A Partial Ordering Of Knots

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Research Supported by NSF Grant DMS-0851721

1. Abstract

Knot theory is the study of the different ways to embed a circle in three-dimensional space. Our research concerns how knots behave under crossing changes. In particular, we investigate a partial ordering of alternating knots. A similar ordering was originally introduced by Kouki Taniyama in the paper "A Partial Order of Knots". We amend Taniyama's partial ordering and present theorems about the structure of our ordering for more complicated knots. Our approach is largely graph theoretic, as we translate each knot diagram into one of two planar graphs by checkerboard coloring the plane. Of particular interest are the class of knots known as pretzel knots, as well as knots that have only one "direct" minor in the partial ordering.

2. Introduction

2.1. **Basic Knot Theory.** A **knot** K is a smooth embedding of a circle S^1 in $\mathbb{R}^3$. Some of our results generalize to links. A **link** L is a smooth embedding of multiple disjoint copies of S^1 in $\mathbb{R}^3$. Knot theorists generally do not want to be walking around with 3-dimensional objects. That is why it is common to use knot diagrams. A **knot diagram** D of K is a way of projecting K on a flat $\mathbb{R}^2$ surface. This projection is one-to-one everywhere except a finite number of points called **crossings** where it is two-to-one. At every crossing there is an unbroken line for the overstrand and a broken line for the understrand. The overstrand corresponds to the arc that was initially closer to the viewer in $\mathbb{R}^3$. In our research, we convert these knot diagrams to graphs.

The main problem in knot theory is that one knot K may have many different diagrams that don't look remotely similar. How do we know when two knot diagrams represent the same knot?

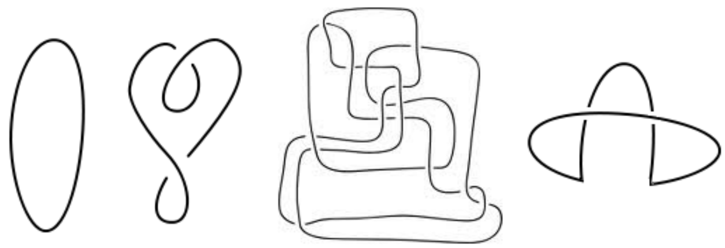


FIGURE 1. Knot diagrams of the unknot

The **Reidemeister Moves** are a set of moves that connect any two diagrams of the same knot. These moves are local, meaning the knot is unchanged outside of the exhibited region. They do not change the underlying knot.

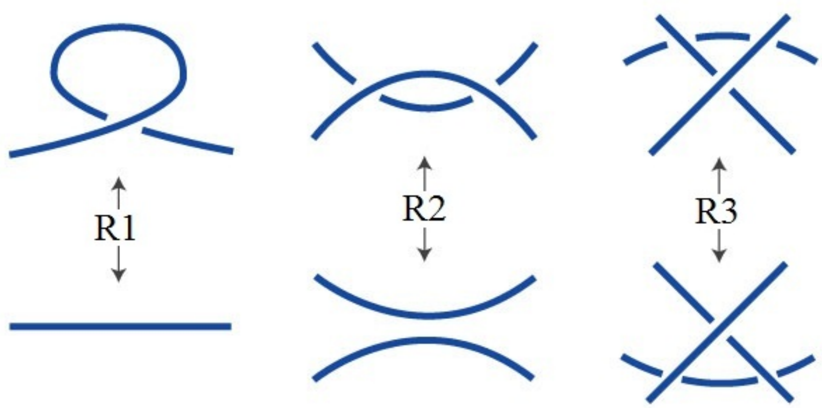


FIGURE 2. Reidemeister Moves (Image courtesy of [2])

Theorem 1 (Reidemeister)**.** *Two diagrams D_1 and D_2 represent the same knot K if and only if they may be connected by a finite number of Reidemeister moves.*

The proof is omitted here since this is a standard result. (See [8] for proof.)

The **crossing number** $c(L)$ of a link L is the minimum number of crossings over all diagrams L. A **minimal knot diagram** is a diagram D where the number of crossings equals $c(L)$.

Our research concerns how knots behave under crossing changes. A **crossing change** is a local operation that changes the role of the overstrand and the understrand at a single crossing. A crossing change may change the underlying knot. An example of a crossing change is shown in Figure 3. As with our images for the Reidemeister moves, it is assumed that the link is unchanged outside of the region shown below.

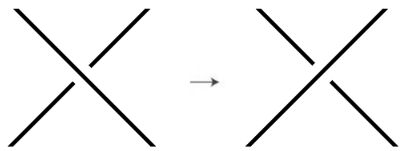


FIGURE 3. Crossing Change

We also focus on prime alternating links, since they have many nice properties that allow for stronger results. An **alternating link** is a link with an alternating diagram, which is a link diagram that alternates between overstrands and understrands as one travels around each of the components in a fixed direction. A prime link is a link that cannot be drawn as a "connect sum" of two non-trivial knots. Below is the Granny Knot, which is a connect sum of two trefoils.

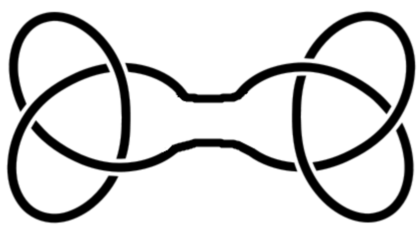


FIGURE 4. A Non-Prime Knot

The following two theorems are important results that make alternating links especially nice to work with. These are both standard results whose proofs can be found in the literature.

Theorem 2 (Kauffman, Murasugi, and Thistlethwaite)**.** *Let L be a prime alternating link with diagram D. Then D is a minimal diagram for L if and only if D is a reduced alternating diagram.*

The proof is omitted here since this is a standard result. (See [1] for proof.)

Reduced means that there are no nugatory crossings. A crossing in a diagram D is a **nugatory (removable) crossing** if removing a neighborhood of that crossing splits the link diagram into two separate pieces. These are the crossings that can obviously be eliminated to lower the crossing number of D.

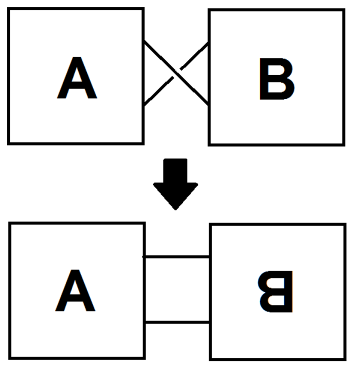


FIGURE 5. A Nugatory Crossing

Theorem 3 (**Tait's Flyping Conjecture**, Menasco & Thistlethwaite). *Let D_1 and D_2 be two minimal diagrams of the same alternating knot K in S^2. Then D_1 can be transformed into D_2 via a series of flypes.*

The proof is omitted here since this is a standard result. (See [7] for proof.)

An example of a flype is shown in Figure 6. A flype is usually a complex combination of Reidemeister moves, but just like the basic Reidemeister moves, it does not change the underlying knot.

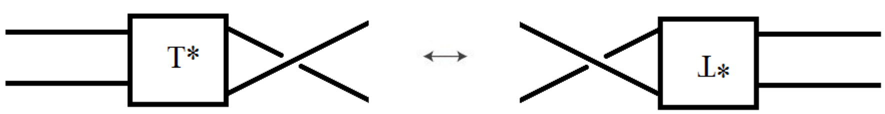


FIGURE 6. Flype Operation

2.2. **Partial Orderings of Knots.** Kouki Taniyama defined an ordering on knots in his paper "A Partial Order of Knots" [9]. We call this partial ordering the T-Order.

Definition 1. *Let K_1 and K_2 be knots. The **T-Order** defines $K_1 \leq K_2$ if every diagram of K_2 can be transformed into some diagram of K_1 via some number of simultaneous crossing changes.*

In this paper we present a modified version of Taniyma's T-Ordering and call it the V-Order.

Definition 2. *Let K_1 and K_2 be prime alternating knots. The* ***V-Order*** *defines K_1 to be a* ***V-minor*** *of K_2 if there exists a minimal diagram of K_2 that can be transformed into some diagram of K_1 via simultaneous crossing changes. We then define a* ***proper sequence*** *of knots (K_n, K_{n-1},. . ., K_2, K_1) if K_i is a V-minor of K_{i+1} for all i. $K_1 \leq K_2$ if there exists a proper sequence containing both K_1 and K_2, where K_1 appears to the right of K_2.*

Notice that if K_1 is a V-minor of K_2, then $K_1 \leq K_2$. We do not differentiate between a knot, its reflection, and its reverse in the V-Order. One can verify that the V-Order defines a partial order of alternating knots, meaning:

(1) $K \leq K$ for all K.

