

# 4–equitable Tree Labelings

Zena Coles<sup>1</sup> Alana Huszar<sup>2</sup> Jared Miller<sup>3</sup> Zsuzsanna Szaniszló<sup>4</sup>

## Abstract

A tree is a vertex-edge graph that is connected and contains no cycles. A 4–equitable labeling of a graph is an assignment of labels  $\{0, 1, 2, 3\}$  to the vertices. The edge labels are the absolute difference of the labels of the vertices that they are incident to. The labels must be distributed as evenly as possible amongst the vertices and they must also be distributed as evenly as possible amongst the edges. We study 4–equitable labelings of different trees; we found that all caterpillars, symmetric generalized  $n$ –stars, and complete  $n$ –ary trees for all  $n \in \mathbb{N}$  are 4–equitable. We believe that proving all trees are 4–equitable will bring us one step closer to proving the famous graceful tree conjecture that has been open for half a century.

## Introduction

In 1964, Ringel conjectured that  $K_{2n+1}$  can be decomposed into copies of  $2n + 1$  isomorphic trees with  $n$  vertices regardless of the structure of the tree [5]. In order to attack this conjecture, Rosa introduced  $\beta$ –valuations. Golomb then called these labelings graceful labelings, a name that is now the standard [4]. A *graceful labeling* of a graph with  $e$  edges is a labeling where each vertex gets a distinct label from the set  $\{0, 1, \dots, e\}$ . Each edge label is the absolute difference of its incident vertices, and all edge labels must also be distinct [6]. Ringel and Kotzig then conjectured that all trees are graceful. Information on all graceful labeling results is collected in Gallian’s *Dynamic Survey of Graph Labeling* [3].

In 1995, Cahit generalized graceful labelings to  $k$ –equitable labelings. A  $k$ –*equitable* labeling is a labeling such that the vertices are labeled with the numbers  $\{0, 1, \dots, k - 1\}$  where the labels are distributed as evenly as possible. The edge labels are defined as the absolute difference of the vertices it is incident to, the same as the graceful labelings. The edge labels must also be distributed evenly. Cahit conjectured that all trees are  $k$ –equitable [1]. He then went on to prove all trees are 2–equitable, and specific trees are 3–equitable [2]. It is known that paths and 1–centipedes are  $k$ –equitable for all  $k \in \mathbb{N}$  and all trees are 3–equitable [7].

We looked at 4–equitable labelings of different sets of trees. In this paper we show that caterpillars, symmetric generalized  $n$ –stars, and complete  $n$ –ary trees are all 4–equitable. However, we used different approaches than those used to prove that all trees are 2– and 3–equitable.

## Caterpillars

**Definition 1.** A labeling of a graph is a function  $f$  from the vertex set to some subset of the natural numbers. The image of a vertex is called its label. We assign the label  $|f(u) - f(v)|$  to the edge incident with vertices  $u$  and  $v$ . In a 4–equitable labeling, the image of  $f$  is the set  $\{0, 1, 2, 3\}$ . We require both the vertex labels and the edge labels to be as equally distributed as possible, i.e., if  $v_i$  denotes the number of vertices labeled  $i$  and  $e_i$  denotes the number of edges labeled  $i$ , we require  $|v_i - v_j| \leq 1$  and  $|e_i - e_j| \leq 1$  for every  $i, j$  in  $\{0, 1, 2, 3\}$ . We will refer to a vertex labeled  $i$  as an  $i$ –vertex. Likewise, we will refer to an edge labeled  $i$  as an  $i$ –edge.

---

<sup>1</sup>Bard College

<sup>2</sup>The College of New Jersey

<sup>3</sup>Bob Jones University

<sup>4</sup>Valparaiso University

**Definition 2.** A caterpillar is a tree where every vertex is at most distance one away from the longest path. This longest path is called the spine. Any leaves not on the spine are called legs.

**Lemma 1.** There exists a 4-equitable labeling of the spine where there are at least as many legs connected to 0- and 3-vertices as there are connected to 1- and 2-vertices.

