Estimating the Volatility in the Black-Scholes Formula

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Abstract

The Black-Scholes formula is one of the most popular option pricing models; however, one of the inputs, volatility, is not deterministic and thus not available for immediate application in the formula. In our research, we examine four different approaches for better estimating the volatility: smoothing, deriving the distribution of the volatility, building time series models, and nonparametric approaches. We employ both single and double exponential smoothing techniques on European call option valuations for the S&P 100 Index. We derive a function for σ^2 and σ and calculate their expected values. Secondly, we derive the probability distributions of the volatility with a transformation technique. The expectations of the volatility from the probability distributions are then applied back to the Black-Scholes formula. Additionally, we extend the cumulative normal distribution functions in the Black-Scholes formula using a Taylor series expansion to arrive at functions of volatility. With time series volatility models, we apply Autoregressive Conditional Heteroscedasticity (ARCH) and Generalized Autoregressive Conditional Heteroscedasticity (GARCH) volatility for application into the formula. Similarly to these two well-discussed volatility models, we purpose three new time series volatility models: Moving Average Conditional Heteroscedasticity (MACH), Autoregressive Moving Average Conditional Heteroscedasticity (ARMACH), and Generalized Autoregressive Moving Average Conditional Heteroscedasticity (GARMACH). With the nonparametric approaches we do not assume any probability distributions and instead calculate volatility both from average sample variance as well as weighted sample variance.

Keywords: Black-Scholes formula, option pricing, volatility models, exponential smoothing

1 Introduction

An option is a type of financial contract where the owner has the right, but not the obligation, to buy or sell a stock at a certain price (strike price) before a certain date (expiration date). The Black-Scholes

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formula is one of the most popular option pricing models due to its compact form and computational ease. It was first introduced by Fischer Black and Myron Scholes in their 1973 paper,"The Pricing of Options and Corporate Liabilities" [2]. From their stochastic partial differential equation model, the Black-Scholes formula can be deduced.

$$C = \Phi(d_1)S - \Phi(d_2)Ke^{-rT}$$
$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

where

$$\begin{aligned}
C &= \text{premium for call option} \\
\Phi &= \text{cumulative density function of a} \\
& \text{standard normal distribution} \\
T &= \text{expiration date} \\
S &= \text{stock price at time } t_0 \\
K &= \text{strike price} \\
r &= \text{risk-free interest rate} \\
\sigma &= \text{volatility}
\end{aligned}$$

It gives a theoretical estimate of the price of a European-style option. The formula's arrival led to an explosion in options trading that shaped the world's financial markets. However, despite its simplicity and widespread use, it is not a perfect formula. One of the inputs, volatility, or σ, is not deterministic and thus not available for immediate application in the formula. A generic approach is to compute the volatility of the stock price, S_t, prior to time t_0. However, this approach may not produce the ideal result. A search for a better estimate for the volatility has generated several different approaches each with varying degrees of success. One of the most popular methods is using time series models that allow for volatility to change over time. Allowing for volatility in one time period to be dependent on the volatility during previous time periods is important in financial modeling due to volatility clustering. Following the groundbreaking introduction of the Autoregressive Conditional Heteroscedasticity (ARCH) model in Engle (1982) [4], subsequent research has explored new models that allow for volatility to change over time.

In this paper, we derive a theoretical apparatus for four different approaches: smoothing, deriving the distribution of the volatility, building time series models, and nonparametric techniques. We test the different approaches numerically using three different numerical comparisons: sum of squared errors of prediction (SSE), root-mean-square error (RMSE), and percent bias. Additionally, we propose three new times series models: Moving Average Conditional Heteroscedasticity (MACH), Autoregressive Moving Average Conditional Heteroscedasticity (ARMACH), and Generalized Autoregressive Moving Average Conditional Heteroscedasticity (GARMACH).

The following paper is organized as follows: In Section 2, we present new single and double exponential smoothing techniques. Section 3 contains time series models used to estimate the variance from returns data. Section 4 focuses on deriving the distribution of the volatility. In this section we derive the expectation of σ and σ^2 using a Jacobian transformation, the expectation of $\Phi(d_1)$ and $\Phi(d_2)$ using a Taylor series expansion, as well as the expectation for five different volatility models. Section 5 details our nonparametric approach with average as well as weighted sample variances. Section 6 discloses the findings of our study and Section 7 concludes with a summary of those findings as well as recommendations for future work.

2 Smoothing the Data

In this approach, we modify the standard single and double exponential smoothing formulas. Smoothing transforms the data to make it less volatile. Additionally, it makes it easier to compare over time. With exponential smoothing we are able to apply more weight to more recent values. In this section and beyond, instead of working with the stock price, S_t, we will work with the returns, which are defined as $x_t = \log(\frac{S_t}{S_{t-1}})$.

2.1 Single Exponential Smoothing

Consider the standard single exponential formula,

$$A_t = \alpha x_{t-1} + (1-\alpha)A_{t-1}, \text{ with } A_1 = x_1$$

where x_t = observed value at time t, A_t = smoothed value at time t, and α is a coefficient between 0 and 1.

This formula is recursive, but through repeated iterations we are able to generalize the formula. Additionally, we set $A_t = x_t$ rather than $A_1 = x_1$. In this way we are setting the most recent smoothed value equal to the most recent observed value.

$$\begin{aligned}
A_{t-1} &= \frac{A_t - \alpha x_{t-1}}{1-\alpha} \\
A_{t-2} &= \frac{A_{t-1} - \alpha x_{t-2}}{1-\alpha} \\
&= \frac{\frac{A_t - \alpha x_{t-1}}{1-\alpha} - \alpha x_{t-2}}{1-\alpha} \\
&= \frac{A_t}{(1-\alpha)^2} - \frac{\alpha x_{t-1}}{(1-\alpha)^2} - \frac{\alpha x_{t-2}}{1-\alpha} \\
A_{t-3} &= \frac{A_{t-2} - \alpha x_{t-3}}{1-\alpha} \\
&= \frac{A_t}{(1-\alpha)^3} - \frac{\alpha x_{t-1}}{(1-\alpha)^3} - \frac{\alpha x_{t-2}}{(1-\alpha)^2} - \frac{\alpha x_{t-3}}{1-\alpha} \\
&\vdots \\
A_{t-i} &= \frac{A_t - \alpha x_{t-1}}{(1-\alpha)^i} - \frac{\alpha x_{t-2}}{(1-\alpha)^{i-1}} - \frac{\alpha x_{t-3}}{(1-\alpha)^{i-2}} - \cdots - \frac{\alpha x_{t-i}}{(1-\alpha)}
\end{aligned}$$

2.2 Double Exponential Smoothing

When the time series data displays a trend, double exponential smoothing is more appropriate. Consider the standard double exponential formulas ,

$$A_t = \alpha x_t + (1-\alpha)(A_{t-1} + B_{t-1}), \text{ with } A_1 = x_1$$

$$B_t = \beta(A_t - A_{t-1}) + (1-\beta)B_{t-1}, \text{ with } B_1 = x_2 - x_1$$

where x_t = observed value at time t, A_t = smoothed level at time t, B_t = smoothed trend at time t, and α and β are coefficients between 0 and 1.

