\mathcal{R}_{k-2}(1423)| + |\mathcal{R}_{k-1}(1423)| + 2k + 2 \\ + |\mathcal{R}_k(1423)| - |\mathcal{R}_{k-1}(1423)| - |\mathcal{R}_{k-2}(1423)| - 2k,\end{aligned}$$

which simplifies to

$$\begin{aligned}|\mathcal{R}_{k+1}(1423)| &= 2|\mathcal{R}_k(1423)| + |\mathcal{R}_{k-2}(1423)| + 2 \\ &= 2|\mathcal{R}_{(k+1)-1}(1423)| + |\mathcal{R}_{(k+1)-3}(1423)| + 2.\end{aligned}$$

Hence, by the principle of mathematical induction,

$$|\mathcal{R}_n(1423)| = 2|\mathcal{R}_{n-1}(1423)| + |\mathcal{R}_{n-3}(1423)| + 2$$

for $n \geq 5$. $\square$

Finally, using Corollaries 1.2 and 1.4, we obtain the following corollary as it follows from Theorem 3.16.

Corollary 3.18. $|\mathcal{R}_n(1423)| = |\mathcal{R}_n(2314)| = |\mathcal{R}_n(3241)| = |\mathcal{R}_n(4132)|$.

3.3 Final Formulas

Table 2 gives a complete list of the proven formulas for the number of reverse double lists that avoid each pattern of length four.

Pattern ρ	$\lvert\mathcal{R}_n(\rho)\rvert$ Formulas		Theorem
$1234 \sim 4321$	0	$(n \geq 7)$	3.1
$1243 \sim 2134 \sim 3421 \sim 4312$	$\frac{n^3}{3} - \frac{7n}{3} + 4$	$(n \geq 3)$	3.13
$1324 \sim 4231$, $2143 \sim 3412$	$2\lvert\mathcal{R}_{n-1}(\rho)\rvert + 4$	$(n \geq 4)$	3.11, 3.9
$1423 \sim 2314 \sim 3241 \sim 4132$	$2\lvert\mathcal{R}_{n-1}(\rho)\rvert + \lvert\mathcal{R}_{n-3}(\rho)\rvert + 2$	$(n \geq 5)$	3.16
$1432 \sim 2341 \sim 3214 \sim 4123$	$2\lvert\mathcal{R}_{n-1}(\rho)\rvert + \lvert\mathcal{R}_{n-2}(\rho)\rvert$	$(n \geq 5)$	3.3
$1342 \sim 2431 \sim 3124 \sim 4213$	$2\lvert\mathcal{R}_{n-1}(\rho)\rvert + \lvert\mathcal{R}_{n-2}(\rho)\rvert + 2$	$(n \geq 4)$	3.5
$2413 \sim 3142$	$2\lvert\mathcal{R}_{n-1}(\rho)\rvert + 2\lvert\mathcal{R}_{n-2}(\rho)\rvert$	$(n \geq 3)$	3.7

Table 2: Formulas to calculate $|\mathcal{R}_n(\rho)|$.

Notice that while two of the seven formulas are closed-form, the remaining five are recursive. To more easily compute terms late in the sequences represented by these formulas, it makes sense to create generating functions. Following the general process for turning recursive formulas into generating functions, we obtain Table 3.

Pattern ρ	Generating Function
$1324 \sim 4231$, $2143 \sim 3412$	$\dfrac{2x^4+2x^3-x^2+x}{2x^2-3x+1}$
$1423 \sim 2314 \sim 3241 \sim 4132$	$\dfrac{x^5-x^4-2x^3+x^2-x}{-x^4+x^3-2x^2+3x-1}$
$1432 \sim 2341 \sim 3214 \sim 4123$	$\dfrac{2x^4+x^3+x}{-x^2-2x+1}$
$1342 \sim 2431 \sim 3124 \sim 4213$	$\dfrac{x^4+x^3-x^2+x}{x^3+x^2-3x+1}$
$2413 \sim 3142$	$\dfrac{x}{-2x^2-2x+1}$

Table 3: Generating functions for the 18 patterns of length four with recursive formulas.