**Proof:**

Given any caterpillar, find its spine. Label this path with the pattern 3-0-2-1-1-2-0-3, repeating as necessary. If the original path labeling led to more legs from 1- and 2-vertices than from 0- and 3-vertices, then we use the pattern 1-2-0-3-3-0-2-1. We know that there are at least as many legs connected to 0- and 3-vertices as 1- and 2-vertices, because the second pattern switches every 1 and 2 with 3 and 0, respectively.

**Theorem 1.** All Caterpillars are 4-equitable

**Proof:**

Label the caterpillar's spine so that there are more legs from 0- and 3-vertices than 1- and 2-vertices. If the path is of length 1 (mod 4) or of length 3 (mod 4), do not label the last vertex. In these two cases, we will treat the unlabeled edge and vertex as a leg. This means paths of length 0 (mod 4) and 1 (mod 4) will be treated the same way, and paths of length 2 (mod 4) and 3 (mod 4) will be treated another way. Notice that this path labeling pattern is 4-equitable at each step; however, this pattern is not unique; for example, the shifted pattern 1-2-0-3-3-0-2-1 also provides a labeling of a path that is 4-equitable in each step. Our original pattern will be modified as necessary.

Now that we have the spine labeled 4-equitably, we must label the legs. Once we have labeled the spine to have at least as many legs from 0- and 3-vertices as 1- and 2-vertices, we note the remaining unlabeled vertices and what they are connected to. We then label the legs in groups of four using the following labelings if possible. Start with the first pattern and use as many times as possible before starting with the next pattern.

1. If there is 1 unlabeled leg available on each vertex label 1, 0, 2, 3, then create edges 1-0, 0-3, 2-2, and 3-1
2. If there are 2 unlabeled legs available on 0-vertices and 2 on 2-vertices, then create edges 0-3, 0-1, 2-2, and 2-0.
3. If there are 2 unlabeled legs available on 2-vertices and 2 on 3-vertices, then create edges 2-3, 2-2, 3-1, and 3-0.
4. If there are 2 unlabeled legs available on 0-vertices and 2 on 1-vertices, then create edges 0-3, 0-2, 1-1, and 1-0.
5. If there are 2 unlabeled legs available on 1-vertices and 2 on 3-vertices, then create edges 1-3, 1-1, 3-0, and 3-2.
6. If there is 1 unlabeled leg available on a 2-vertex and 3 on 0-vertices, then create edges 0-0, 0-2, 0-3, and 2-1.
7. If there is 1 unlabeled leg available on a 2-vertex, 1 on a 0-vertex, and 2 on 3-vertices, then create edges 2-1, 0-2, 3-3, and 3-0.
8. If there is 1 unlabeled leg available on a 1-vertex and 3 on 3-vertices, then create edges 3-0, 3-1, 3-3, and 1-2.
9. If there is 1 unlabeled leg available on a 1-vertex, 1 on a 3-vertex, and 2 on 0-vertices, then create edges 1-2, 3-1, 0-3, and 0-0.

10. If there are 2 unlabeled legs available on 0-vertices and 2 on 3-vertices, then create edges  $0-0$ ,  $0-3$ ,  $3-1$ , and  $3-2$ .
11. If there are 4 unlabeled legs available on 0-vertices, then create edges  $0-0$ ,  $0-1$ ,  $0-2$ , and  $0-3$ .
12. If there are 4 unlabeled legs available on 3-vertices, then create edges  $3-0$ ,  $3-1$ ,  $3-2$ , and  $3-3$ .

We have labeled legs in groups of 4, adding 4 distinct vertex labels and 4 distinct edge labels in each step. This way, the 4-equitability of the caterpillar will be the same as the 4-equitability of the spine. Hence the distribution of extra edge and vertex labels does not change. If the spine was of length  $0 \pmod{4}$ , either path labeling pattern results in a perfectly even set of vertex labels, and one less 0-edge. If the spine was of length  $2 \pmod{4}$ , using the pattern results in either an extra 0- and 3-vertex, and an extra 3-edge, or results in an extra 1- and 2-vertex, and an extra 1-edge. After reducing the amount of unlabeled legs as much as possible by the previous groupings, we will be left with up to 4 unlabeled vertices. We may have up to 1 unlabeled leg from a 1-vertex, up to 1 from a 2-vertex, up to 3 from 0-vertices, and up to 3 from 3-vertices. It could also be the case that we may have 2 legs from 1-vertices, 1 from a 0-vertex, and 1 from a 3-vertex, or 2 legs from 2-vertices, 1 from a 0-vertex, and 1 from a 3-vertex. Unfortunately, even though there are no more than 4 vertices to label, we will need to consider over 20 possibilities. In the rest of this section, we will present all of these for the sake of completeness.