The formula is similar to single exponential smoothing but with an added term, B_t to account for trend. However, despite this added term we can use a similar process of repeated iterations to arrive at a generalized, rather than recursive, formula.

$$
\begin{aligned}
A_{t-1} &= \frac{A_t - \alpha x_t}{(1-\alpha)} - B_{t-1} \\
A_{t-2} &= \frac{A_{t-1} - \alpha x_{t-1}}{1-\alpha} - B_{t-2} \\
&= \frac{[\frac{A_t - \alpha x_t}{1-\alpha} - B_{t-1}] - \alpha x_{t-1}}{1-\alpha} - B_{t-2} \\
&= \frac{A_t - \alpha x_t}{(1-\alpha)^2} - \frac{B_{t-1} + \alpha x_{t-1}}{1-\alpha} - B_{t-2} \\
&\vdots \\
A_{t-i} &= \frac{A_t - \alpha x_t}{(1-\alpha)^i} - \frac{B_{t-1} + \alpha x_{t-1}}{(1-\alpha)^{i-1}} - \frac{B_{t-2} + \alpha x_{t-2}}{(1-\alpha)^{i-2}} - \cdots - \frac{B_{t-i+1} + \alpha x_{t-i+1}}{(1-\alpha)} - B_{t-i}
\end{aligned}
$$

$$
\begin{aligned}
B_{t-1} &= \frac{B_t - \beta(A_t - A_{t-1})}{1-\beta} \\
B_{t-2} &= \frac{B_{t-1} - \beta(A_{t-1} - A_{t-2})}{1-\beta} \\
&= \frac{[\frac{B_t - \beta(A_t - A_{t-1})}{1-\beta}] - \beta(A_{t-1} - A_{t-2})}{1-\beta} \\
&= \frac{B_t - \beta(A_t - A_{t-1})}{(1-\beta)^2} - \frac{\beta(A_{t-1} - A_{t-2})}{1-\beta} \\
&\vdots \\
B_{t-i} &= \frac{B_t - \beta(A_t - A_{t-1})}{(1-\beta)^i} - \frac{\beta(A_{t-1} - A_{t-2})}{(1-\beta)^{i-1}} - \frac{\beta(A_{t-2} - A_{t-3})}{(1-\beta)^{i-2}} - \cdots - \frac{\beta(A_{t-i-1} - A_{t-i})}{1-\beta}
\end{aligned}
$$

Additionally, we can set $A_t = x_t$ and $B_t = x_t - x_{t-1}$ so that the most recent observed value will be consistent with the most recent smoothed value. Although the new single and double exponential smoothing formulas have been proposed, the estimation of the smoothing coefficients, α and β, are still under development. For purposes of illustration, in the numerical comparison, we use the conventional single and double exponential smoothing formulas with $A_1 = x_1$ and $B_1 = x_2 - x_1$.

3 Modeling the Returns

Considering the return, x_t, we can apply some time series models to derive the volatility. The Autoregressive Moving Average (ARMA) model is a typical time series model. In this section, we apply three different ARMA models to obtain an estimate of the volatility.

3.1 ARMA(1,1)

We begin with the ARMA(1,1) model. The ARMA(1,1) model accounts for the most recent return and a random component.

$$x_t = \phi_1 x_{t-1} + \theta_1 e_{t-1} + e_t$$

where $e_t \sim N(0, \sigma_e^2)$.

To find the variance in this model we start with $cov(x_t, x_t)$, put the ARMA(1,1) model in for x_t, and use the following properties to solve:

$$\begin{aligned} cov(x_t, e_t) &= \sigma^2 \\ cov(x_s, e_t) &= 0 \; if \; s < t \\ var(x_t) &= var(x_s) \end{aligned}$$

After several steps, we are able to arrive at a formula for the variance of returns:

$$\begin{aligned} var(x_t) &= cov(x_t, x_t) \\ &= cov(\phi_1 x_{t-1} + e_t + \theta_1 e_{t-1}, \phi_1 x_{t-1} + e_t + \theta_1 e_{t-1}) \\ &= cov(\phi_1 x_{t-1}, \phi_1 x_{t-1}) + cov(\phi_1 x_{t-1}, e_t) + cov(\phi_1 x_{t-1}, \theta_1 e_{t-1}) + cov(e_t, \phi_1 x_{t-1}) + cov(e_t, e_t) \\ &\quad + cov(e_t, \theta_1 e_{t-1}) + cov(\theta_1 e_{t-1}, \phi_1 x_{t-1}) + cov(\theta_1 e_{t-1}, e_t) + cov(\theta_1 e_{t-1}, \theta_1 e_{t-1}) \\ &= \phi_1^2 cov(x_{t-1}, x_{t-1}) + cov(e_t, e_t) + \theta_1^2 cov(e_{t-1}, e_{t-1}) + 2cov(\phi_1 x_{t-1}, \theta_1 e_{t-1}) \\ &= \phi_1^2 var(x_{t-1}) + var(e_t) + \theta_1^2 var(e_{t-1}) + cov(e_{t-1}, e_{t-1}) + cov(e_{t_1}, \phi_1 x_{t-2}) + cov(e_{t-1}, \theta_1 e_{t-2}) \\ &= \phi_1^2 var(x_t) + var(e_t) + \theta_1^2 var(e_t) + 2\theta_1 \phi_1 \sigma^2 \\ &= \phi_1^2 var(x_t) + \sigma^2 + \theta_1^2 \sigma^2 + 2\theta_1 \phi_1 \sigma^2 \end{aligned}$$

So

$$var(x_t) = \frac{\sigma^2(1 + \theta_1^2 + 2\theta_1\phi_1)}{1 - \phi_1^2}$$

3.2 Higher-Order ARMA Models

In similar fashion to the ARMA(1,1), we also examine the ARMA(2,2):

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$$

and the ARMA(3,3):

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \theta_3 e_{t-3} + e_t$$

and arrive at

$$var(x_t) = \frac{\sigma_e^2[(\theta_1^2 + \theta_2^2 + 1) + 2(\phi_1\theta_1 + \phi_2\theta_2 + \phi_1\theta_1\theta_2)]}{1 - \phi_2^2 - \phi_2^2}$$

for the variance of the ARMA(2,2) model and

$$var(x_t) = \frac{\sigma_e^2[(1 + \sum_{i=1}^{3} \theta_i^2) + 2[(\sum_{i=1}^{3} \phi_i\theta_i) + (\phi_1\theta_1)(\phi_1\theta_2 + \phi_2\theta_1 + \phi_1^2\theta_3) + (\phi_1\phi_2\theta_3 + \phi_1\theta_2\theta_3)]]}{1 - \phi_1^2 - \phi_2^2 - \phi_3^2}$$

for the variance of the ARMA(3,3) model. Due to the number of steps required to derive these formulas, the steps are shown in Appendix 1. Given the complexity beyond the ARMA(3,3) model, in practice, time series data will not be fit to a higher order ARMA model than ARMA(3,3). The variance of ARMA(3,3) can be downgraded to any lower order model, by simply setting the corresponding ϕ and θ equal to 0.

4 Deriving the Distribution of the Volatility

In this section we derive the expected value for σ and σ^2, $\Phi(d_1)$ and $\Phi(d_2)$, as well as five volatility models. Each expected value is then plugged back into the Black-Scholes formula to yield numerical results.

4.1 Functions of Volatility

One method for estimating σ^2 in the Black-Scholes formula is to start by deriving the probability density function for σ^2. Then, we can find the expected value of this function and apply the result back to the Black-Scholes formula. Besides using this method with σ^2, we also work with σ.

4.1.1 Expected value of σ^2

Given the setting of the Black-Scholes formula, the returns, x_t, follow a normal distribution. Thus, we know $\frac{(t-1)s^2}{\sigma^2}$ follows a chi-squared distribution with $t-1$ degrees of freedom [5], where s^2 is the sample variance of the returns and t is the total number of returns.