The coefficient of x^n in the Taylor series of the function corresponding to ρ is $|\mathcal{R}_n(\rho)|$. Using algebraic techniques, we can also determine the rate of growth of these sequences using the highest root of the corresponding characteristic polynomial, which are shown in Table 4. In other words, $|\mathcal{R}_n(\rho)| \sim \alpha^n$.

Pattern ρ	Highest Root α
$1324 \sim 4231$, $2143 \sim 3412$	2
$1423 \sim 2314 \sim 3241 \sim 4132$	$\frac{2}{3}+\frac{1}{3}\sqrt[3]{\frac{1}{2}(43-3\sqrt{177})}+\frac{1}{3}\sqrt[3]{\frac{1}{2}(43+3\sqrt{177})}$
$1432 \sim 2341 \sim 3214 \sim 4123$	$1+\sqrt{2}$
$1342 \sim 2431 \sim 3124 \sim 4213$	$1+\sqrt{2}$
$2413 \sim 3142$	$1+\sqrt{3}$

Table 4: Rate of growth for the 18 patterns of length 4 with recursive formulas.

4 Avoiding Longer Ascending and Descending Patterns

In addition to our research on length four patterns, we also extended our results for avoiding strictly ascending and descending patterns of length four (1234 and 4321) to strictly ascending and descending subsequences of any length. To do this, we generalized the Erdős-Szekeres theorem.

The Erdős-Szekeres theorem states that for $r, s \geq 1$, every permutation of length at least $(r-1)(s-1)+1$ must contain the pattern $12\cdots r$ or the pattern $s\cdots 21$.

Seidenberg [6] proved this formula in 1959 using the pigeonhole principle. In his proof, he takes an arbitrary permutation of length n and labels each digit in the permutation with a coordinate pair (a_i, b_i), where a_i is the length of the longest ascending subsequence that terminates at the i^{th} digit and b_i is the length of the longest descending subsequence that terminates at the i^{th} digit. He then proves that each coordinate pair assigned to a digit of the permutation must be unique. We will call these coordinate pairs labels. Next, he shows that there are only $(r-1)(s-1)$ labels that allow the permutation to avoid the patterns $12\cdots r$ and $s\cdots 21$ since there are $(r-1)$ valid x-coordinates and $(s-1)$ valid y-coordinates the label can take. Thus, there are not enough valid labels to allow a permutation of length $(r-1)(s-1)+1$ to avoid both an ascending subsequence of length r and a descending subsequence of length s by the pigeonhole principle.

We now apply the ideas of Erdős, Szekeres, and Siedenberg to reverse double lists that avoid ascending subsequences and descending subsequences of any length, simultaneously. So far, we have only avoided one pattern at a time. However, due to the symmetric structure of reverse double lists, Corollary 1.2 can be used to show that $\mathcal{R}_n(\rho) = \mathcal{R}_n(\rho^r)$ (since $(\pi\pi^r)^r = \pi\pi^r$). As a result, generalizing the Erdős-Szekeres theorem is the same thing as generalizing Theorem 2.1 and Theorem 3.1.

Let's first work an example. How long can a reverse double list be and still avoid 1234 and 4321? There are nine valid labels that the digits of the reverse double list can have, which are the labels that contain an x-coordinate of less than 4 and a y-coordinate of less than 4. Here are the nine labels:

$$\begin{array}{c}(1,1)(1,2)(1,3)\\(2,1)(2,2)(2,3)\\(3,1)(3,2)(3,3)\end{array}$$

Figure 21: The nine labels with x and y-coordinates less than 4.

Now, we only know that for a permutation that each label must be unique. Some labels in a reverse double list must be repeated - for example, π_n will always have the same label as $(\pi^r)_1$, since the two digits are adjacent and have the same magnitude. To understand which labels can be repeated, it is helpful to arrange the labels by their magnitude, or in other words, the sum of their x-coordinate and y-coordinate. Ordering Figure 21 by magnitude, we get the following:

magnitude	**labels**
2	$(1,1)$
3	$(1,2)(2,1)$
4	$(2,2)(1,3)(3,1)$
5	$(2,3)(3,2)$
6	$(3,3)$

Figure 22: Figure 21 organized by magnitude.