**Case 1.** *Longest path is  $0 \pmod{4}$ .*

1. If there is one unlabeled leg from a 2-vertex, one from a 1-vertex and two from 0-vertices, create the edges  $1-1$ ,  $2-3$ ,  $0-2$ ,  $0-0$ .
2. If there is one unlabeled leg from a 2-vertex, one from a 1-vertex and two from 3-vertices, create the edges  $2-2$ ,  $1-0$ ,  $0-1$ ,  $0-3$ .

Otherwise: If there exists a leg on a 1-vertex, create a 0-edge by labeling the leaf 1. If there exists a leg on a 2-vertex, create a 0-edge by labeling the leaf 2. If all legs are on 0- and 3-vertices, create edge  $0-0$  and/or  $3-3$ , then use other vertex labels to complete the labeling.

**Case 2.** *Longest path is  $2 \pmod{4}$ , with extra 3-edge.*

If there is exactly one unlabeled leg on both a 0-vertex and a 3-vertex, we will need to go back to the labeling of the caterpillar's spine, which we labeled with the pattern  $3-0-2-1-1-2-0-3$ . In order to 4-equitably label the graph, we will remove the label on the first 3-vertex on the spine, and relabel it with a 1. Then, we will use the unlabeled legs to create edges  $0-2$  and  $3-3$ , which will result in a 4-equitable labeling. Otherwise do one of the following:

1. If there are only unlabeled legs from 0-vertices, create edges in the order  $0-1$ ,  $0-2$ , then  $0-0$  as necessary.
2. If there are only unlabeled legs from 3-vertices, create edges in the order  $3-1$ ,  $3-2$ , then  $3-3$  as necessary.
3. If there is one unlabeled leg from a 3-vertex, and the rest (at least two) are from 0-vertices, create the edges in the order  $3-3$ ,  $0-1$ ,  $0-2$ ,  $0-0$ .
4. If there is one unlabeled leg from a 0-vertex, and the rest (at least two) are from 3-vertices, create the edges in the order  $0-0$ ,  $3-1$ ,  $3-2$ ,  $3-3$ .
5. If there is one unlabeled leg from a 2-vertex, and the rest (if any) are from 0-vertices, create the edges in the order  $2-1$ ,  $0-2$ ,  $0-0$ .
6. If there is one unlabeled leg from a 2-vertex, and the rest are from 3-vertices, create the edges in the order  $2-2$ ,  $3-1$ ,  $3-2$ ,  $3-3$ .

7. If there is one unlabeled leg from a 2–vertex, one from a 3–vertex and the rest are from 0–vertices, create the edges in the order  $2 - 2, 3 - 1, 0 - 1, 0 - 2$ .
8. If there is one unlabeled leg from a 1–vertex, and the rest are from 0–vertices, create the edges in the order  $1 - 1, 0 - 2, 0 - 1, 0 - 0$ .
9. If there is one unlabeled leg from a 1–vertex, and the rest (if any) are from 3–vertices, create the edges in the order  $1 - 2, 3 - 1, 3 - 3$ .
10. If there is one unlabeled leg from a 0–vertex, one from a 1–vertex and the rest are from 3–vertices, create the edges in the order  $0 - 0, 1 - 2, 3 - 1, 3 - 2$ .
11. If there is one unlabeled leg from a 2–vertex, one from a 1–vertex and two from 0–vertices, create the edges  $1 - 3, 2 - 2, 0 - 1, 0 - 2$ .
12. If there is one unlabeled leg from a 2–vertex, one from a 1–vertex and two from 3–vertices, create the edges  $2 - 2, 1 - 0, 3 - 1, 3 - 2$ .