Let's assume $y = \frac{(t-1)s^2}{\sigma^2}$. Then the probability density function of y, $f_y(y)$, can be written as

$$f_y(y) = \frac{y^{(\frac{v}{2}-1)}e^{(\frac{-y}{2})}}{2^{(\frac{v}{2})}\Gamma(\frac{v}{2})}$$

where $v = t-1$. Employing a Jacobian transformation [7], we can derive the probability density function for σ^2:

$$f(\sigma^2) = \frac{[\frac{(t-1)s^2}{\sigma^2}]^{(\frac{t-1}{2}-1)}e^{(\frac{-(t-1)s^2}{2\sigma^2})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})} \cdot \frac{(t-1)s^2}{(\sigma^2)^2}$$

. Proof. Let $u = \sigma^2$. By Jacobian transformation we have,

$$f_u(u) = f_y[h^{-1}(u)]|\frac{dh^{-1}}{du}|$$

where $h^{-1}(u) = \frac{(t-1)s^2}{u}$ and $\frac{dh^{-1}}{du} = \frac{-(t-1)s^2}{u^2}$,

After plugging all factors back into the transformation equation, we have the probability density function of $f(\sigma^2)$ as shown above.

The expectation of σ^2 can be derived by integration techniques [7]:

$$E[\sigma^2] = \frac{(t-1)s^2}{(t-3)}$$

. Work is shown in Appendix 2.

4.1.2 Expected value of σ

Using the same process as with $E[\sigma^2]$ but allowing $u = \sigma$ rather than $u = \sigma^2$, we are able to arrive at an expected value for σ:

$$E[\sigma] = \frac{\sqrt{t-1}s(\frac{t-4}{2})!}{\sqrt{2}(\frac{t-3}{2})!}$$

Steps for this process are also shown in Appendix 2.

4.2 Taylor Series Approximations

Rather than finding the expected value of σ, we can derive the expected value of $\Phi(d_1)$ and $\Phi(d_2)$ which are both functions of σ. We start by deriving a general form for the expected value of $\Phi(d_1)$ and $\Phi(d_2)$ by using a Taylor series expansion. We can obtain numerical results by looking at some specific functions of s^2.

4.2.1 General Formulas

The generic Taylor series is given by:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + R_n$$

We modify the formula to solve for $\Phi(d_1)$ and allow a to equal any function of s^2, the sample variance:

$$\Phi(d_1) = \Phi(h(t)(s^2)) + \frac{\Phi'(h(t)(s^2))}{1!}(\sigma^2 - (h(t)(s^2))) + \frac{\Phi''(h(t)(s^2))}{2!}(\sigma^2 - (h(t)(s^2))^2) + R_n$$

Thus,

$$\begin{aligned} E[\Phi(d_1)] &= E[\Phi(h(t)(s^2)) + \frac{\Phi'(h(t)(s^2))}{1!}(\sigma^2 - (h(t)(s^2))) + \frac{\Phi''(h(t)(s^2))}{2!}(\sigma^2 - (h(t)(s^2))^2)] \\ &= \Phi(h(t)(s^2)) + \frac{\Phi'(h(t)(s^2))}{1!}E[(\sigma^2 - (h(t)(s^2)))] + \frac{\Phi''(h(t)(s^2))}{2!}E[(\sigma^2 - (h(t)(s^2))^2)] \end{aligned}$$

We can simplify this formula by solving for $E[(\sigma^2 - (h(t)(s^2)))^2]$,

$$\begin{aligned} E[(\sigma^2 - (h(t)(s^2)))^2] &= E[\sigma^4 - 2\sigma^2(h(t)(s^2)) + (h(t)(s^2))^2] \\ &= E[\sigma^4] - 2(h(t)(s^2))E[\sigma^2] + (h(t)(s^2))^2 \end{aligned}$$

Thus, we can arrive at a simplified general Taylor series expansion for functions of s^2:

$$E[\Phi(d_1)] = \Phi(h(t)(s^2)) + \Phi'(h(t)(s^2))[E[\sigma^2] - (h(t)(s^2))] + \frac{1}{2}\Phi''(h(t)(s^2))[E[\sigma^4] - 2(h(t)(s^2))E[\sigma^2] + (h(t)(s^2))^2]$$

where

$$\begin{aligned} \Phi(d_1) &= \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_1^2/2} \, \mathrm{d}d_1 \quad [8] \\ \Phi'(d_1) &= \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \\ \Phi''(d_1) &= \frac{-d_1}{\sqrt{2\pi}} e^{-d_1^2/2} * d_1' \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_1' &= \frac{\sigma^2 T - 2\ln(\frac{S}{K}) - 2rT}{4\sigma^2\sqrt{\sigma^2 T}} \\ E[\sigma^2] &= \frac{t-1}{t-3}s^2 \\ E[\sigma^4] &= \frac{(t-1)^2 s^4}{(t-3)(t-5)} \end{aligned}$$

For the derivation of $E[\sigma^4]$, see Appendix 3. We can use a similar method to find

$$E[\Phi(d_2)] = \Phi(h(t)(s^2)) + \Phi'(h(t)(s^2))[E[\sigma^2] - (h(t)(s^2))] + \frac{1}{2}\Phi''(h(t)(s^2))[E[\sigma^4] - 2(h(t)(s^2))E[\sigma^2] + ((h(t)(s^2))^2]$$

Work is also shown in Appendix 3.

4.2.2 Two Possible Selections

To yield numerical results, we allow $h(t)(s^2)$ to equal two different functions of s^2. First, we can set $h(t)(s^2) = s^2$, and this way:

$$\begin{aligned} E[\Phi(d_1)] &= E[\Phi(s^2) + \frac{\Phi'(s^2)}{1!}(\sigma^2 - s^2) + \frac{\Phi''(s^2)}{2!}(\sigma^2 - s^2)^2 + R_n] \\ &= \Phi(s^2) + \Phi'(s^2)E[\sigma^2 - s^2] + \frac{\Phi''(s^2)}{2!}E[(\sigma^2 - s^2)^2] \end{aligned}$$

To simplify further we can solve for $E[\sigma^2 - s^2]$ and $E[(\sigma^2 - s^2)^2]$:

$$E[\sigma^2 - s^2] = E[\sigma^2] - s^2 = \frac{t-1}{t-3}s^2 - s^2 = \frac{2s^2}{t-3}$$

$$\begin{aligned}
E[(\sigma^2 - s^2)^2] &= E[\sigma^4] - 2s^2E[\sigma^2] + s^4 \\
&= E[\sigma^4] - (2s^2)(\frac{t-1}{t-3}s^2) + s^4 \\
&= E[\sigma^4] - \frac{t-1}{t-3}2s^4 + s^4 \\
&= E[\sigma^4] - \frac{t+1}{t-3}s^4 \\
&= \frac{(t-1)^2}{(t-3)(t-5)}s^4 - \frac{t+1}{t-3}s^4 \\
&= \frac{2s^4(t+3)}{(t-3)(t-5)}
\end{aligned}$$

Thus, we arrive at a simplified form for $E[\Phi(d_1)]$:

$$E[\Phi(d_1)] = \Phi(s^2) + \Phi'(s^2)(\frac{2s^2}{t-3}) + \Phi''(s^2)\frac{s^4(t+3)}{(t-3)(t-5)}$$

A similar process can be used to solve for $E[\Phi(d_2)]$:

$$E[\Phi(d_2)] = \Phi(s^2) + \Phi'(s^2)(\frac{2s^2}{t-3}) + \Phi''(s^2)\frac{s^4(t+3)}{(t-3)(t-5)}$$