(2) If $K_1 \leq K_2$ and $K_2 \leq K_3$, then $K_1 \leq K_3$.

(3) If $K_1 \leq K_2$ and $K_2 \leq K_1$, then $K_1 = K_2$.

The third condition in the partial ordering definition requires that this is an ordering on alternating knots. (See Theorem 5 in Section 3.2.)

We represent the V-Order with a Hasse Diagram, which is a graphical way to represent the relationships in the partial ordering. If two knots K_1 and K_2 are connected by an edge on the diagram and K_1 is below K_2, then $K_1 \leq K_2$. We verified that the V-Order is identical to the T-Order through 7_1, yielding the following Hasse diagram.

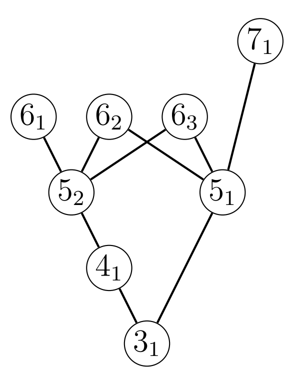


Figure 7. Partial ordering for the first eight prime knots.

Note that our ordering restricts our attention to prime alternating knots and requires that we check only one minimal diagram of a knot K_2

to verify $K_1 \leq K_2$, while Taniyama's ordering requires that we check all diagrams of K_2. Also notice that if $K_1 \leq K_2$ in the T-order, then $K_1 \leq K_2$ in the V-order. The converse is not necessarily true a priori, although we conjecture that it is true for prime alternating knots. (See Conjecture 1 in Section 5.) A similar ordering to our V-Order was introduced by Diao, Ernst, and Stasiak in "A Partial Ordering Of Knots And Links Through Diagrammatic Unknotting" [5]. Their ordering allows for only one crossing change, while ours allows for multiple simultaneous crossing changes.

We are especially interested in direct V-minors.

Definition 3. *L_1 is a* ***direct V-minor*** *of L_3 if $L_1 \leq L_3$ and there does not exists L_2 such that $L_1 \leq L_2 \leq L_3$.*

Definition 4. *L_1 is a* ***remote V-minor*** *of L_3 if $L_1 \leq L_3$ and there exists L_2 such that $L_1 \leq L_2 \leq L_3$.*

For example, in our Hasse Diagram, 3_1 is a remote V-minor of 7_1 because $3_1 \leq 5_1 \leq 7_1$. However, 3_1 is a direct V-minor of 5_1 since there does not exist a knot K such that $3_1 \leq K \leq 5_1$.

2.3. **Graph Theoretical Methods in Knot Theory.** By Theorem 1, we know that we can connect the infinite amount of diagrams of one knot via Reidemeister Moves, but it is tedious to constantly redraw the diagram for a knot every time we perform a Reidemeister Move. For this reason, we converted knot diagrams to signed planar graphs.

The procedure for converting a knot diagram to a graph is as follows:

(1) "Checkerboard" color the knot diagram so that every crossing borders two "white" regions and two "gray" regions. Then mark the crossings by dropping a mark counterclockwise from the overstrand.

(2) Pick a color and place a vertex inside each region of this fixed color.

(3) If two of the chosen regions share a crossing, add an edge between the corresponding vertices in the graph. This edge is solid if the marking falls within the chosen regions and dotted if the marking falls within the regions of the other color.

Since we had two choices in Step 2, we get two graphs. These graphs are signed duals.

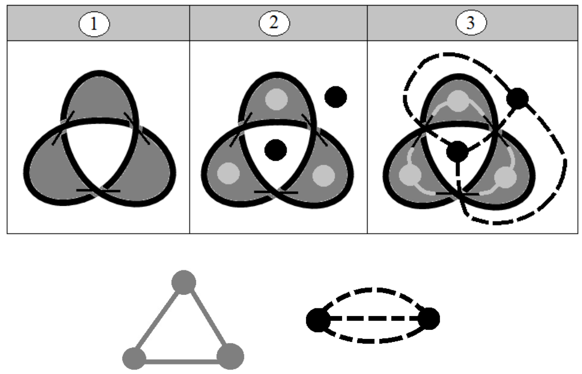


FIGURE 8. Checkerboard Graphs of 3_1

We need to know how the Reidemeister moves translate from knot diagrams to checkerboard graphs. The important thing to note here is that every Reidemeister move translates to two "graph Reidemeister moves" that are duals of one another. In Figure 9 below, E and F represent nodes that may have other edges entering them, while the small black vertices may only have the edges shown.

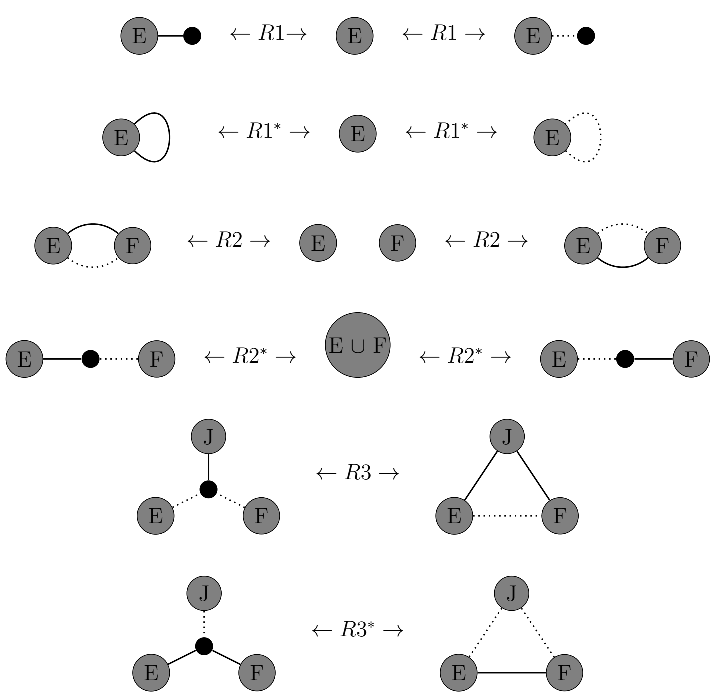


FIGURE 9. Reidemeister Moves for Graphs

An important thing to note is that the graph representation of an alternating knot diagram has either all solid or all dotted edges.

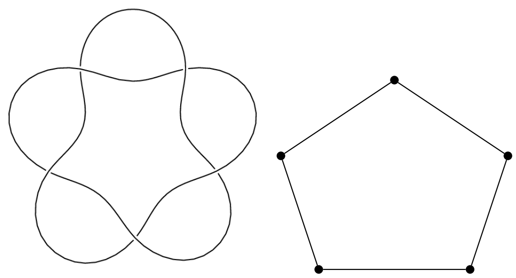


FIGURE 10. Diagram and Graph of 5_1

Since our research deals with how knots behave under crossing changes, we need to translate what a crossing change does to the graph of a knot. Crossing changes switch the roles of the overstrand and understrand. In the graph, this changes the marking on the associated edge (dotted to solid and vice versa).

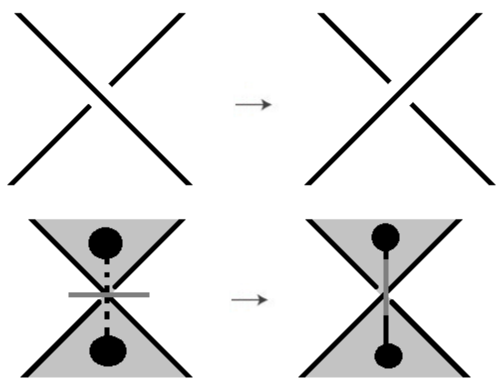


FIGURE 11. A Crossing Change and its Graph Representation

We also rely on flypes in many of our proofs. Below are the graph representations of flypes. Just as with the Reidemeister move graphs, a flype has two different graph representations that are duals of one another.