After analyzing which labels can be found in π, we get a very interesting result. Given the labels in Figure 22, any label (x,y) assigned to a digit in π so that $x+y\geq 5$ will force the reverse double list to have a length four ascending subsequence. This occurs because if a digit p has label (x, y), then there are $x-1$ numbers strictly less than p in an ascending subsequence that precedes p. Similarly, there are $y-1$ numbers strictly greater than p in a descending subsequence that precedes p. However, this means that in π^r, these $y-1$ numbers will form an ascending subsequence using digits strictly greater than p. Thus, in the entire reverse double list, we are guaranteed an ascending subsequence that contains at least $x-1$ numbers less than p, p itself, and at least $y-1$ numbers greater than p. So if $x+y \geq 5$, then there is an ascending subsequence of at least length $(x-1)+1+(y-1) \geq 4$.

In this specific example, only six valid labels for digits in π remain. While this does not guarantee that $|\mathcal{R}_6(1234, 4321)| > 0$, it does guarantee that for $n \geq 7$, $|\mathcal{R}_n(1234, 4321)| = 0$ since there are not enough pattern-avoiding labels for π when $n \geq 7$.

This result generalizes to the following theorem, which we will now prove:

Theorem 4.1. $|\mathcal{R}_n(12\cdots r, r\cdots 21)| = 0$, *for* $n \geq T_{r-1}+1$, *where* T_{r-1} *is the* $(r-1)^{th}$ *triangular number.*

Proof. Consider $\pi \in \mathcal{R}_n(12\cdots r, r\cdots 21)$. Observe that no pattern avoiding labels (x, y) exists of magnitude $x+y \geq r+1$. If there existed such a label, then the reverse double list would have an ascending subsequence of at least length $(x-1)+1+(y-1) \geq r$. So the list would have an ascending subsequence of length r.

This means that the only valid labels to label the digits of π are the labels (x, y), such that $x+y \leq r$. There are $i-1$ labels (x, y) where $x, y \geq 1$ and $x+y = i$. Thus, there are

$$\sum_{i=2}^{r}(i-1) = \sum_{i=1}^{r-1} i = T_{r-1}$$

valid labels. This means that the maximum length of π is T_{r-1} since this is the maximum number of pattern-avoiding labels π can use. (Recall that every label in π must be unique.)

Thus, $|\mathcal{R}_n(12\cdots r, r\cdots 21)| = 0$ for all $n \geq T_{r-1}+1$, since this is the first n where a label that does not avoid the patterns must be used. □

Now that a lower bound for when $|\mathcal{R}_n(12\cdots r, r\cdots 21)| = 0$ has been established, the next reasonable question to ask is, "Is this the lowest possible bound?" In other words, does $|\mathcal{R}_n(12\cdots r, r\cdots 21)| \neq 0$ for all $n \leq T_{r-1}$? It happens, and can be proven, that the previously established bound is in fact sharp. Let's first consider the critical case, where $n = T_{r-1}$, and prove that $|\mathcal{R}_n(12\cdots r, r\cdots 21)| \neq 0$ for this n. Specifically, we will show that $\mathcal{R}_n(12\cdots r, r\cdots 21)| \geq 1$ for $n = T_{r-1}$, or in other words, when the number of pattern-avoiding labels is exactly equal to the length of a set of reverse double lists, a reverse double list that avoids ascending and descending subsequences of length r can be constructed using these labels.

First, it needs to be shown that it is possible to construct a permutation π so that π utilizes all n labels.

It turns out that π can always be reverse engineered in a way such that π uses the coordinate pairs so that labels with lower x-coordinates precede labels with higher x-coordinates, and for labels with the same x-coordinate, labels with lower y-coordinates precede labels with higher y-coordinates. For example, if $r = 4$, there are six labels to build π with, shown below:

x-coordinate	labels
1	$(1,1)(1,2)(1,3)$
2	$(2,1)(2,2)$
3	$(3,1)$

These labels, now sorted by x-coordinate, need to be assigned by row in the figure above, going left to right in each row. Let $J_a = a \cdots 1$. Since there are three labels with x-coordinate 1, two labels with x-coordinate 2, and one label with x-coordinate 3, we can create the permutation $\pi = J_3 \oplus J_2 \oplus J_1 = 321546$. Now, it can be verified through brute force that $\pi\pi^r$ does not contain an ascending or descending subsequence of length four.