Work is shown in Appendix 4. Another selection of $h(t)(s^2)$ is $\frac{(t-1)}{(t-3)}s^2$. Since $E[\sigma^2] = \frac{(t-1)}{(t-3)}s^2$, as proved previously, the second component of the right side of the Taylor series is zero and the computation is simplified. Thus, $E[\Phi(d_1)]$ is given by:

$$\begin{aligned}
E[\Phi(d_1)] &= E[\Phi(\frac{t-1}{t-3}s^2) + \frac{\Phi'(\frac{t-1}{t-3}s^2)}{1!}[\sigma^2 - (\frac{t-1}{t-3}s^2)] + \frac{\Phi''(\frac{t-1}{t-3}s^2)}{2!}[\sigma^2 - (\frac{t-1}{t-3}s^2)]^2] \\
&= \Phi(\frac{t-1}{t-3}s^2) + \frac{\Phi''(\frac{t-1}{t-3}s^2)}{2!}E[(\sigma^2 - (\frac{t-1}{t-3}s^2))^2]
\end{aligned}$$

To simplify further we only need to solve for $E[(\sigma^2 - (\frac{t-1}{t-3}s^2))^2]$:

$$\begin{aligned}
E[(\sigma^2 - (\frac{t-1}{t-3}s^2))^2] &= E[(\sigma^2 - (\frac{t-1}{t-3}s^2))(\sigma^2 - (\frac{t-1}{t-3}s^2))] \\
&= E[\sigma^4 - 2(\frac{t-1}{t-3}s^2)E[\sigma^2] + \frac{(t-1)^2}{(t-3)^2}s^4] \\
&= E[\sigma^4 - 2(\frac{t-1}{t-3}s^2)(\frac{t-1}{t-3}s^2) + \frac{(t-1)^2}{(t-3)^2}s^4] \\
&= E[\sigma^4 - \frac{(t-1)^2 s^4}{(t-3)^2}] \\
&= \frac{(t-1)^2 s^4}{(t-3)(t-5)} - \frac{(t-1)^2 s^4}{(t-3)^2} \\
&= \frac{2s^4(t-1)^2}{(t-3)^2(t-5)}
\end{aligned}$$

Thus, we arrive at a simplified form for $E[\Phi(d_1)]$:

$$E[\Phi(d_1)] = \Phi(\frac{t-1}{t-3}s^2) + \phi''(\frac{t-1}{t-3}s^2)\frac{s^4(t-1)^2}{(t-3)^2(t-5)}$$

A similar process can be used to solve for $E[\Phi(d_2)]$:

$$E[\Phi(d_2)] = \Phi(\frac{t-1}{t-3}s^2) + \phi''(\frac{t-1}{t-3}s^2)\frac{s^4(t-1)^2}{(t-3)^2(t-5)}$$

Work is shown in Appendix 4.

4.3 Modeling Volatility [3]

Due to the highly volatile data, the volatility of time series data usually tends to change as time progresses. Hence, we would like to set the volatility as a volatility model. Meanwhile, the volatility model can be used to account for volatility clustering too.

4.3.1 ARCH

A widely used volatility model is the Autoregressive Conditional Heteroscedasticity (ARCH) model. The general ARCH model is of the form:

$$ARCH(p) : \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i z_{t-i}^2$$

where $z_t \sim N(0, \sigma_t^2)$ and α is a coefficient between 0 and 1. The volatility at time t depends on some random effects from the past.

We are able to solve for $E[\sigma_t^2]$ and arrive at a solution dependent on only the coefficients [6].

$$E[\sigma_t^2] = E[\alpha_0 + \sum_{i=1}^{p} \alpha_i z_{t-i}^2] = \alpha_0 + \sum_{i=1}^{p} \alpha_i \sum_{i=1}^{p} E[z_{t-i}^2] = \alpha_0 + \sum_{i=1}^{p} \alpha_i \sum_{i=1}^{p} E[\sigma_{t-i}^2]$$

$$= \alpha_0 + \sum_{i=1}^{p} \alpha_i \sum_{i=1}^{p} E[\sigma_t^2] = \alpha_0 + (\sum_{i=1}^{p} \alpha_i)E[\sigma_t^2] = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i}$$

The values of the coefficients, α_i, can be computed by the statistical analysis software R.

For numerical testing, we derived $E[\sigma_t^2]$ for the ARCH(1) model, $\sigma_t^2 = \alpha_0 + \alpha_1 z_{t-1}^2$: $E[\sigma_t^2] = \frac{\alpha_0}{1-\alpha_1}$

4.3.2 GARCH

The ARCH model can be expanded to include the previous volatility. This model is named the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model. The general form is given by:

$$GARCH(p,q) : \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

where $z_t \sim N(0, \sigma_t^2)$ and α_i and β_j are coefficients between 0 and 1.

In similar fashion to the ARCH model, we are able to derive the expected value of the GARCH model:

$$\begin{aligned}
E[\sigma_t^2] &= E[\alpha_0 + \sum_{i=1}^{p} \alpha_i z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2] \\
&= \alpha_0 + \sum_{i=1}^{p} \alpha_i \sum_{i=1}^{p} E[z_{t-i}^2] + \sum_{j=1}^{q} \beta_j \sum_{j=1}^{q} E[\sigma_{t-j}^2] \\
&= \alpha_0 + \sum_{i=1}^{p} \alpha_i \sum_{i=1}^{p} E[\sigma_t^2] + \sum_{j=1}^{q} \beta_j \sum_{j=1}^{q} E[\sigma_t^2] \\
&= \alpha_0 + (\sum_{i=1}^{p} \alpha_i) E[\sigma_t^2] + (\sum_{j=1}^{q} \beta_j) E[\sigma_t^2] \\
&= \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}
\end{aligned}$$

For numerical testing, we derived $E[\sigma_t^2]$ for the GARCH(1,1) model, $\sigma_t^2 = \alpha_0 + \alpha_1 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$:

$$E[\sigma_t^2] = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

4.3.3 MACH

The ARCH and GARCH models have been well discussed in the literature. Correspondingly, in this paper, we propose three new volatility models. The first proposed model is the Moving Average Conditional Heteroscedasticity (MACH) model defined as:

$$MACH(r) : \sigma_t^2 = \alpha_0 + e_t^2 + \sum_{i=1}^{r} \theta_i e_{t-i}^2$$

where $e_t \sim N(0, \lambda^2)$ and is i.i.d and α_0 and θ_i are coefficients between 0 and 1.

Similar to the ARCH and GARCH models, we can derive the expression of $E[\sigma_t^2]$.

$$
\begin{aligned}
E[\sigma_t^2] &= E[\alpha_0 + e_t^2 + \sum_{i=1}^{r} \theta_i e_{t-i}^2] = \alpha_0 + E[e_t^2] + \sum_{i=1}^{r} \theta_i \sum_{i=1}^{r} E[e_{t-i}^2] \\
&= \alpha_0 + E[e_t^2] + \sum_{i=1}^{r} \theta_i \sum_{i=1}^{r} E[e_t^2] = \alpha_0 + E[e_t^2] + (\sum_{i=1}^{r} \theta_i) E[e_t^2] \\
&= \alpha_0 + (1 + (\sum_{i=1}^{r} \theta_i)) E[e_t^2] = \alpha_0 + \lambda^2 (1 + \sum_{i=1}^{r} \theta_i)
\end{aligned}
$$

4.3.4 ARMACH

By adding random components, the MACH model can be rewritten as the Autoregressive Moving Average Conditional Heteroscedasticity (ARMACH) model. The general expression is shown as:

$$
ARMACH(p, r) : \sigma_t^2 = \alpha_0 + e_t^2 + \sum_{i=1}^{r} \theta_i e_{t-i}^2 + \sum_{j=1}^{p} \alpha_j z_{t-j}^2
$$

where $z_t \sim N(0, \sigma_t^2)$, $e_t \sim N(0, \lambda^2)$ and is i.i.d, and α_j and θ_i are coefficients between 0 and 1.