**Case 3.** *Longest path is  $2 \pmod{4}$ , with extra 1–edge.*

If there is exactly one unlabeled leg on both a 0–vertex and a 3–vertex, we will need to go back to the labeling of the caterpillar’s spine, which we labeled with the pattern  $1 - 2 - 0 - 3 - 3 - 0 - 2 - 1$ . In order to 4–equitably label the graph, we will remove the label on the first 1–vertex on the spine, and relabel it with a 0. Then, we will use the unlabeled legs to create edges  $0 - 1$  and  $3 - 3$ , which will result in a 4–equitable labeling. Otherwise, do one of the following:

1. If there are only unlabeled legs from 0–vertices, create edges in the order  $0 - 3, 0 - 0$ , then  $0 - 2$  as necessary.
2. If there are only unlabeled legs from 3–vertices, create edges in the order  $3 - 0, 3 - 3$ , then  $3 - 1$  as necessary.
3. If there is one unlabeled leg from a 3–vertex, and the rest (at least two) are from 0–vertices, create the edges in the order  $3 - 1, 0 - 0, 0 - 3, 0 - 2$ .
4. If there is one unlabeled leg from a 0–vertex, and the rest (at least two) are from 3–vertices, create the edges in the order  $0 - 2, 3 - 0, 3 - 3, 3 - 1$ .
5. If there is one unlabeled leg from a 2–vertex, and the rest (if any) are from 0–vertices, create the edges in the order  $2 - 0, 0 - 3, 0 - 0$ .
6. If there is one unlabeled leg from a 2–vertex, and the rest are from 3–vertices, create the edges in the order  $2 - 0, 3 - 3, 3 - 0, 3 - 1$ .
7. If there is one unlabeled leg from a 2–vertex, one from a 3–vertex and the rest are from 0–vertices, create the edges in the order  $2 - 0, 3 - 3, 0 - 3, 0 - 1$ .
8. If there is one unlabeled leg from a 1–vertex, and the rest are from 0–vertices, create the edges in the order  $1 - 3, 0 - 0, 0 - 3, 0 - 1$ .
9. If there is one unlabeled leg from a 1–vertex, and the rest (if any) are from 3–vertices, create the edges in the order  $1 - 3, 3 - 0, 3 - 3$ .
10. If there is one unlabeled leg from a 0–vertex, one from a 1–vertex and the rest are from 3–vertices, create the edges in the order  $0 - 0, 1 - 3, 3 - 0, 3 - 3$ .
11. If there is one unlabeled leg from a 2–vertex, one from a 1–vertex and two from 0–vertices, create the edges  $1 - 1, 2 - 0, 0 - 3, 0 - 2$ .

12. If there is one unlabeled leg from a 2-vertex, one from a 1-vertex and two from 3-vertices, create the edges  $2-2$ ,  $1-3$ ,  $3-0$ ,  $3-1$ .

Now that all cases are accounted for, we can conclude that all caterpillars are 4-equitable.

## Symmetric Generalized $n$ -stars

**Definition 3.** A symmetric generalized  $n$ -star for  $n \geq 2$  is a tree in which all leaves are the same distance from a central vertex of degree  $n$ . The number of leaves in the tree is equal to the maximum degree of the graph. A level is the set of all vertices the same distance away from the root vertex.

**Lemma 2.** All symmetric generalized 2-stars are 4-equitable.

**Proof**

A symmetric generalized 2-star is simply a path. We know that all paths are 4-equitable by following a path labeling pattern that keeps the path 4-equitable at each step, such as  $3-0-2-1-1-2-0-3$  repeating. We could also use the alternate path labeling pattern  $0-3-1-2-2-1-3-0$  repeating. Notice that for both, the same sequence of edge labels are produced, namely,  $3, 2, 1, 0, 1, 2, 3, 0$  repeating.

**Lemma 3.** All symmetric generalized 3-stars are 4-equitable.

**Proof:**

Let us consider a symmetric generalized 3-star as a tree rooted at the vertex of maximum degree. We use a labeling pattern where we label each level at a time such that the tree stays 4-equitable at each step of the pattern. To do this, we use the 4 subsets of  $\{0, 1, 2, 3\}$  with a cardinality of three as vertex labels and label the vertices in a way such that the edge differences also produce the 4 subsets of  $\{0, 1, 2, 3\}$ . We start by labeling the root vertex 0. We then go in a clockwise direction following the pattern  $3-2-1, 3-0-2, 0-1-2, 3-1-0$ . This takes care of the first four levels of the symmetric generalized 3-star. Now for the next four levels, the concept is essentially the same, but since we are not starting from a root vertex of zero, but are continuing from our  $3-1-0$  leaves, we must start from the beginning, switching path one with path three and keeping the center path the same. Namely, the vertex labels on the next four levels of the complete generalized 3-star are  $1-2-3, 2-0-3, 2-1-0, 0-1-3$  moving in the same direction. These two patterns alternate as we label further and further along a symmetric generalized 3-star. At each level, the vertex labels produce three different edge labels using three different vertex labels, and the star remains 4-equitable at each level.