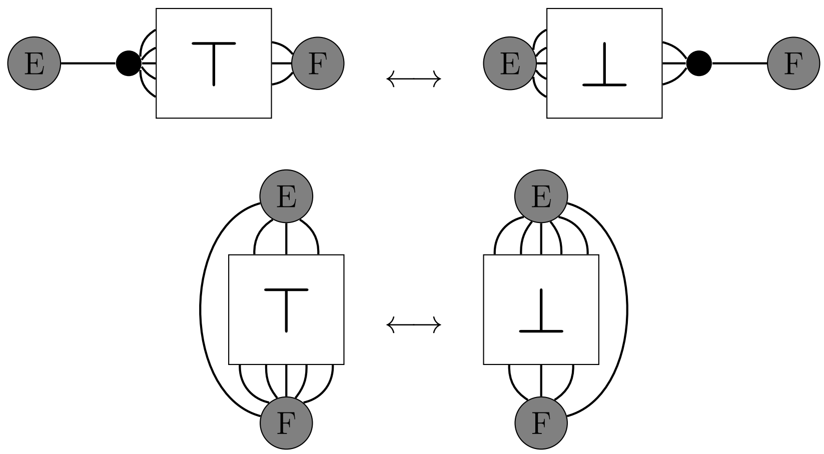
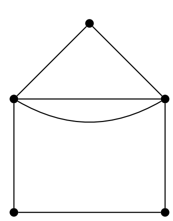


FIGURE 12. Graph equivalents of a flype

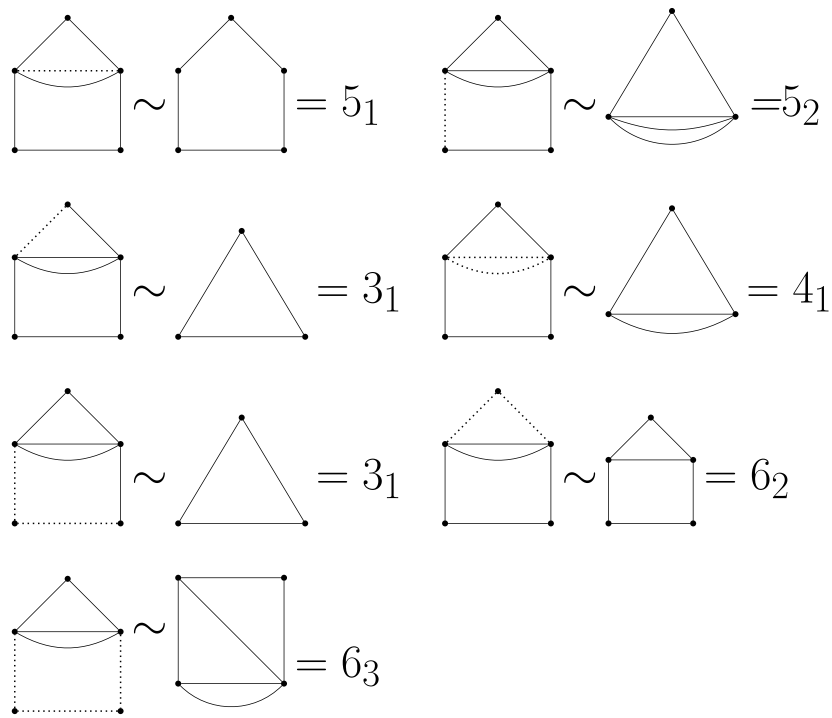
3. OUR RESULTS

3.1. **Methods for Expanding the V-Order.** Our first goal is to expand the Hasse Diagram of Subsection 2.2 to 7 crossing prime alternating knots. In order to determine which knots are V-minors of a particular knot K, we exhaustively check all possible ways to make simultaneous crossing changes on K's graph. We check all the different ways to make one crossing change at a time, and then two crossing changes at a time up to half of the crossing number of K. The reason we do not need to change more than half the crossings is because we are not distinguishing between a knot and its reflection. If changing some set of crossings yields a diagram of K, then changing the compliment of that set gives a diagram of the reflection of K. We also save time by not checking different sets that are combinatorially identical.

Example 1. *Take the graph of* 7_5*:*



Below are all of the fundamentally distinct combinations of crossing changes.



Using this graph theoretic technique, we have updated our Hasse Diagram for up to 7 crossing knots.

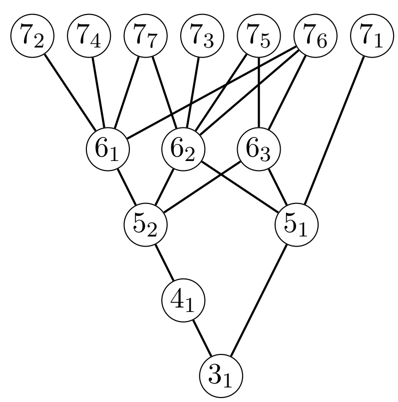


FIGURE 13. The V-Order for Prime Alternating Knots Through 7_7

See Appendix B for calculations similar to the one in Example 1 that yield this Hasse Diagram.

3.2. **Invariants and the V-Order.** From Subsection 3.1, one can see that there are many cases to check. In order to quickly eliminate many possible relationships in the V-order, we prove several results about the ordering that involve knot invariants. A **knot invariant** is a function $i : \kappa \to \alpha$ from the set of all knots κ to some type of algebraic structure α. The important thing to note here is that distinct diagrams of the same knot must get sent to the same value by the invariant, so if an invariant gives different values for two diagrams, they must not be the same knot.

The knot invariants we work with are crossing number $c(K)$, bridge index $br(K)$, and braid index $b(K)$.

Recall that the crossing number $c(L)$ of a link L is the minimum number of crossings over all diagrams L.

Theorem 4. *Let K_1 and K_2 be knots with $K_1 \leq K_2$, then $c(K_1) \leq c(K_2)$*

Proof. Let K_1 and K_2 be knots, where $K_1 \leq K_2$ and $c(K_2) = n$. Then there exists a minimal diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 via some number of simultaneous crossing changes. D_1 has n crossings. Then the crossing number of K_1 can be at most n. □

The following theorem is more specific to our research since our V-order restricts its attention to alternating knots.

Theorem 5 (Taniyama)**.** *Let K_1 and K_2 be alternating knots with $K_1 \leq K_2$, then $c(K_1) < c(K_2)$*

Proof. Let K_1 and K_2 be alternating knots, where $K_1 \leq K_2$ and $c(K_2) = n$. Then there exists a minimal diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 by simultaneously changing some but not all of the crossings in D_2. D_1 has n crossings. Note that D_1 is not a minimal diagram. By Theorem 2, $c(K_1) < n$. □

The second invariant we work with is the bridge number. The bridge number of a diagram D of a knot K is the number of local maxes in D. The **bridge number** $br(K)$ of a knot K is the minimal bridge number over all diagrams of K. Note that there is a local minimum for every local maximum.

An example of a knot diagram D with $br(D) = 4$ is shown in Figure 14. Here the box represents some (possibly complex) part of the knot diagram, as long as it contains no local maxima or minima.

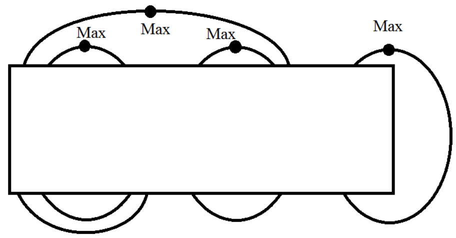


FIGURE 14. A Knot Diagram With $br(D) = 4$

Theorem 6. *If $K_1 \leq K_2$, then $br(K_1) \leq br(K_2)$.*

Proof. Let K_1 and K_2 be knots, where $K_1 \leq K_2$ and $br(K_2) = n$. Then there exists a minimal bridge diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 via some number of simultaneous crossing changes. D_1 has n local maxes. Then the bridge number of K_1 can be at most n. □

The last invariant we work with is the braid index. The **braid index** or $b(K)$ is the minimal number of"strands" over all braid representations of a knot.

An example of a braid representation is shown in Figure 15. As with our figure for bridge number, the box represents some (possibly complex) part of the knot diagram, as long as it contains no local maxima or minima.

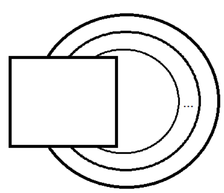


FIGURE 15. A General Braid Representation

Theorem 7. *If $K_1 \leq K_2$, then $b(K_1) \leq b(K_2)$.*

Proof. Let K_1 and K_2 be knots, where $K_1 \leq K_2$ and $b(K_2) = n$. Then there exists a minimal braid diagram D_2 of K_2 that can be transformed into a diagram D_1 of K_1 via some number of simultaneous crossing changes. D_1 has n strands. Then the braid index of K_1 can be at most n. □

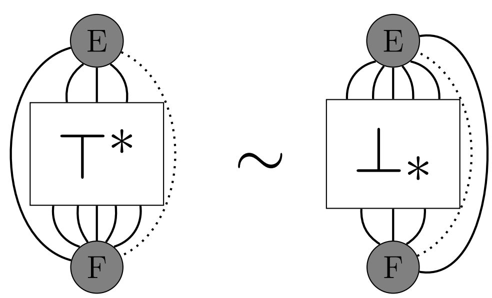
3.3. **Direct V-Minors.** We turn our attention to finding direct V-minors. Recall that K_1 is a direct V-minor of K_3 if $K_1 \leq K_3$ and there does not exist K_2 such that $K_1 \leq K_2 \leq K_3$. In particular, we search for V-minors such that $c(K_1) = c(K_2) - 1$ because for alternating knots, that means that nothing can be "in between" by Theorem 5. Hence, K_1 must be a direct V-minor of K_2.