The expectation of the model, $E[\sigma_t^2]$, is still of interest to us. The derivation process is similar.

$$
\begin{aligned}
E[\sigma_t^2] &= E[\alpha_0 + e_t^2 + \sum_{i=1}^{r} \theta_i e_{t-i}^2 + \sum_{j=1}^{p} \alpha_j z_{t-j}^2] \\
&= \alpha_0 + E[e_t^2] + \sum_{i=1}^{r} \theta_i \sum_{i=1}^{r} E[e_{t-i}^2] + \sum_{j=1}^{p} \alpha_j \sum_{j=1}^{p} E[z_{t-j}^2] \\
&= \alpha_0 + E[e_t^2] + \sum_{i=1}^{r} \theta_i \sum_{i=1}^{r} E[e_t^2] + \sum_{j=1}^{p} \alpha_j \sum_{j=1}^{p} E[\sigma_t^2] \\
&= \alpha_0 + E[e_t^2] + (\sum_{i=1}^{r} \theta_i) E[e_t^2] + (\sum_{j=1}^{p} \alpha_j) E[\sigma_t^2] \\
&= \frac{\alpha_0 + \lambda^2 (1 + \sum_{i=1}^{r} \theta_i)}{1 - \sum_{j=1}^{p} \alpha_j}
\end{aligned}
$$

4.3.5 GARMACH

Combining the GARCH model and the ARMACH model, our third proposed model is the Generalized Autoregressive Moving Average Conditional Heteroscedasticity (GARMACH) model. It accounts for previous random components with fixed variance, previous volatilities, and previous random components with variance of volatility for the volatility. This model is defined as:

$$
GARMACH(p, q, r) : \sigma_t^2 = \alpha_0 + e_t^2 + \sum_{i=1}^{r} \theta_i e_{t-i}^2 + \sum_{j=1}^{p} \alpha_j z_{t-j}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2
$$

where $z_t \sim N(0, \sigma_t^2)$, $e_t \sim N(0, \lambda^2)$ and is i.i.d, and α_j, β_k, and θ_i are coefficients between 0 and 1.

We are able to solve for $E[\sigma_t^2]$ in a similar way as the MACH(r) and ARMACH(p,r) models.

$$
\begin{aligned}
E[\sigma_t^2] &= \alpha_0 + e_t^2 + \sum_{i=1}^{r} \theta_i e_{t-i}^2 + \sum_{j=1}^{p} \alpha_j z_{t-j}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2 \\
&= \alpha_0 + E[e_t^2] + \sum_{i=1}^{r} \theta_i \sum_{i=1}^{r} E[e_{t-i}^2] + \sum_{j=1}^{p} \alpha_j \sum_{j=1}^{p} E[z_{t-j}^2] + \sum_{k=1}^{q} \beta_k \sum_{k=1}^{q} E[\sigma_{t-k}^2] \\
&= \alpha_0 + E[e_t^2] + \sum_{i=1}^{r} \theta_i \sum_{i=1}^{r} E[e_t^2] + \sum_{j=1}^{p} \alpha_j \sum_{j=1}^{p} E[\sigma_t^2] + \sum_{k=1}^{q} \beta_k \sum_{k=1}^{q} E[\sigma_t^2] \\
&= \alpha_0 + E[e_t^2] + (\sum_{i=1}^{r} \theta_i) E[e_t^2] + (\sum_{j=1}^{p} \alpha_j) E[\sigma_t^2] + (\sum_{k=1}^{q} \beta_k) E[\sigma_t^2] \\
&= \alpha_0 + (1 + (\sum_{i=1}^{r} \theta_i)) E[e_t^2] + ((\sum_{j=1}^{p} \alpha_j) + (\sum_{k=1}^{q} \beta_k)) E[\sigma_t^2] \\
&= \frac{\alpha_0 + \lambda^2 (1 + \sum_{i=1}^{r} \theta_i)}{1 - \sum_{j=1}^{p} \alpha_j - \sum_{k=1}^{q} \beta_k}
\end{aligned}
$$

5 Nonparametric Approaches

Our last approach is nonparametric. The approaches discussed in the previous sections all assume the use of parametric statistics. In this section, we choose a nonparametric approach where we do not make any assumption about the data's probability distribution.

5.1 Average Sample Variance

Rather than making an assumption about the data's probability distribution, we calculate both average and weighted sample variances. We start by calculating the average sample variance given by the equation:

$$\widehat{\sigma_t^2} = \frac{s_1^2 + s_2^2 + \cdots + s_{t-1}^2}{t-1}$$

where s_i^2 = sample variance of x_1 through x_{i+1}, t = number of returns, and $t-1$ = number of sample variances.

5.2 Weighted Sum of Sample Variances

Although the previous method is simple to work with, it may not be best since sample variances that include only the earliest data are weighted the same as sample variances that include the most recent data. In order to weight more heavily sample variances that include more recent data, we propose to estimate the volatility as a function of weighted sample variances.

$$\widehat{\sigma_t^2} = \alpha_1 s_1^2 + \alpha_2 s_2^2 + \cdots + \alpha_{t-1} s_{t-1}^2$$

where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{t-1}$, $\sum_{i=1}^{t-1} \alpha_i = 1$, and s_t is defined as in the previous subsection.

There are plenty of selections for the weight of α_i. For numerical illustration, we propose two series of weights, which satisfy both conditions with a large enough number of data points:

$$\alpha_1 = \frac{1}{2^{t-1}},\ \alpha_2 = \frac{1}{2^{t-2}},\ \ldots,\ \alpha_{t-1} = \frac{1}{2^1}$$

$$\alpha_1 = \frac{9}{10^{t-1}},\ \alpha_2 = \frac{9}{10^{t-2}},\ \ldots,\ \alpha_{t-1} = \frac{9}{10^1}$$

6 Numerical Analysis

In this section, we provide numerical comparisons between our different approaches to determine which approach estimates the price of the options the best. We start with a dataset of S&P 100 daily index series from January 2, 1991 through December 29, 2000. However, since our second dataset contains 36 different S&P 100 call options dating back to June 11, 1997, we only use data from the S&P 100 daily index series from January 2, 1991 through June 11, 1997. Thus, we are left with 1630 observations. The strike price, stock price, and expiration date vary between different options; however, we use the same three month treasury yield rate of 4.98% for each option [1].

The observed call prices are compared to the estimated prices obtained from our different approaches. Most of our approaches do not perform as well as the sample variance method that used an annualized variance of .01309; however, the GARCH(1,1), single exponential smoothing, and double exponential smoothing outperform the sample variance method. The strength of each of our approaches is also dependent on the expiration date and value of the call option. Our approaches improve as the duration of the option contract decreases since the RMSE is lower for 24 day expiration dates than it is for either 87 or 115 day expiration dates. This is an expected finding since a shorter time period leads to less uncertainty. Also, as the value of the call increases, our approaches perform better.