**Lemma 4.** All symmetric generalized 4-stars are 4-equitable.

**Proof:**

With a similar argument, we can also see that all symmetric generalized 4-stars are also 4-equitable. We begin by labeling the root vertex 0. We then label, following a clockwise motion, the vertices at each level with  $0-1-2-3$ , then  $2-1-3-0$ . Since we are working with exactly four vertices at each level, we can simply alternate between these two labelings. What is essential is that we use four different vertex labels and four different edge labels at each level. Thus, the tree remains 4-equitable as the symmetric generalized 4-star gets larger and larger. Specifically, the edges are distributed evenly and there is one extra zero vertex.

**Lemma 5.** All symmetric generalized 5-stars are 4-equitable.

**Proof:**

Notice that we can add a path to a symmetric generalized 4-star to create a symmetric generalized 5-star. In order to keep the new generalized 5-star 4-equitable at each level, we must choose a vertex label that will be repeated at each level as well as an edge label to be repeated at each level. To do this, we can simply

label this extra path following the path labeling pattern that starts with 0 that is already the label of the root vertex and we continue with  $3 - 1 - 2 - 2 - 1 - 0 - 3$  which we know produces different edge labels and vertex labels at each step in groups of 4, which is consistent with the labels for the complete generalized 4–star. This results in a 4–equitable labeling of the symmetric generalized star with five leaves.

**Lemma 6.** *All symmetric generalized 6–stars are 4–equitable.*

**Proof:**

With a similar argument, we can add two paths to the symmetric generalized 4–star to create a symmetric generalized 6–star. Notice in the alternate path labeling pattern :  $3 - 0 - 2 - 1 - 1 - 2 - \underline{0} - 3 - 3 - 0 - 2 - 1 - 1$  that the vertices and edges are asymmetric about the indicated 0. We will take this vertex label as the label of the root vertex. The asymmetry of the path labeling pattern about the root vertex is essential for keeping the symmetric generalized 6–star 4–equitable at each step.

**Theorem 2.** *All symmetric generalized stars are 4–equitable.*

To create a 4–equitable labeling of a symmetric generalized  $n$ –star for all  $n \geq 2$  we will connect copies of a generalized 4–star with one copy of a symmetric generalized  $m$ –star where  $m < 4$ . i.e, if  $n = 4k + m$ , then we will use  $k$  copies of our symmetric generalized 4–star and one copy of our symmetric generalized  $m$ –star if  $m \neq 1$ . If  $m = 1$ , we use  $k - 1$  copies of the symmetric generalized 4–star and one copy of our symmetric generalized 5–star.

## Complete $n$ –ary Trees

In order to prove all complete  $n$ –ary trees are 4–equitable, we first handle binary, ternary, quaternary, and 5–ary trees.

**Lemma 7.** *All Complete Binary Trees are 4–equitable*

**Proof:**

Given any complete binary tree, label the root vertex 3. Label the children of each 0– and 2–vertices 2 and 3; label the children of each 1– and 3–vertices 0 and 1. On level 2 the 4 vertices are labeled 0, 1, 2, 3. The number of vertices on every future level is a multiple of 4, and the vertex labels are evenly distributed on each level. The edges between any 3– or 0–vertex and its children are 3 and 2; the edges between any 2– or 1–vertex and its children are 0 and 1. The edge labels between levels 1 and 2 are completely evenly distributed. The number of edges after every future level is a multiple of 4, and the edge labels are evenly distributed on each level. Therefore, all complete binary trees are 4–equitable.