Theorem 8. *Let K_1 and K_2 be alternating knots with $K_1 \leq K_2$. Let G_2 be any minimal graph of K_2.*

(1) In G_2, if we switch some but not all of the edges connecting two vertices, then $c(K_1) \leq c(K_2) - 2$.

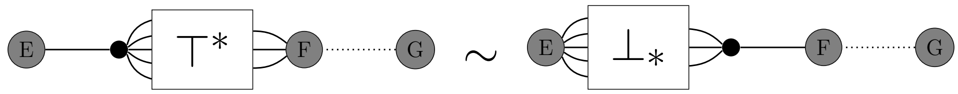
(2) In G_2, if we switch some but not all of the edges separating two regions, then $c(K_1) \leq c(K_2) - 2$.

Proof. We are given that K_1 and K_2 are alternating knots with $K_1 \leq K_2$. Let G_2 be an alternating graph of K_2. We will switch some but not all of the edges connecting two vertices. Make some number of crossing changes to G_2 such that one edge is dotted and one edge is solid between two vertices. In general these edges need not be directly adjacent. If they are not directly adjacent, we can perform a flype as shown below:



This leads to an $R2$ move, which will reduce the graph to a graph with at least two edges less than the original G_2. Thus, K_1 has at most $c(K_2) - 2$ crossings. Therefore, $c(K_1) \leq c(K_2) - 2$.

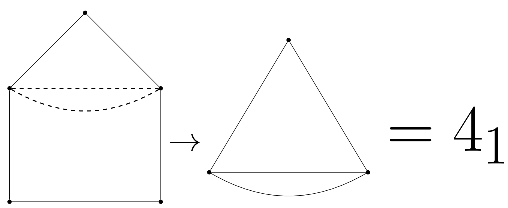
Now we will switch some but not all of the edges separating two regions. First, take the graph G_2 of K_2. Make some number of crossing changes such that one edge is dotted and one edge is solid between two regions. In general these edges need not be directly adjacent either. If they are not directly adjacent, we can perform a flype as shown below:



This leads to an $R2^*$ move, which will reduce the graph to a graph with at least two edges less than the original G_2. Thus, K_1 has at most $c(K_2) - 2$ crossings. Therefore, $c(K_1) \leq c(K_2) - 2$. □

It should be noted that this condition, although necessary for obtaining a direct V-minor with $c(K_1) = c(K_2) - 1$, is not sufficient to guarantee that $c(K_1) = c(K_2) - 1$. Example 2 gives one situation where the conditions are not sufficient.

Example 2. *If we change both of the middle edges of the graph of* 7_5*, we drop to the graph of* 4_1*, which has* $c(4_1) = c(7_5) - 3$*, even though we followed Theorem 8.*



4. Polygonal Graphs and Pretzel Links

4.1. **Basic Properties.** A particularly simple class of links are pretzel links. A link is a **pretzel link** if it has a diagram that takes the form in Figure 16. Here the boxes represent "twist boxes" full of half-twists in either direction.

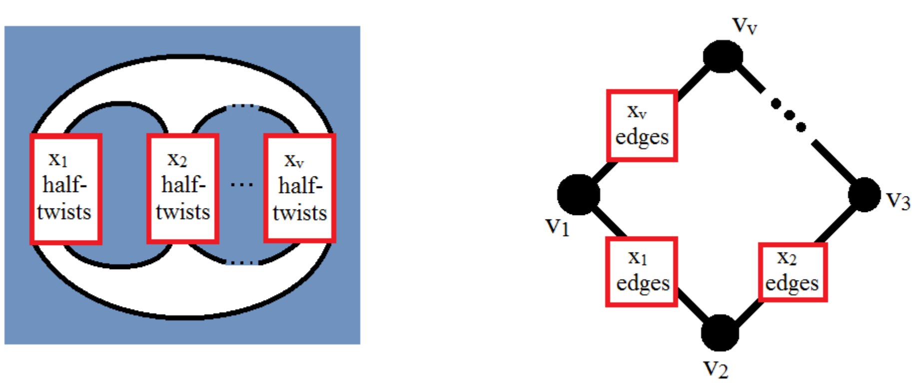


Figure 16. Pretzel Knot Diagram and Its Graph

A pretzel link always has a graph of the form in Figure 16, where the half-twists translate into parallel edges between adjacent vertices. We refer to graphs of this type as **polygonal graphs**. We denote the pretzel link of Figure 16 by $P_v(x_1, x_2, x_3, ..., x_v)$, where v is the number of vertices in the polygonal graph and x_i is the number of edges connecting the consecutive vertices, v_i and v_{i+1}. We define v_v to precede v_1.

For our partial ordering, we are only concerned with alternating knots and are considering the knot and its reflection to be essentially the same, so we have no need to differentiate between solid and dotted edges in these types of graphs as it relates to our partial ordering. However, to differentiate, we use a negative x_i when the edges are dotted and a positive x_i when the edges are solid.

Our procedure for drawing a graph given $P_v(x_1, x_2, x_3, ..., x_v)$ is as follows:

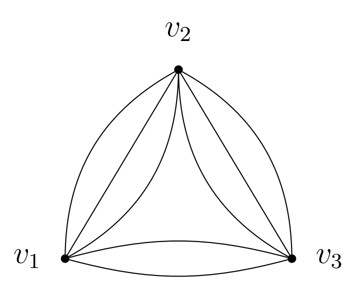
(1) Draw v vertices.

(2) Number each vertex in a counterclockwise fashion.

(3) Start with vertex 1 and draw x_1 edges between vertices 1 and 2. Then put x_2 edges between vertices 2 and 3. Repeat in this fashion until x_v edges have been drawn between vertices v and 1. Draw each edge according to the rule:

- If the x_i coordinate is negative, make the edges dotted.
- If the x_i cooridinate is positive, make the edges solid.

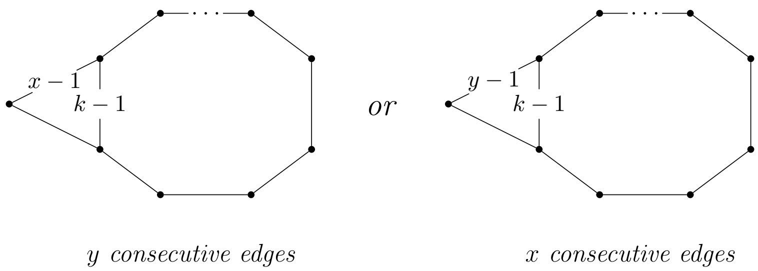
If, for example, we are given that the polygonal representation of 8_5 is $P_3(3, 3, 2)$, we draw 3 vertices, v_1, v_2, v_3. We then draw 3 edges between v_1 and v_2, 3 edges between v_2 and v_3, and 2 edges between v_3 and v_1.



$8_5 = P_3(3, 3, 2)$

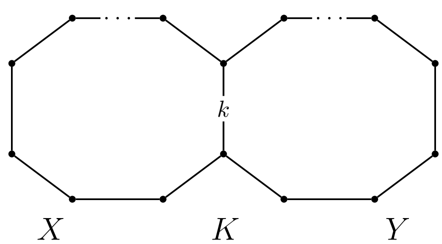
4.2. **Pretzel Links and Our Partial Order.** We are interested in finding knots that have one or two direct V-minors. Pretzel links are a special class of links that contain many knots with this property. A particularly simple type of pretzel link that includes most knots with this property are of the form $P_{k+2}(x, y, 1, 1, 1, \ldots)$.

Theorem 9. *If L is a pretzel link with polygonal graph of the form $P_{k+2}(x, y, 1, 1, 1, \ldots)$, where $k \neq 1$, then L has two direct V-minors L' with $c(L') = c(L) - 1$. These V-minors have graphs:*



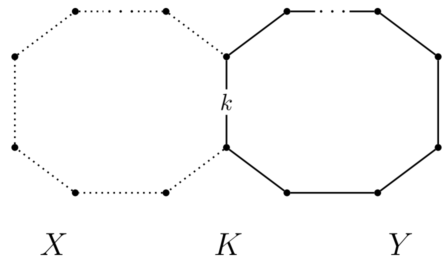
Here the $x-1$, $y-1$, and $k-1$ refer to the number of parallel strands in the given location. In the case that $x = 1$, L has only one V-minor of the form $P_{k+1}(y, 2, 1, 1, 1, \ldots)$. Equivalently, if $y = 1$ then L has only one V-minor of the form $P_{k+1}(x, 2, 1, 1, 1, \ldots)$.