Table 1 displays the sum of squared errors of prediction (SSE) for each approach as well as the root-mean-square error (RMSE) for each expiration date where:

$$SSE = \sum_{i=1}^{36} [(actual\ call\ price)_i - (predicted\ call\ price)_i]^2$$

$$RMSE(T) = \sqrt{\frac{SSE}{\#\ of\ data\ points\ with\ expiration\ date\ T}}$$

Table 1: SSE and RMSE

Approaches	*Methods*	SSE	$RMSE(24)$	$RMSE(87)$	$RMSE(115)$
Baseline	*Sample Variance*	983.79	2.12	5.59	7.33
Smoothing	*Single*	331.83	1.19	3.21	4.33
	Double	326.21	1.02	3.14	4.42
Time Series Models	$ARMA(3)$	993.70	2.13	5.62	7.36
	$ARMA(2)$	991.85	2.13	5.61	7.36
	$ARMA(1)$	992.87	2.13	5.62	7.36
Deriving the Distribution	$E(\sigma^2)$	985.40	2.12	5.60	7.33
	$E(\sigma)$	985.17	2.12	5.59	7.33
	$\Phi(s^2)$	983.79	2.12	5.59	7.33
	$\Phi[E(\sigma^2)]$	985.40	2.12	5.60	7.33
	$ARCH(1)$	991.62	2.13	5.61	7.36
	$GARCH(1,1)$	867.67	1.96	5.24	6.92
	$ARCH(2)$	995.55	2.14	5.63	7.37
	$GARCH(2,2)$	982.64	2.12	5.59	7.33
Nonparametric Approaches	*Average Sample Variance*	1350.36	2.59	6.59	8.48
	Weighted ($\alpha_{t-1} = .5$)	984.25	2.12	5.59	7.33
	Weighted ($\alpha_{t-1} = .9$)	984.02	2.12	5.59	7.33

Table 2 displays percent bias for each option for the most relevant methods where:

$$\%bias = \frac{|observed - predicted|}{observed}$$

Table 2: Percent Bias for Most Relevant Methods Only

T	S	K	*Observed*	*Sample Var*	% *bias*	$GARCH(1,1)$	% *bias*	*Single*	% *bias*	*Double*	% *bias*
24	425.73	395	30.75	32.66	6.21	32.64	6.15	32.60	6.01	32.60	6.01
24	425.73	400	25.88	27.78	7.35	27.74	7.20	27.62	6.73	27.62	6.74
24	425.73	405	21.00	23.03	9.67	22.95	9.30	22.65	7.84	22.65	7.87
24	425.67	410	16.50	18.44	11.76	18.31	10.98	17.61	6.73	17.66	7.02
24	425.68	415	11.88	14.26	20.08	14.07	18.51	12.64	6.47	12.87	8.41
24	425.65	420	7.69	10.52	36.86	10.28	33.73	7.64	0.65	8.45	9.91
24	425.65	425	4.44	7.40	66.69	7.13	60.53	2.92	34.19	4.85	9.13
24	425.68	430	2.10	4.94	135.71	4.66	122.53	0.34	83.90	2.35	12.06
24	425.65	435	0.78	3.09	295.79	2.84	263.87	0.00	99.39	0.92	17.82
24	425.16	440	0.25	1.71	585.43	.52	507.17	0.00	100.00	0.25	1.22
24	424.78	445	0.10	0.89	836.78	0.75	693.17	0.00	100.00	0.05	43.40
24	425.19	450	0.10	0.48	406.57	0.39	309.94	0.00	100.00	0.01	87.82
87	425.73	380	46.75	52.47	12.24	52.40	12.09	52.21	11.67	52.21	11.68
87	425.73	385	42.00	47.73	13.64	47.63	13.40	47.29	12.60	47.30	12.61
87	425.73	390	37.50	43.07	14.86	42.93	14.48	42.38	13.01	42.39	13.04
87	425.73	395	33.00	38.53	16.76	38.34	16.19	37.46	13.52	37.50	13.63
87	425.73	400	28.50	34.14	19.81	33.90	18.95	32.55	14.20	32.64	14.54
87	425.73	405	24.13	29.95	24.10	29.64	22.84	27.63	14.52	27.87	15.48
87	425.26	410	20.38	25.60	25.63	25.23	23.82	22.25	9.20	22.79	11.84
87	425.86	415	16.13	22.36	38.66	21.94	36.06	17.93	11.22	18.92	17.35
87	425.68	420	12.82	18.81	46.76	18.34	43.11	12.85	0.30	14.69	14.65
87	425.42	425	9.32	15.55	66.90	15.04	61.48	7.84	15.79	10.89	16.90
87	425.62	430	6.51	12.91	98.44	12.38	90.36	3.93	39.57	7.95	22.29
87	425.82	435	4.51	10.58	134.86	10.05	123.08	1.39	69.26	5.57	23.55
87	425.68	440	2.75	8.42	206.01	7.90	187.13	0.27	90.15	3.60	30.97
87	425.75	445	1.60	6.66	317.71	6.17	286.73	0.03	98.00	2.25	41.23
87	425.78	450	0.85	5.19	513.80	4.73	459.73	0.00	99.76	1.33	57.28
87	425.39	455	0.44	3.88	781.02	3.47	688.68	0.00	99.99	0.70	59.26
115	425.73	380	47.25	54.75	15.87	54.64	15.63	54.27	14.85	54.27	14.86
115	425.73	390	38.13	45.57	19.50	45.37	18.98	44.49	16.69	44.52	16.77
115	425.73	400	29.38	36.88	25.52	36.57	24.47	34.72	18.17	34.88	18.73
115	425.73	410	21.19	28.91	36.42	28.48	34.39	24.94	17.71	25.61	20.84
115	425.41	420	13.88	21.65	56.00	21.11	52.14	14.86	7.11	16.94	22.12
115	425.63	430	8.13	15.86	95.14	15.26	87.78	5.96	26.69	10.25	26.21
115	425.28	440	3.88	10.91	181.50	10.30	165.86	0.78	79.91	5.22	34.72
115	425.13	450	1.50	7.23	382.09	6.67	344.80	0.02	98.66	2.30	53.16

7 Conclusion

In this paper, we explored different approaches to obtain the volatility embedded in the Black-Scholes formula: smoothing, deriving the distribution of the volatility, building time series models, and using nonparametric techniques. Some numerical results were computed to compare the closeness of the estimated call prices with the observed call prices. We discovered that the GARCH(1,1), single exponential smoothing, and double exponential smoothing methods are better methods for estimating the volatility than immediately plugging the sample variance back into the Black-Scholes formula. This paper adds to the literature by proposing new formulas for single and double exponential smoothing as well as three new time-series models: the MACH, ARMACH, and GARMACH. However, due to complexity and page limitation at the current stage, the estimation procedures for the parameters in these models are still under development and were not discussed in this paper. We would like to present the updates of our proposed models in another paper.

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Appendix 1

Computing variance for ARMA(2,2)

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$$

$$\begin{aligned}
var(x_t) &= cov(\phi_1 x_{t-1}, \phi_1 x_{t-1}) + cov(\phi_1 x_{t-1}, \phi_2 x_{t-2}) + cov(\phi_1 x_{t-1}, \theta_1 e_{t-1}) \\
&\quad + cov(\phi_1 x_{t-1}, \theta_2 e_{t-2}) + cov(\phi_1 x_{t-1}, e_t) + cov(\phi_2 x_{t-2}, \phi_1 x_{t-1}) \\
&\quad + cov(\phi_2 x_{t-2}, \phi_2 x_{t-2}) + cov(\phi_2 x_{t-2}, \theta_1 e_{t-1}) + cov(\phi_2 x_{t-2}, \theta_2 e_{t-2}) + cov(\phi_2 x_{t-2}, e_t) \\
&\quad + cov(\theta_1 e_{t-1}, \phi_1 x_{t-1}) + cov(\theta_1 e_{t-1}, \phi_2 x_{t-2}) + \\
&\quad + cov(\theta_1 e_{t-1}, \theta_1 e_{t-1}) + cov(\theta_1 e_{t-1}, \theta_2 e_{t-2}) + cov(\theta_1 e_{t-1}, e_t) \\
&\quad + cov(\theta_2 e_{t-2}, \phi_1 x_{t-1}) + cov(\theta_2 e_{t-2}, \phi_2 x_{t-2}) + cov(\theta_2 e_{t-2}, \theta_1 e_{t-1}) \\
&\quad + cov(\theta_2 e_{t-2}, \theta_2 e_{t-2}) + cov(\theta_2 e_{t-2}, e_t) + cov(e_t, \phi_1 x_{t-1}) + cov(e_t, \phi_2 x_{t-2}) \\
&\quad + cov(e_t, \theta_1 e_{t-1}) + cov(e_t, \theta_2 e_{t-2}) + cov(e_t, e_t) \\
&= cov(\phi_1 x_{t-1}, \phi_1 x_{t-1}) + cov(\phi_1 x_{t-1}, \theta_1 e_{t-1}) + cov(\phi_1 x_{t-1}, \theta_2 e_{t-2}) \\
&\quad + cov(\phi_2 x_{t-2}, \phi_2 x_{t-2}) + cov(\phi_2 x_{t-2}, \theta_2 e_{t-2}) + cov(\theta_1 e_{t-1}, \phi_1 x_{t-1}) \\
&\quad + cov(\theta_1 e_{t-1}, \theta_1 e_{t-1}) + cov(\theta_2 e_{t-2}, \phi_1 x_{t-1}) + cov(\theta_2 e_{t-2}, \phi_2 x_{t-2}) \\
&\quad + cov(\theta_2 e_{t-2}, \theta_2 e_{t-2}) + cov(e_t, e_t) \\
&= \phi_1^2 var(x_t) + \phi_2^2 var(x_t) + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 + \sigma^2 + 2\phi_1 \theta_1 \sigma^2 \\
&\quad + 2\phi_2 \theta_2 \sigma^2 + 2\phi_1^2 \theta_2 \sigma^2 + 2\phi_1 \theta_1 \theta_2 \sigma^2
\end{aligned}$$