We will make use of the following facts to prove all ternary trees are 4–equitable. If a number has the form  $(8a + 3)$ , then  $9(8a + 3) = (8 + 1)(8a + 3) = 64a + 8a + 24 + 3 = 8B + 3$ . This implies that every odd exponent of 3 is of the form  $(8B + 3)$  and is congruent to 3 (mod 4). If a number has the form  $(8a + 1)$ , then  $9(8a + 1) = (8 + 1)(8a + 1) = 8B + 1$ . This implies that every even exponent of 3 is of the form  $8B + 1$  and is congruent to 1 (mod 4).

**Claim 1.**  $\frac{3^{k+1}-1}{2} \equiv 1 \pmod{4}$  when  $k$  is even.

**Proof:**

If  $k$  is even, then  $k + 1$  is odd, so  $3^{(k+1)}$  is of the form  $8B + 3$ , and  $\frac{8B+3-1}{2} = 4B + 1 \equiv 1 \pmod{4}$ .

**Claim 2.**  $3^{k+1} - 1 \equiv 0 \pmod{4}$  when  $k$  is odd.

**Proof:**

If  $k$  is odd, then the exponent is even, so  $3^{k+1}$  is of the form  $8B+1$ , and  $8B+1-1 = 8B$ , which is divisible by 4.

**Lemma 8.** *All complete ternary trees are 4-equitable.*

**Proof:**

Given any complete ternary tree, label the root vertex 3. Label the children of each 3-vertex with one 0-vertex, one 1-vertex, and one 2-vertex. Label the children of each 2-vertex, with one 0-vertex, one 1-vertex, and one 2-vertex. Label the children of each 1-vertex, with one 1-vertex, one 2-vertex, and one 3-vertex. Label the children of each 0-vertex with two of the vertices 3 and one of the vertices 0.

The total number of vertices in the tree is given by  $\sum_{i=0}^k 3^i = \frac{3^{k+1}-1}{2}$ . We know that either the number of vertices or number of edges in the tree is divisible by 4, because each power of 3 is congruent to either  $1 \pmod{4}$  or  $0 \pmod{4}$ , and the total number of edges is one less than the total number of vertices.

Let  $v_{l,i}$  be the number of vertices labeled  $i$  in level  $l$ , and  $e_{l,i}$  be the number of edges labeled  $i$  in level  $l$ . The distribution of vertex labels in level  $l+1$  is as follows: The number of 0 vertices in level  $l+1$  is equal to  $v_{l,0} + v_{l,2} + v_{l,3}$ ; the number of 1 vertices in level  $l+1$  is equal to  $v_{l,1} + v_{l,2} + v_{l,3}$ ; The number of 2 vertices in level  $l+1$  is equal to  $v_{l,1} + v_{l,2} + v_{l,3}$ ; and the number of 3 vertices in level  $l+1$  is equal to  $2v_{l,0} + v_{l,2}$ . In level 1 we have  $v_{1,3} = 0$  and  $v_{1,1} = v_{1,2} = v_{1,0} = 1$  and, with the equations for the vertex label distribution in level  $l+1$  when  $l = 1$ , we see there exists an extra 3-vertex in level 2. Substituting  $v_{2,0}$  into the previous formulas we get  $v_{3,0} = v_{3,1} = v_{3,2} = v_{2,0} + v_{2,0} + (v_{2,0} + 1) = 3v_{2,0} + 1$ . Also,  $v_{3,3} = 2v_{2,0} + v_{1,0} = 3v_{2,0}$ , so level 3 has one less 3-vertex than 0-, 1-, and 2-vertices. Since every odd exponent of 3 is congruent to  $3 \pmod{4}$ , and every even exponent of 3 is congruent to  $1 \pmod{4}$ , the pattern of having one extra 3-vertex in a level and one extra 0, 1, and 2-vertices in the next level continues to repeat infinitely. This means if the vertex labels include an extra 3 in one level, then the labels of the next level include an extra 0, 1, and 2, and the reverse also holds true. Furthermore, because the total number of vertices in the tree is congruent to either  $1 \pmod{4}$  or  $0 \pmod{4}$ , the pattern for total vertices in the tree of an extra 3-vertex and then completely even vertex labels repeats infinitely. A similar argument for the total number of edge labels shows a repeating pattern of one less 0-edge followed by the edge labels being distributed completely evenly in the whole graph. Therefore, all complete ternary trees are 4-equitable.