Proof. Given L as defined above, the dual of $P_{k+2}(x, y, 1, 1, 1, \ldots)$ is:

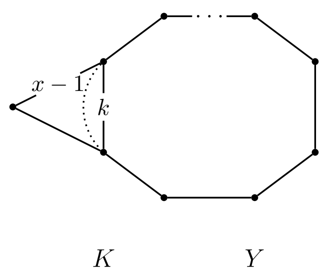


Here X is the subgraph that contains the x consecutive edges on the left and Y is the subgraph that contains the y consecutive edges on the right. K is the subgraph that contains the k parallel strands in between X and Y.

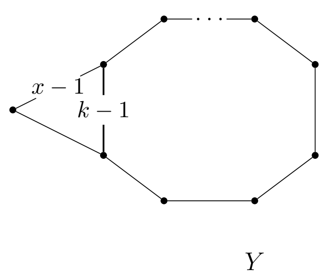
Performing crossing changes on the X subgraph will yield:



This graph will reduce with the addition of an edge by performing an $R1$ move on a two valent vertex between any two dotted edges in X and then a series of $R3$ moves yielding:

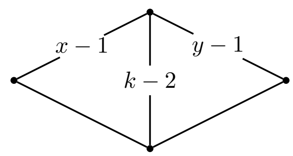


This will not change the number of edges in K. After performing an $R2$ move on K we get the graph:



Changing this graph back to the dual will not give us a polygonal graph unless $x \leq 2$. If $x > 2$, we will get a graph that looks exactly like this one, except we have $y - 1$ instead of $x - 1$. When $x = 2$, we have only one edge on the side marked with $x - 1$, so it can easily be shown that when we take the dual we will get $P_{k+1}(y, 2, 1, 1, \ldots)$ with $k - 1$ coordinates that are 1. If we wanted to change the Y subgraph, we could just switch the role of x and y in $P_{k+2}(x, y, 1, 1, 1, \ldots)$.

The only other change we can make that is significantly different from this case and follows the Theorem 8 is to change all crossings in K or all crossings in both X and Y. Doing so will yield the following graph after crossing changes and Reidemeister moves:



This graph obviously has $c(L) = x-1+y-1+k-2+2 = x+y+k-2$ crossings, which is two fewer than the original $c(L) = x+y+k$. There are no other cases that will follow Theorem 8 and yield a knot with one fewer crossing than L. □

Note that when $x = y$ for this theorem, we will only have one V-minor with one fewer crossing than L. Also note that if $k = 1$, then we have $P_3(x, y, 1) \to P_{y+2}(x-1, 1, 1, \ldots)$, whose dual is $P_x(y+1, 1, 1, \ldots)$.

The $(p, 2)$-torus links and twist knots are special cases of Theorem 9. When $x = 1$ and $y = 1$ we have a **$(\mathbf{p}, \mathbf{2})$-torus link**, which has polygonal graph $P_{c(L)}(1, 1, 1, \ldots)$. When $x = 2$ and $y = 1$, L is a **twist knot**, which has polygonal graph $P_{c(L)-1}(2, 1, 1, \ldots)$.

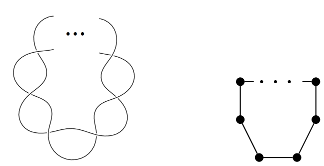


FIGURE 17. $(p, 2)$ Torus Knot (diagram courtesy of [4]).

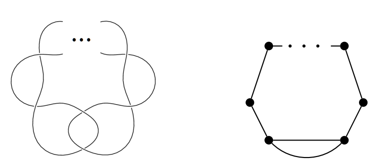


FIGURE 18. Twist Knot (diagram courtesy of [4]).

Theorem 10. $(p, 2)$ *torus knots have only* $(p, 2)$ *torus knots as their V-minors.*

Proof. Consider the graph $P_{c(K)}(1, \ldots, 1)$ of the torus knot. If we change some, but not all of the crossings, say m crossings, there will always lie a solid edge next to a dotted edge. This means that we can always perform an $R2$ move until the edges are all solid or all dotted. Every time we perform an $R2$ move, we lose two edges. Then the graph will always be $P_{|c(K)-2m|}(1, \ldots, 1)$, which is a $(p, 2)$ torus knot. It can easily be checked that this result holds for the dual. □

Theorem 11. *Twist knots have only twist knots as their V-minors.*

Proof. Since a twist knot K_2 has the graph $P_{c(K_2)-1}(2, 1, 1, \ldots, 1)$, it follows from Theorem 9 that the only direct V-minor of K_2 is a twist knot K_1 which has the graph $P_3(k-1, 1, 1)$ where k was the number of 1's in K_2's graph. K_1 is a twist knot. Therefore, since any twist knot has only one direct V-minor that is a twist knot of one fewer crossing, the only V-minors of any given twist knot are twist knots. □

In Figure 13 in Section 3.1, the $(p, 2)$ torus knots are on the rightmost branch of the Hasse Diagram, and the twist knots are on the leftmost branch of the Hasse Diagram.

5. Future Work

Our work revealed several questions that we were not able to address. The biggest technical problem we face is what we call "The Minimal Conjecture."

Conjecture 1. ***The Minimal Conjecture*** *Given a prime, alternating knot K_2 and some knot K_1, if there exists a minimal diagram of K_2 that can be transformed into a diagram of K_1 via some number of simultaneous crossing changes, then every diagram of K_2 can be transformed into K_1 via some number of simultaneous crossing changes.*

Note that Conjecture 1 implies that the V-Order and T-Order are equivalent for prime alternating knots. This means that our work is a direct refinement over Taniyama's original methods.

In Section 4, we produced many knots with only one direct V-minor. The only knots we found with one direct V-minor were pretzel knots. This begs the following conjecture.

Conjecture 2. *Pretzel knots are the only prime alternating knots with one direct V-minor.*

Below are a few more general topics that we may address in future research.

Future Topic 1. *All $(p, 2)$ torus knots K lack direct V-minors K' with $c(K') = c(K) - 1$. Most other knots seem to have at least one V-minor with $c(K') = c(K) - 1$, but there are still examples of non $(p, 2)$ torus knots that fail in this regard.* 8_5 *and* 8_{16} *are non-$(p, 2)$ torus knots K that have no direct V-minors K' with $c(K') = c(K) - 1$. Is there something special about these knots that we can generalize?*

Notice that both 8_5 and 8_16 also have non-prime V-minors.

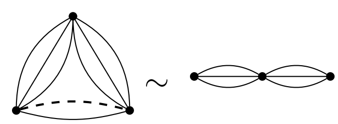


FIGURE 19. $8_5 \sim 3_1\#3_1$

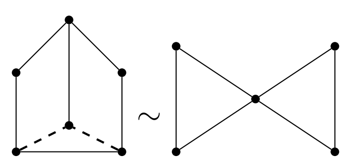


FIGURE 20. $8_{16} \sim 3_1\#3_1$

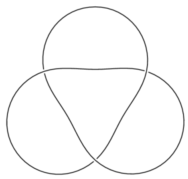
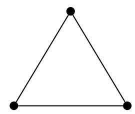
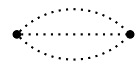
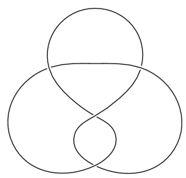
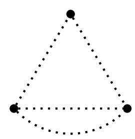
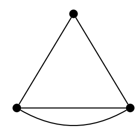
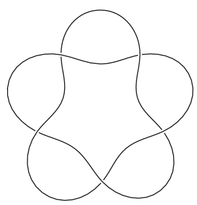
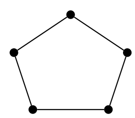
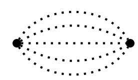
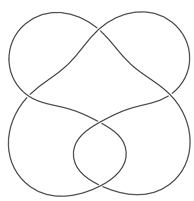
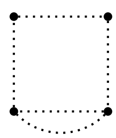
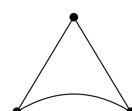
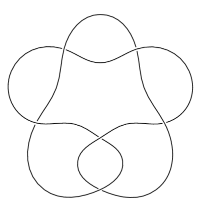
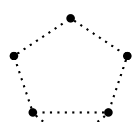
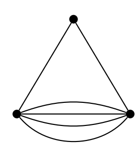
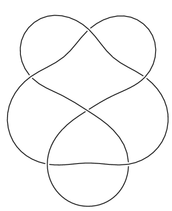
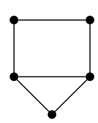
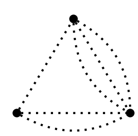
Future Topic 2. *We would like to expand our work to non-prime or non-alternating links, or at least find what prime alternating knots have non-prime or non-alternating knots directly beneath them in our ordering.*