$$var(x_t) = \frac{\sigma^2[(\theta_1^2 + \theta_2^2 + 1) + 2(\phi_1 \theta_1 + \phi_2 \theta_2 + \phi_1 \theta_1 \theta_2)]}{1 - \phi_1^2 - \phi_2^2}$$

Computing variance for ARMA(3,3)

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \theta_3 e_{t-3} + e_t$$

$$\begin{aligned}
var(x_t) &= cov(\phi_1 x_{t-1}, \phi_1 x_{t-1}) + cov(\phi_1 x_{t-1}, \phi_2 x_{t-2}) + cov(\phi_1 x_{t-1}, \phi_3 x_{t-3}) + cov(\phi_1 x_{t-1}, \theta_1 e_{t-1}) \\
&\quad + cov(\phi_1 x_{t-1}, \theta_2 e_{t-2}) + cov(\phi_1 x_{t-1}, \theta_3 e_{t-3}) + cov(\phi_1 x_{t-1}, e_t) + cov(\phi_2 x_{t-2}, \phi_1 x_{t-1}) \\
&\quad + cov(\phi_2 x_{t-2}, \phi_2 x_{t-2}) + cov(\phi_2 x_{t-2}, \phi_3 x_{t-3}) + cov(\phi_2 x_{t-2}, \theta_1 e_{t-1}) + cov(\phi_2 x_{t-2}, \theta_2 e_{t-2}) \\
&\quad + cov(\phi_2 x_{t-2}, \theta_3 e_{t-3}) + cov(\phi_2 x_{t-2}, e_t) + cov(\phi_3 x_{t-3}, \phi_1 x_{t-1}) + cov(\phi_3 x_{t-3}, \phi_2 x_{t-2}) \\
&\quad + cov(\phi_3 x_{t-3}, \phi_3 x_{t-3}) + cov(\phi_3 x_{t-3}, \theta_1 e_{t-1}) + cov(\phi_3 x_{t-3}, \theta_2 e_{t-2}) + cov(\phi_3 x_{t-3}, \theta_3 e_{t-3}) \\
&\quad + cov(\phi_3 x_{t-3}, e_t) + cov(\theta_1 e_{t-1}, \phi_1 x_{t-1}) + cov(\theta_1 e_{t-1}, \phi_2 x_{t-2}) + cov(\theta_1 e_{t-1}, \phi_3 x_{t-3}) \\
&\quad + cov(\theta_1 e_{t-1}, \theta_1 e_{t-1}) + cov(\theta_1 e_{t-1}, \theta_2 e_{t-2}) + cov(\theta_1 e_{t-1}, \theta_3 e_{t-3}) + cov(\theta_1 e_{t-1}, e_t) \\
&\quad + cov(\theta_2 e_{t-2}, \phi_1 x_{t-1}) + cov(\theta_2 e_{t-2}, \phi_2 x_{t-2}) + cov(\theta_2 e_{t-2}, \phi_3 x_{t-3}) + cov(\theta_2 e_{t-2}, \theta_1 e_{t-1}) \\
&\quad + cov(\theta_2 e_{t-2}, \theta_2 e_{t-2}) + cov(\theta_2 e_{t-2}, \theta_3 e_{t-3}) + cov(\theta_2 e_{t-2}, e_t) + cov(\theta_3 e_{t-3}, \phi_1 x_{t-1}) \\
&\quad + cov(\theta_3 e_{t-3}, \phi_2 x_{t-2}) + cov(\theta_3 e_{t-3}, \phi_3 x_{t-3}) + cov(\theta_3 e_{t-3}, \theta_1 e_{t-1}) + cov(\theta_3 e_{t-3}, \theta_2 e_{t-2}) \\
&\quad + cov(\theta_3 e_{t-3}, \theta_3 e_{t-3}) + cov(\theta_3 e_{t-3}, e_t) + cov(e_t, \phi_1 x_{t-1}) + cov(e_t, \phi_2 x_{t-2}) + cov(e_t, \phi_3 x_{t-3}) \\
&\quad + cov(e_t, \theta_1 e_{t-1}) + cov(e_t, \theta_2 e_{t-2}) + cov(e_t, \theta_3 e_{t-3}) + cov(e_t, e_t) \\
&= cov(\phi_1 x_{t-1}, \phi_1 x_{t-1}) + cov(\phi_2 x_{t-2}, \phi_2 x_{t-2}) + cov(\phi_3 x_{t-3}, \phi_3 x_{t-3}) + cov(\theta_1 e_{t-1}, \theta_1 e_{t-1}) \\
&\quad + cov(\theta_2 e_{t-2}, \theta_2 e_{t-2}) + cov(\theta_3 e_{t-3}, \theta_3 e_{t-3}) + cov(e_t, e_t) + 2cov(\phi_1 x_{t-1}, \phi_2 x_{t-2}) \\
&\quad + 2cov(\phi_1 x_{t-1}, \phi_3 x_{t-3}) + 2cov(\phi_1 x_{t-1}, \theta_1 e_{t-1}) + 2cov(\phi_1 x_{t-1}, \theta_2 e_{t-2}) + 2cov(\phi_1 x_{t-1}, \theta_3 e_{t-3}) \\
&\quad + 2cov(\phi_2 x_{t-2}, \phi_3 x_{t-3}) + 2cov(\phi_2 x_{t-2}, \theta_2 e_{t-2}) + 2cov(\phi_2 x_{t-2}, \theta_3 e_{t-3}) + 2cov(\phi_3 x_{t-3}, \theta_3 e_{t-3})
\end{aligned}$$

$$
\begin{aligned}
&= \phi_1^2 var(x_{t-1}) + \phi_2^2 var(x_{t-2}) + \phi_3^2 var(x_{t-3}) + \theta_1^2 var(e_{t-1}) + \theta_2^2 var(e_t t-2) + \theta_3^2 var(e_{t-3}) \\
&\quad + var(e_t) + 2\phi_1\theta_1\sigma^2 + 2\phi_2\theta_2\sigma^2 + \phi_3\theta_3\sigma^2 + 2\phi_1\theta_2\sigma^2(\phi_1+\theta_1) \\
&\quad + 2\phi_2\theta_3\sigma^2(\phi_1+\theta_1) + 2\phi_1\theta_3\sigma^2(\phi_1(\phi_{1\theta 1}) + \phi_2 + \theta_2)
\end{aligned}
$$

$$
var(x_t) = \frac{\sigma^2[(\theta_1^2+\theta_2^2+\theta_3^2+1) + 2[(\phi_1\theta_1+\phi_2\theta_2+\phi_3\theta_3) + [(\phi_1\theta_1)(\phi_1\theta_2+\phi_2\theta_1+\phi_1^2\theta_3)] + (\phi_1\phi_2\theta_3+\phi_1\theta_2\theta_3)]]}{1-\phi_1^2-\phi_2^2-\phi_3^2}
$$