**Lemma 9.** *All complete quaternary trees are 4-equitable.*

**Proof:**

Given any complete quaternary tree, let  $v_{l,i}$  be the number of vertices labeled  $i$ , and  $e_{l,i}$  be the number of edges labeled  $i$  in level  $l$ . Label the root vertex 3. Label the children of each 3-vertex use one 0-vertex, 1-vertex, 2-vertex, and 3-vertex. Label the children of each 2-vertex label three of the vertices 0 and label one of the vertices 2. Label the children of each 1-vertex label half of the vertices 1 and the other half of the vertices 2. For children of each 0-vertex label three of the vertices 3 and one of the vertices 1. On level 2 the vertices are evenly distributed, and the number of children is always divisible by 4. The number of 3-vertices in level  $l+1$  is equal to  $(v_{l,3} + 3v_{l,0})$ ; the number of 2-vertices in level  $l+1$  is equal to  $(v_{l,3} + v_{l,2} + 2v_{l,1})$ ; the number of 1 vertices in level  $l+1$  is equal to  $(v_{l,3} + 2v_{l,2} + v_{l,0})$ ; and the number of 0 vertices in level  $l+1$  is equal to  $(v_{l,3} + 3v_{l,2})$ . We know that  $v_{2,0} = v_{2,1} = v_{2,2} = v_{2,3}$  and  $v_{3,0} = v_{3,1} = v_{3,2} = v_{3,3}$  because the vertex labels on level 2 are distributed as evenly as possible. Therefore, the equation for each vertex label in level  $l+1$  can be written as  $(4v_{l,0})$ . This means if the vertex labels are equally distributed on level  $l$ , then they are also equally distributed on level  $l+1$ . This means that the vertex labels are distributed as evenly as possible at every level for any complete quaternary tree, so then the labels are evenly distributed for the whole graph. A similar argument shows that the edge labels are distributed as evenly as possible. Therefore, all complete quaternary trees are 4-equitable.

**Lemma 10.** *All complete 5-ary trees are 4-equitable.*

**Proof:**

Given any complete 5-ary tree, label the root vertex 3. Label the vertices in level 1 using two 0's, one 1, one 2, and one 3. Treat one 0-vertex, the 1-vertex, the 2-vertex, and the 3-vertex as a group, this will be referred to in the future as group 1. Label the children of the 0-vertex 3, the children of the 1-vertex 0, the children of the 2-vertex 2, and the children of the 3-vertex 1; the vertex and edge labels of this group are distributed completely evenly. Follow the same pattern of vertex labeling for the children of this group, and the labels will continue to be distributed completely evenly. Then, from the 0-vertex that was not put in the group of 4, label the children with two 2-vertices, a 1-vertex, a 3-vertex, and a 0-vertex. Treat the 1-vertex, 2-vertices, and 3-vertex as a separate group. Label the children of the 1-vertex 3, the children of one 2-vertex 1, the children of the other 2-vertex 2, and the children of the 3-vertex 0; the vertex and edge labels of this group are distributed completely evenly. From level 3 on, label the children of this group using the pattern from group 1, labeling the children of the 0-vertices 3, the children of the 1-vertices 0, the children of the 2-vertices 2, and the children of the 3-vertices 1, so the vertex and edge labels of this group will be distributed completely evenly. Then, from the 0-vertex in level 2 that was not put in the group of 4, label the children with two 1-vertices, a 0-vertex, a 2-vertex, and a 3-vertex. Treat the two 1-vertices, the 0-vertex, and the 2-vertex as a separate group. Label the children of the 0-vertex 3, the children of one 1-vertex 1, the children of the other 1-vertex 2, and the children of the 2-vertex 0; the vertex and edge labels of this group are distributed completely evenly. From level 4 on, label the children of this group using the pattern from group 1, so the vertex and edge labels of this group will be distributed completely evenly. Then, from the 3-vertex in level 3 that was not put in the group of 4, label the children with two 3-vertices, a 0-vertex, a 1-vertex, and a 2-vertex. Treat one 3-vertex, the 0-vertex, the 1-vertex, and the 2-vertex as a group. Label the children of this group using the pattern of group 1; the vertex and edge labels of this group are distributed completely evenly. Then, label the children of the 3-vertex in level 4 that was not put in the group of 4 with two 0-vertices, a 1-vertex, a 2-vertex, and a 3-vertex. Starting at level 5, repeat the pattern that began at level 1. Using this pattern, an extra 3-vertex exists at level 0; an extra 3-edge, an extra 0-vertex, and an extra 3-vertex exist at level 1; an extra 3-edge, 2-edge, 0-vertex, 2-vertex, and 3-vertex exist at level 2; an extra 3-edge, 2-edge, and 1-edge exist at level 3 with completely even vertex labels; an extra 3-vertex exists at level 4 with completely even edge labels; and this pattern continues to repeat. Therefore, all complete 5-ary trees are 4-equitable.