Now, this method of labeling will be generalized to show that we can always build a permutation π such that when π is built into a reverse double list, it avoids ascending and descending subsequences of length r. More concisely, we will prove the following theorem:

Theorem 4.2. $|\mathcal{R}_n(12\cdots r, r\cdots 21)| \geq 1,$ *for* $n \leq T_{r-1}$.

Proof. Suppose $n = T_{r-1}$. So there are T_{r-1} labels to assign. The labels available are shown below:

$$\begin{array}{l} (1,1) \\ (1,2)(2,1) \\ \quad\vdots \qquad\quad \ddots \\ (1,r-1)\cdots(r-1,1) \end{array}$$

Figure 23: The T_{r-1} labels available, all of which will be used to construct π.

So there are $r-1$ labels that have an x-coordinate of 1, $r-2$ labels that have an x-coordinate of 2, and so on and so forth, until the 1 final label that has an x-coordinate of $r-1$ is reached.

There is one specific class of permutation that can always use all T_{r-1} available labels. Let $J_a = a \cdots 1$. Construct the permutation $\pi = J_{r-1} \oplus J_{r-2} \oplus \cdots \oplus J_2 \oplus J_1$, which is of length T_{r-1}. This permutation does indeed contain all the available labels in the order detailed before.

Let's first show that the subsequence J_{r-1} has the first $r-1$ labels in the correct order. Each of the digits in J_{r-1} is the smallest digit seen yet, so it has an x-coordinate of 1. Now, since the digits labeled are a strictly descending subsequence, each label will have a y-coordinate one larger than the previous label, giving the correct y-coordinates.

Now, check that the subsequence J_{r-2} has the next $r-2$ labels in the correct order. Each digit in J_{r-2} is larger than any digit in J_{r-1}. However, since J_{r-1} is strictly descending, only an ascending subsequence of correct length 2 can be formed between a digit in J_{r-1} and J_{r-2} because the digits in J_{r-2} are strictly descending. Now, since the digits in J_{r-2} are the largest digits seen yet, and strictly descend, each label will have a y-coordinate one larger than the previous label, giving the correct y-coordinates.

In general, each digit in J_a, $1 \leq a \leq r-1$, will be part of an ascending subsequence of correct length $r-a$ formed by a single digit in each J_b, $1 \leq b < a$. As before, J_a's digits are the biggest set of digits seen yet and strictly descend, so the digits in J_a have the correct

y-coordinates as well. Thus, all the available labels appear, and they do so in the correct order.

It now remains to show that when the reverse double list $\pi\pi^r$ is formed from $\pi = J_{r-1} \oplus J_{r-2} \oplus \cdots \oplus J_2 \oplus J_1$, $\pi\pi^r$ avoids both ascending subsequences of length r and descending subsequences of length r. Observe that when I_a is denoted as $1 \cdots a$, $\pi^r = I_1 \ominus I_2 \ominus \cdots \ominus I_{r-2} \ominus I_{r-1}$.

This reverse double list will always have the following dot diagram form:

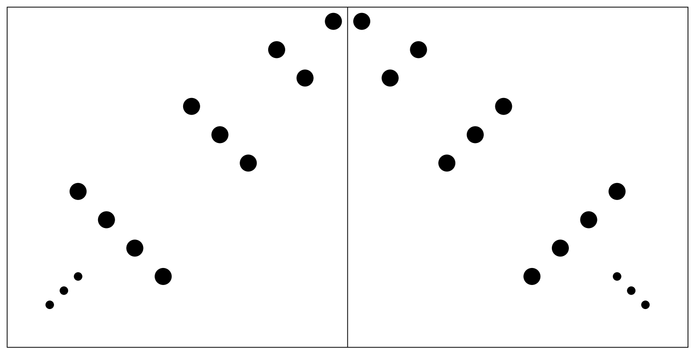

Figure 24: Generalized dot diagram of the constructed reverse double list.

Notice that each J_a is a block of π, and that each I_a is a block of π^r. Each of these blocks are a maximal set of adjacent digits with consecutive values.