We already have one result about the placement of non-alternating knots within the V-order:

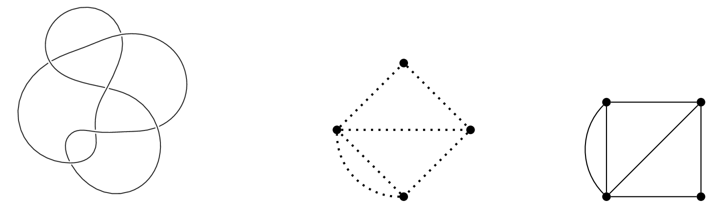
Theorem 12. *Let L_1 be a non-alternating link with $c(L_1) = n$. Then there exists an alternating link L_2 where $c(L_2) = n$, such that $L_1 \leq L_2$.*

Proof. If L_1 is a non-alternating link with $c(L_1) = n$, the minimal graph for L_1 will have both dotted and solid edges, with n edges. If we change all the dotted edges to solid, we now have a graph of a link L_2, with all solid edges, which, since this projection is alternating, implies that this graph is a reduced alternating graph of L_2. This is equivalent to saying this graph of L_2 is minimal. So we have a minimal graph of L_2 with crossing number n. We also can see that $L_1 \leq L_2$ since we are able to transform a minimal diagram of L_2 into L_1 via crossing changes. Thus, there does exist an alternating link L_2 with $c(L_2) = n$ for each non-alternating link L_1 with $c(L_1) = c(L_2) = n$, where $L_1 \leq L_2$. $\square$

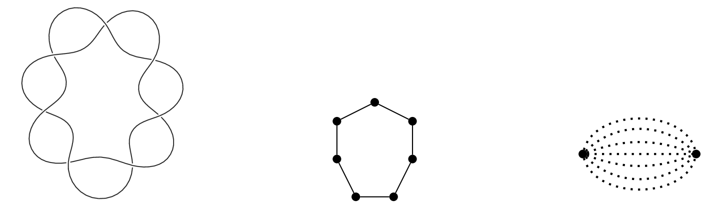
Appendix A. Prime Alternating Knots Up to $c(K) = 8$

Knot	Knot Diagram	Graph 1	Graph 2
3_1			
4_1			
5_1			
5_2			
6_1			
6_2			

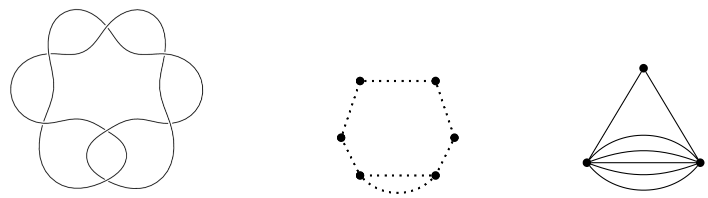
6_3



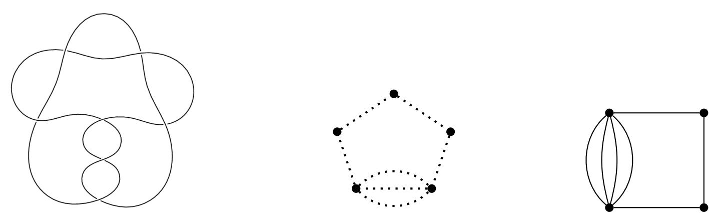
7_1



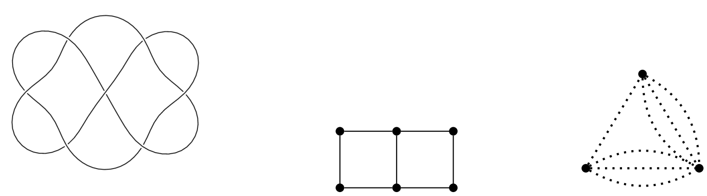
7_2



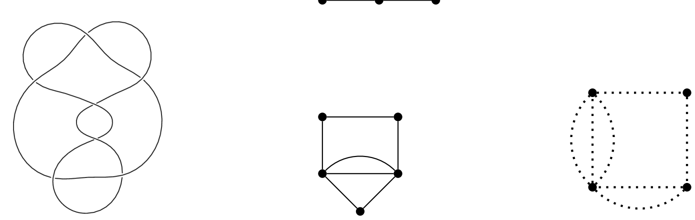
7_3



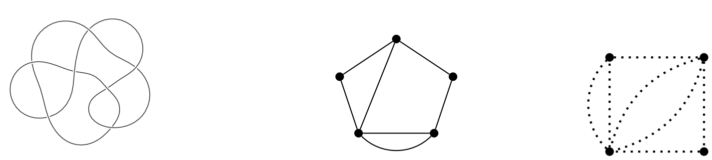
7_4



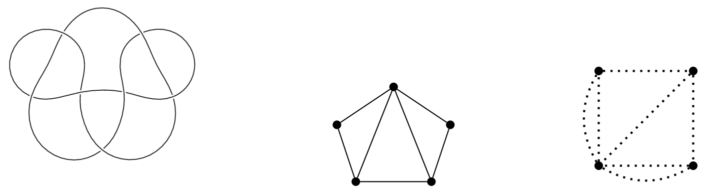
7_5



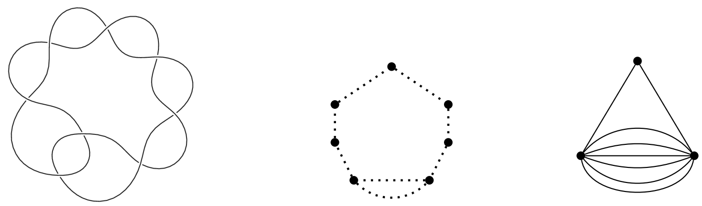
7_6



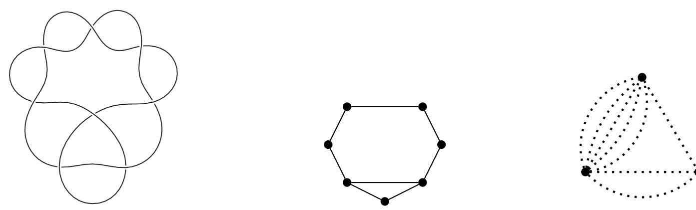
7_7



8_1



8_2



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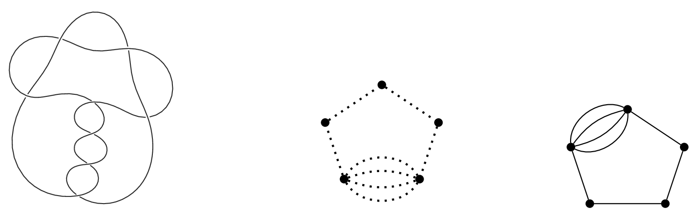

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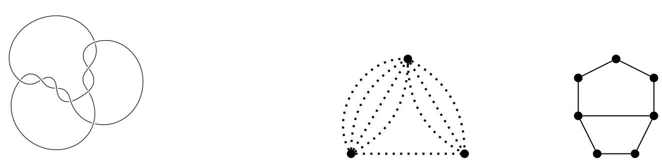

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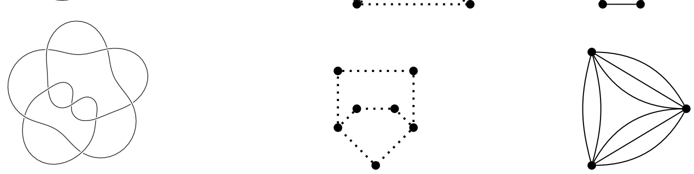
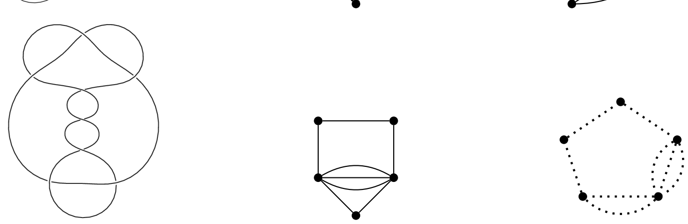
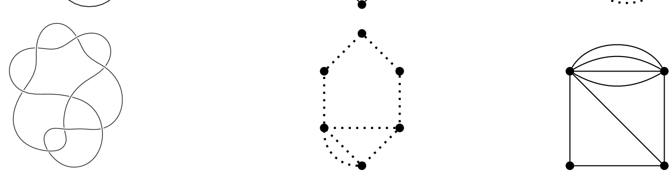