**Theorem 3.** *All complete  $n$ -ary trees are 4-equitable.*

**Proof:**

Note the congruency between  $\frac{a^{k+1}-1}{a-1}$  and  $\frac{(a+4)^{k+1}-1}{a+3}$ . We know that  $\frac{a^{k+1}-1}{a-1} = \sum_{i=1}^k a^{k-i}$ , and  $\frac{(a+4)^{k+1}-1}{a+3} = \sum_{i=0}^k (a+4)^i = \sum_{i=0}^k (c_i a^j)$  where  $c_i = 1 + 4A_j$  for some  $A_j$ . These equations show that the coefficients of  $\frac{a^{k+1}-1}{a-1}$  are congruent to the coefficients of  $\frac{(a+4)^{k+1}-1}{a+3} \pmod{4}$ , so if we increase the number of children by 4 we do not change the congruency  $\pmod{4}$  of the number of total vertices of the graph.

We know that we can add  $4m$  children to each vertex in a complete  $n$ -ary tree, where  $m \in \mathbb{N}$  because the congruency  $\pmod{4}$  does not change. We label the new children of each vertex in sets of  $4m$  as if they were in a complete quaternary tree; labeling them based of the parent vertex. Since the vertex and edge labels of quaternary trees are distributed completely evenly for each level, adding these labels to any 4-equitably labeled complete  $n$ -ary tree will result in a 4-equitable labeling of a complete  $n+4$ -ary tree. For any complete  $n$ -ary tree with a 4-equitable labeling, there exists a 4-equitable labeling for any complete  $n+4m$ -ary tree.

Every number is congruent to either  $0 \pmod{4}$ ,  $1 \pmod{4}$ ,  $2 \pmod{4}$ , or  $3 \pmod{4}$ , and, as previously shown, adding 4 to a number that is congruent to  $x \pmod{4}$  results in a number that is still congruent to  $x \pmod{4}$ . Since complete binary, ternary, quaternary and 5-ary trees are 4-equitable, all complete  $n$ -ary trees are 4-equitable.

## Conclusion

While these three sets of trees have been proven to be 4–equitable, the question still remains: Are all trees 4–equitable? Of course, the ultimate goal is proving all trees are graceful. The conjecture that all trees are graceful has been accepted, but no significant progress has been made towards proving this conjecture, even after half a century.

## Acknowledgments

This research was supported by National Science Foundation Grant DMS-1262852.

## References

- [1] I. Cahit. Status of graceful tree conjecture in 1989. In *Topics in combinatorics and graph theory (Oberwolfach, 1990)*, pages 175–184. Physica, Heidelberg, 1990.
- [2] Ibrahim Cahit. Recent results and open problems on cordial graphs. In *Contemporary methods in graph theory*, pages 209–230. Bibliographisches Inst., Mannheim, 1990.
- [3] Joseph A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, 5:Dynamic Survey 6, 43 pp. (electronic), 1998.
- [4] Solomon W. Golomb. How to number a graph. In *Graph theory and computing*, pages 23–37. Academic Press, New York, 1972.
- [5] G. Ringel. Extremal problems in the theory of graphs. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, pages 85–90. Publ. House Czechoslovak Acad. Sci., Prague, 1964.
- [6] A. Rosa. On certain valuations of the vertices of a graph. In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pages 349–355. Gordon and Breach, New York, 1967.
- [7] David E. Speyer and Zsuzsanna Szaniszló. Every tree is 3–equitable. *Discrete Math.*, 220(1-3):283–289, 2000.