Now, Corollary 1.2 states that if a list avoids a pattern, then the reverse of that list avoids the reverse of that pattern. However, the reverse of a reverse double list is exactly the same reverse double list. So it is sufficient to show that a reverse double list avoids $1 \cdots r$ in order to also show that the reverse double list also avoids $(1 \cdots r)^r = r \cdots 1$.

The synthesized reverse double list does indeed avoid ascending subsequences of length r. Consider an arbitrary digit d in $\pi\pi^r$. What is the length of the longest ascending subsequence of numbers less than or equal to d that terminates at d? What is the length of the longest ascending subsequence of numbers strictly greater than d that is found after d? If the magnitude of these answers combined is less than r, then an ascending subsequence of length r cannot be found. Condition on whether d is found in an ascending block (J_i) or descending block (I_i).

Case 1. Suppose $d \in J_i$.

The length of the longest ascending subsequence of numbers less than or equal to d that terminates at d is simply the number of descending blocks leading up to and including J_i. Because the blocks in π are strictly descending subsequences, a maximum of one digit can be taken from each block to create this ascending subsequence. So this subsequence will have length at most $(r-1)-(i-1)$, since there are $r-1$ total J_i blocks, $i-1$ of which are blocks higher than J_i.

The length of the longest ascending subsequence of numbers strictly greater than d that is found after d is the larger of the following:

- the number of digits larger than d in J_i.

- the number of digits in the block J_{i-1}, which is directly above the block J_i,

Both of these cases can maximally yield $i-1$ digits. For example, if d is in J_3, there are at most two digits in J_3 that are greater than d, and two digits in the J_2 block directly above d.

So all together, combining the two parts of this case, d can be part of an ascending subsequence with length that is at most $(r-1)-(i-1)+(i-1)=(r-1)$. Therefore, an ascending subsequence of length r cannot be formed.

Case 2. Suppose $d \in I_i$.

The length of the longest ascending subsequence of numbers less than or equal to d that terminates at d is the number of digits less than or equal to d in I_i, plus the number of J_j blocks, such that $j > i$. There are at most i digits less than or equal to d in I_i, and at most $(r-1)-i$ number of I_j blocks, yielding an ascending subsequence of maximal length $i+(r-1)-i=r-1$.

So, if just a single digit could be found after d such that the digit is greater than d, an ascending subsequence of length r can be formed. However, the only way such a digit exists is if d is not the largest digit in I_i. But if d is not the largest digit in I_i, then there are not i digits less than or equal to d in I_i. Specifically, if there are a digits greater than d found in I_i, then there are $i-a$ digits less than or equal to d found in I_i.

As a result, the maximum length of an ascending subsequence using d in this case is also $r-1$. Thus, an ascending subsequence of length r cannot be formed.

Therefore, it is possible to construct a specific permutation that uses all T_{r-1} available labels, and the reverse double list built out of this permutation avoids ascending and descending subsequences of length r.

Thus, $|\mathcal{R}_n(12\cdots r, r\cdots 21)| \geq 1$ for $n = T_{r-1}$.

Now that the critical case has been proved, consider the synthesized reverse double list again, except this time remove both digits of magnitude n, the highest magnitude in this list. These two copies of n only served to help build ascending and descending subsequences, so when they are removed, the remainder of the reverse double list (which is of length $T_{r-1}-1$ now), will still avoid ascending and descending subsequences of length r. This process can be iterated successfully until all digits of the reverse double list have been removed, and at each step the resulting smaller reverse double list will always avoid ascending and descending subsequences of length r.

So $|\mathcal{R}_n(12\cdots r, r\cdots 21)| \geq 1$ for all $n \leq T_{r-1}$. $\square$

So now it has been shown that our bound for reverse double lists avoiding $1\cdots r$ and $r\cdots 1$ is indeed the lowest possible bound.

5 Permutation Statistics

After computing formulas for the number of reverse double lists avoiding patterns of up to length four, it is worth taking a look at how permutation statistics can be applied to reverse double lists. We will generalize the definitions of three particular permutation statistics so that they can be applied to reverse double lists.

The number of **left to right maxima** in a list, denoted $lrmax(\pi)$, is the number of times a number higher than all numbers encountered previously is found while scanning the list from left to right.