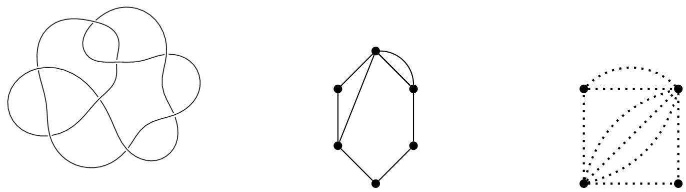
8_6



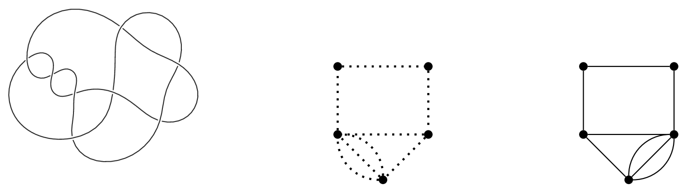
8_7



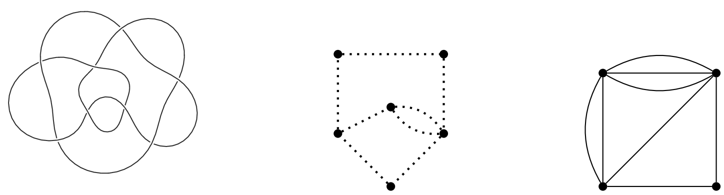
8_8



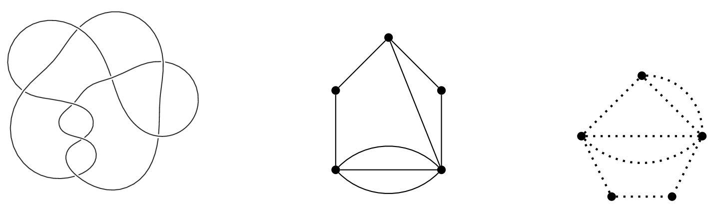
8_9



8_{10}



8_{11}



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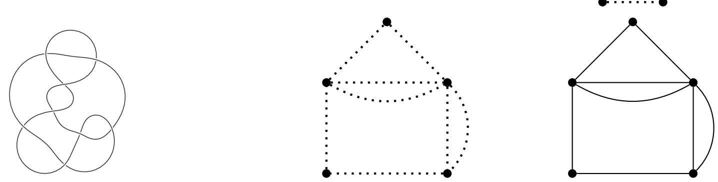
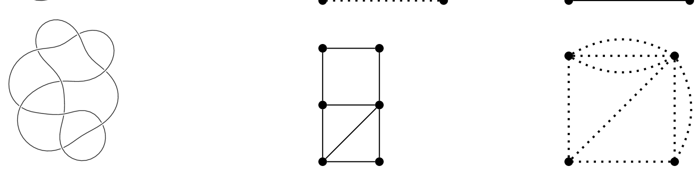


8_{13}



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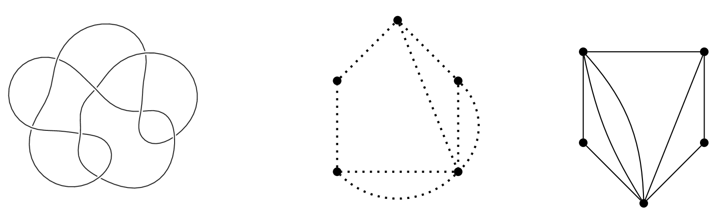
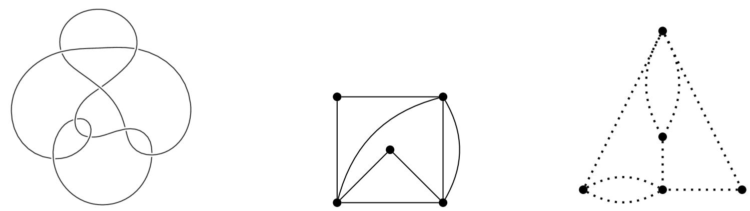
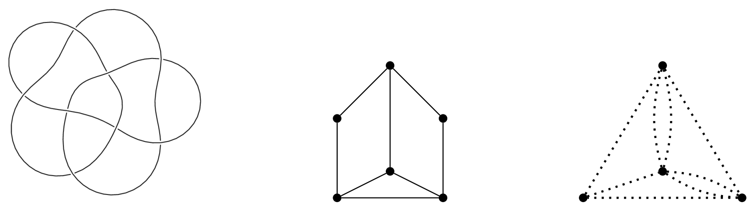


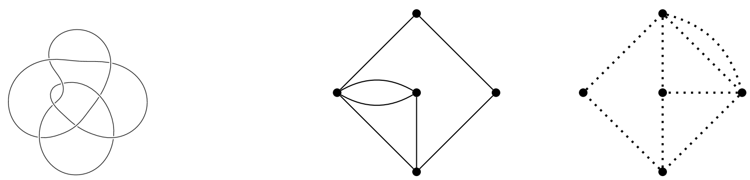
8_{15}



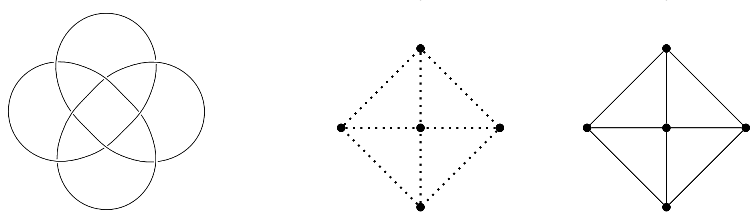
8_{16}



8_{17}



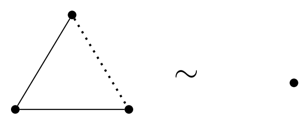
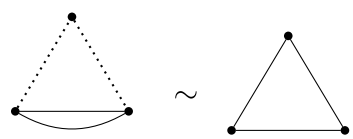
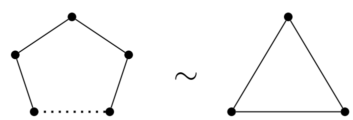
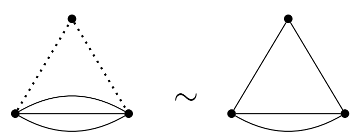
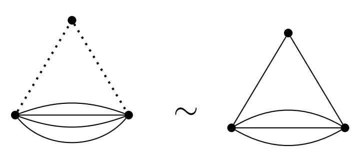
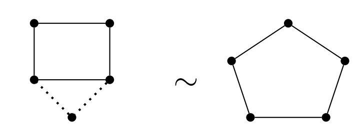
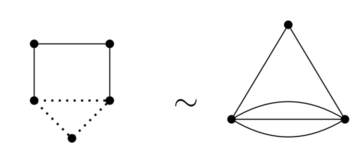
8_{18}

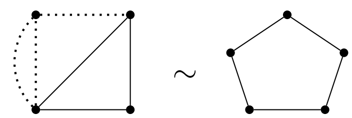


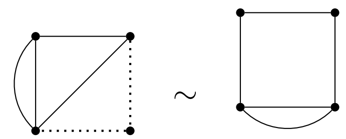
Knot diagrams courtesy of [4].

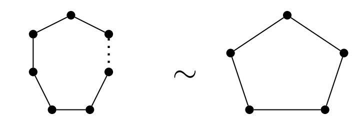
Appendix B. Expansion of the Hasse Diagram

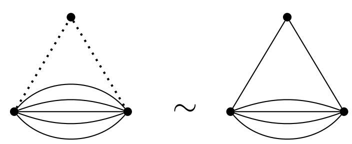
We perform crossing changes on the initial knot K_2 by switching the solid edges to dotted edges. These are the cases that get us to their direct V-minors K_1.