The number of **descents** in a list, denoted $des(\pi)$, is the number of times $\pi_i > \pi_{i+1}, 1 \leq i \leq n-1$.

The number of **inversions** in a list, denoted $inv(\pi)$, is $|\{(i,j)|\pi_i > \pi_j, i < j\}|$, or in other words, the number of pairs of digits in π so that a bigger digit precedes a smaller digit.

Due to the symmetric structure of reverse double lists, permutation statistics follow fairly trivial patterns.

Proposition 1. *$lrmax(\pi) = lrmax(\pi\pi^r)$*

Proof. Suppose we have $\pi \in \mathcal{S}_n$, and $lrmax(\pi) = a$. Since the digit $n \in \pi$, there is no number $m \in \pi^r$ such that $m > n$, since n is the highest digit in $\pi\pi^r$. As a result, no m exists to increase the $lrmax$ above a. So $lrmax(\pi) = lrmax(\pi\pi^r)$. □

Proposition 2. *$des(\pi\pi^r) = n-1$, where n is the length of π.*

Proof. Suppose we have $\pi\pi^r \in \mathcal{R}_n$. Consider π. Since the only places descents can occur in π are between two consecutive numbers, there are then $n-1$ spots in π where a descent can occur. Similarly, there are $n-1$ spots in π^r. Each of these $n-1$ spots must be an ascent or a descent. Consider the a^{th} spot, $1 \leq a \leq n-1$. If the a^{th} spot is a descent, then the a^{th} spot (while reading from right to left) in π^r is an ascent. If the a^{th} spot is an ascent, then the a^{th} spot (while reading from right to left) in π^r is a descent. So there will be exactly one descent when you consider the a^{th} spot in π (from left to right) and the the a^{th} spot in π^r (from right to left). However, there is one spot we have not considered - the spot created between π_n and π^r_1. However, $\pi_n = \pi^r_1$, so this spot is neither an ascent or descent. Since there are $n-1$ spots in π, there are in total $n-1$ descents in $\pi\pi^r$. □

Proposition 3. *$inv(\pi\pi^r) = 2T_{n-1}$, where T_{n-1} is the $(n-1)^{th}$ triangular number.*

Proof. Consider each of the digits 2 through n, and consider an arbitrary digit b in this range. There are $b-1$ digits less than b. Each of these $b-1$ digits can be found after the two b's exactly twice. Consider the digit $c, c < b$. Then, c either comes before the first b or after the first b in π. If c comes before the first b in π, then the second c will come after the second b in π^r. So the first c does not create an inversion relative to b, but the second c creates two inversions. If the first c comes after the first b in π, then the second c comes before the second b. So the first c creates one inversion relative to b, as does the second c. So regardless of the positions of the numbers smaller than b, each digit will create exactly two inversions relative to b. So each b creates $2(b-1)$ inversions. Since $b = 1$ creates no inversions, the total number of inversions of a reverse double list of length n is

$$\sum_{b=2}^{n} 2(b-1) = 2\sum_{b=2}^{n}(b-1) = 2\sum_{b=1}^{n-1} b = 2T_{n-1}.$$

□

When calculating permutation statistics using an entire reverse double list, the results are fairly straightforward. To expand on this concept, one could choose to look at the permutation statistics on only the first half of a reverse double list that avoids a specific pattern.

6 Final Thoughts

In this paper, we analyzed the structure of reverse double lists that avoided patterns of up to length four, created a bound for when zero reverse double lists avoid ascending/descending patterns, and briefly looked into permutation statistics. There were several questions resulting from our work that could be investigated for future research:

1. Does looking at the permutation statistics for the first half of the reverse double lists lead to more interesting results?

2. We saw that when avoiding 1234 and 4321, $|\mathcal{R}_5(1234, 4321)| = |\mathcal{R}_6(1234, 4321)|$. We noticed the same behavior in Sage when avoiding 12345 and 54321 with $n = 9$ and $n = 10$. Why do the last two numbers repeat before the number of ascending/descending avoiding reverse double lists becomes zero?

3. In many of the proofs, we saw that the last two numbers in π could be interchanged. Why is this so?

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