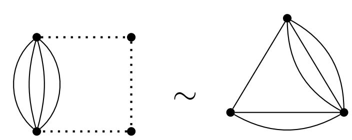
K_2	Crossing Changes Performed	K_1
3_1		0_1
4_1		3_1
5_1		3_1
5_2		4_1
6_1		5_2
6_2		5_1
6_2		5_2

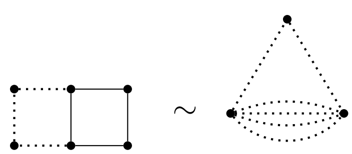
6_3  5_1

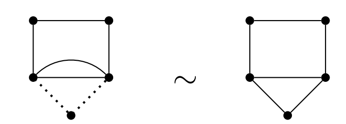
6_3  5_2

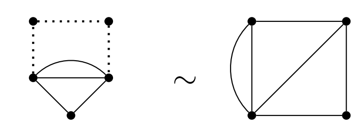
7_1  5_1

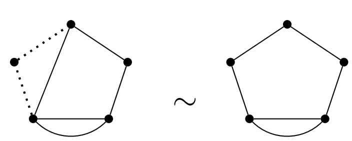
7_2  6_1

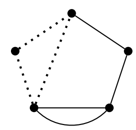
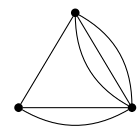
7_3  6_2

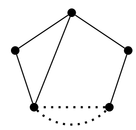
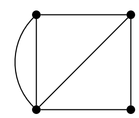
7_4  6_1

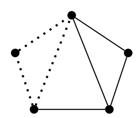
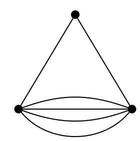
7_5  6_2

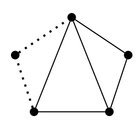
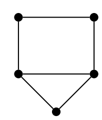
7_5  6_3

7_6  6_1

7_6  $\sim$  6_2

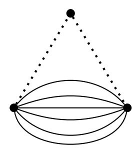
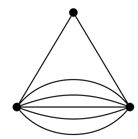
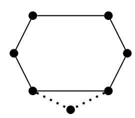
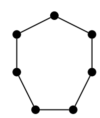
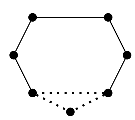
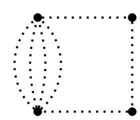
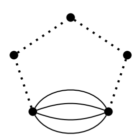
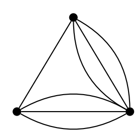
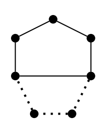
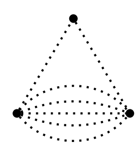
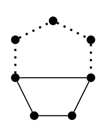
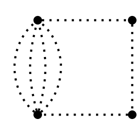
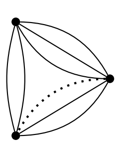
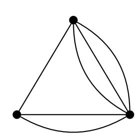
7_6  $\sim$  6_3

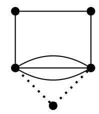
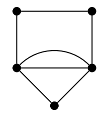
7_7  $\sim$  6_1

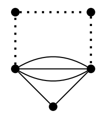
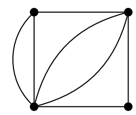
7_7  $\sim$  6_2

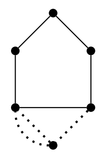
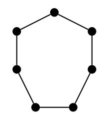
Appendix C. Incomplete Expansion of the Hasse Diagram

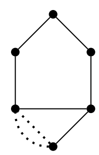
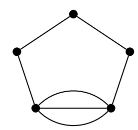
This is an incomplete list of what alternating knots are direct V-minors of knots K with $c(K) = 8$ through 8_{15}. Note that flypes may be necessary to obtain the final knot K_1 from the initial knot K_2.

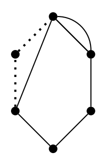
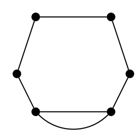
K_2	Crossing Changes Performed	K_1
8_1	 $\sim$ 	7_2
8_2	 $\sim$ 	7_1
8_2	 $\sim$ 	7_3
8_3	 $\sim$ 	7_4
8_4	 $\sim$ 	7_2
8_4	 $\sim$ 	7_3
8_5	 $\sim$ 	6_2

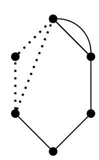
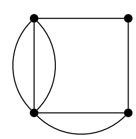
8_6  $\sim$  7_5

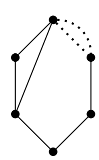
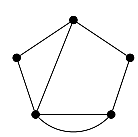
8_6  $\sim$  7_6

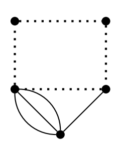
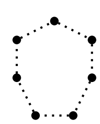
8_7  $\sim$  7_1

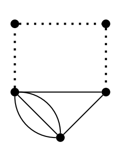
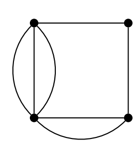
8_7  $\sim$  7_3

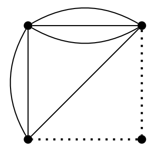
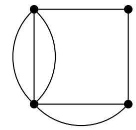
8_8  $\sim$  7_2

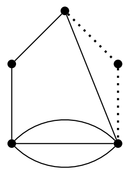
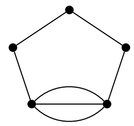
8_8  $\sim$  7_5

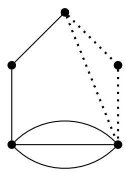
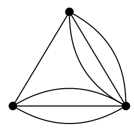
8_8  $\sim$  7_6

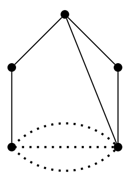
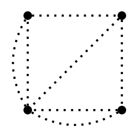
8_9  $\sim$  7_1

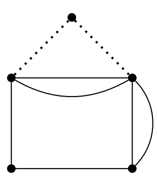
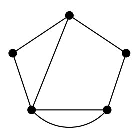
8_9  $\sim$  7_5

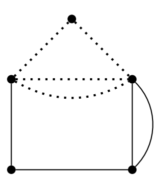
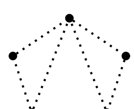
8_{10}  $\sim$  7_5

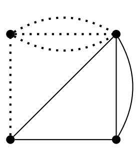
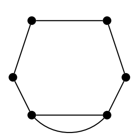
8_{11}  $\sim$  7_3

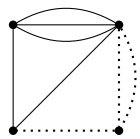
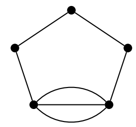
8_{11}  $\sim$  7_4

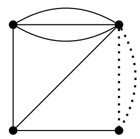
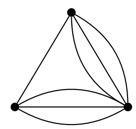
8_{11}  $\sim$  7_7

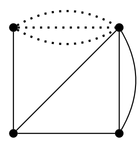
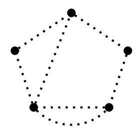
8_{12}  $\sim$  7_6

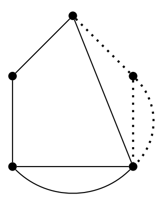
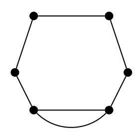
8_{12}  $\sim$  7_7

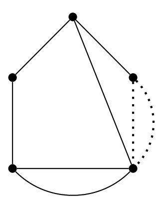
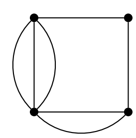
8_{13}  $\sim$  7_2

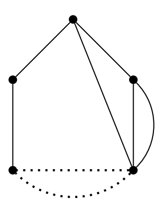
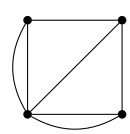
8_{13}  $\sim$  7_3

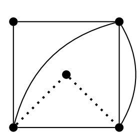
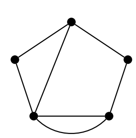
8_{13}  $\sim$  7_4

8_{13}  $\sim$  7_6

8_{14}  $\sim$  7_2

8_{14}  $\sim$  7_5

8_{14}  $\sim$  7_7

8_{15}  $\sim$  7_6

References

[1] Adams, Colin Conrad. The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots. Providence, RI: American Mathematical Society, 2004. Print.

[2] Cohen, Arjeh."First Week of Knot Theory in MasterMathCourse on Quantum Groups and Knot Theory." First Week of Knot Theory in MasterMath Course on Quantum Groups and Knot Theory. N.p., 12 Sept. 2007. Web. 28 July 2012. <http://www.mathadore.nl/mathadore/knots/old/les1.md>.

[3] Bar-Natan, Dror, and Scott Morrison. "The Knot Atlas." N.p., n.d. Web. 22 July 2012. <http://katlas.math.toronto.edu/wiki/Main_Page>.

[4] "Knotilus." Knotilus. N.p., 2003. Web. 28 July 2012. <http://knotilus.math.uwo.ca/>.

[5] Diao Yuanan, Claus Ernst, and Andrzej Stasiak. "A Partial Ordering Of Knots And Links Through Diagrammatic Unknotting." Journal of Knot Theory and Its Ramifications 18.04 (2009): 505. Print.

[6] Livingston, Chuck, and Jae C. Cha. "Table of Knot Invariants." KnotInfo. N.p., n.d. Web. 22 July 2012.<http://www.indiana.edu/~knotinfo/>.

[7] Menasco, W. and Thistlethwaite, M. "The Classification of Alternating Links." Ann. of Math. 138 (1993) 113-171.

[8] Reidemeister, K."Knotten und Gruppen" Abh. Math. Sem. Univ. Hamburg 5, 7-23, 1927.

[9] Taniyama, Kouki. "A Partial Order of Knots." Tokyo Journal of Mathematics 12.1 (1989): 205-29. Print.