Appendix 2

Finding $E[\sigma^2]$

$$
\begin{aligned}
E[u] &= \int_0^\infty f_u(u)\cdot u\ du \\
&= \int_0^\infty \frac{[\frac{(t-1)s^2}{u}]^{(\frac{t-1}{2})} e^{(\frac{-(t-1)s^2}{2u})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})u}\cdot u\ du \\
&= \int_0^\infty \frac{[\frac{(t-1)s^2}{u}]^{(\frac{t-1}{2})} e^{(\frac{-(t-1)s^2}{2u})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})} du
\end{aligned}
$$

$$
u = \frac{(t-1)s^2}{y}
$$

$$
du = \frac{-(t-1)s^2}{y^2} dy
$$

$$
\begin{aligned}
E[u] &= \int_0^\infty \frac{y^{\frac{t-1}{2}} e^{\frac{-y}{2}}}{2^{\frac{t-1}{2}}\Gamma(\frac{t-1}{2})}\cdot\frac{-(t-1)s^2}{y^2}dy \\
&= -(t-1)s^2\int_0^\infty \frac{y^{\frac{v}{2}} e^{\frac{-y}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}dy \\
&= \frac{-(t-1)s^2}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\int_0^\infty y^{\frac{v}{2}} e^{\frac{-y}{2}}\,dy \\
&= \frac{-(t-1)s^2}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\cdot[-2^{\frac{v}{2}-1}\Gamma(\frac{v}{2}-1)] \\
&= \frac{(t-1)s^2}{2(\frac{v}{2}-1)} \\
&= \frac{(t-1)s^2}{(t-3)}
\end{aligned}
$$

Finding $E[\sigma]$

$$
y = \frac{(t-1)s^2}{\sigma^2}
$$

$$
u = \sigma
$$

$$
v = t-1
$$

$$
f_u(u) = f_y[h^{-1}(u)]|\frac{dh^{-1}}{du}|
$$

$$h^{-1}(u) = \frac{(t-1)s^2}{u^2}$$
$$\frac{dh^{-1}}{du} = \frac{-2(t-1)s^2}{u^3}$$
$$f_y[h^{-1}(u)] = \frac{[\frac{(t-1)s^2}{u^2}]^{(\frac{t-1}{2}-1)}e^{(\frac{-(t-1)s^2}{2u^2})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})}$$

$$\begin{aligned}
f_u(u) &= \frac{[\frac{(t-1)s^2}{u^2}]^{(\frac{t-1}{2}-1)}e^{(\frac{-(t-1)s^2}{2u^2})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})} \cdot \frac{-2(t-1)s^2}{u^3} \\
&= \frac{2[\frac{(t-1)s^2}{u^2}]^{(\frac{t-1}{2})}e^{(\frac{-(t-1)s^2}{2u^2})}}{u2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})}
\end{aligned}$$

$$\begin{aligned}
E[u] &= \int_0^\infty f_u(u) \cdot u\ du \\
&= \int_0^\infty \frac{2[\frac{(t-1)s^2}{u^2}]^{(\frac{t-1}{2})}e^{(\frac{-(t-1)s^2}{2u^2})}}{u2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})} \cdot u\ du \\
&= \int_0^\infty \frac{2[\frac{(t-1)s^2}{u^2}]^{(\frac{t-1}{2})}e^{(\frac{-(t-1)s^2}{2u^2})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})}
\end{aligned}$$

$$u = \frac{\sqrt{(t-1)}s}{\sqrt{y}}$$
$$du = \frac{-\sqrt{t-1}s}{2y\sqrt{y}}dy$$

$$\begin{aligned}
E[u] &= \int_0^\infty \frac{2y^{\frac{t-1}{2}}e^{\frac{-y}{2}}}{2^{\frac{t-1}{2}}\Gamma(\frac{t-1}{2})} \cdot \frac{-\sqrt{t-1}s}{2y\sqrt{y}}dy \\
&= -\sqrt{t-1}s \int_0^\infty \frac{y^{\frac{v}{2}-\frac{3}{2}}e^{\frac{-y}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \\
&= \frac{-\sqrt{t-1}s}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \int_0^\infty y^{\frac{v}{2}-\frac{3}{2}}e^{\frac{-y}{2}}dy \\
&= \frac{-\sqrt{t-1}s}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \cdot [-(2^{\frac{v}{2}-\frac{1}{2}}\Gamma(\frac{v}{2}-\frac{1}{2})] \\
&= \frac{\sqrt{t-1}s\Gamma(\frac{v}{2}-\frac{1}{2})}{\sqrt{2}\Gamma(\frac{v}{2})} \\
&= \frac{\sqrt{t-1}s(\frac{t-4}{2})!}{\sqrt{2}(\frac{t-3}{2})!}
\end{aligned}$$

$$E[\sigma] = \frac{\sqrt{t-1}s(\frac{t-4}{2})!}{\sqrt{2}(\frac{t-3}{2})!}$$

Appendix 3

Derivation of $E[\sigma^4]$

$$E[\sigma^4] = E[(\sigma^2)^2] = E[u^2]$$

$$\begin{aligned}
E[u^2] &= \int_0^\infty f_u(u) \cdot u^2 \; du \\
&= \int_0^\infty \frac{[\frac{(t-1)s^2}{u}]^{(\frac{t-1}{2})} e^{(\frac{-(t-1)s^2}{2u})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})u} \cdot u^2 \; du \\
&= \int_0^\infty \frac{u[\frac{(t-1)s^2}{u}]^{(\frac{t-1}{2})} e^{(\frac{-(t-1)s^2}{2u})}}{2^{(\frac{t-1}{2})}\Gamma(\frac{t-1}{2})} du
\end{aligned}$$

$$u = \frac{(t-1)s^2}{y}$$

$$du = \frac{-(t-1)s^2}{y^2} dy$$

$$\begin{aligned}
E[u^2] &= \int_0^\infty \frac{y^{\frac{t-1}{2}} e^{\frac{-y}{2}}}{2^{\frac{t-1}{2}}\Gamma(\frac{t-1}{2})} \cdot \frac{(t-1)s^2}{y} \cdot \frac{-(t-1)s^2}{y^2} dy \\
&= -((t-1)s^2)^2 \int_0^\infty \frac{y^{\frac{v}{2}-3} e^{\frac{-y}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \\
&= \frac{-((t-1)s^2)^2}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \int_0^\infty y^{\frac{v}{2}-3} e^{\frac{-y}{2}} dy \\
&= \frac{-((t-1)s^2)^2}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \cdot [-2^{\frac{v}{2}-2}\Gamma(\frac{v}{2}-2)]
\end{aligned}$$

$$E[\sigma^4] = \frac{(t-1)^2 s^4}{(t-3)(t-5)}$$

General Formula for $E[\Phi(d_2)]$

$$\Phi(d_2) = \Phi(h(t)(s^2)) + \frac{\Phi'(h(t)(s^2))}{1!}(\sigma^2 - (h(t)(s^2))) + \frac{\Phi''(h(t)(s^2))}{2!}(\sigma^2 - (h(t)(s^2))^2) + R_n$$

$$\begin{aligned}
E[\Phi(d_2)] &= E[\Phi(h(t)(s^2)) + \frac{\Phi'(h(t)(s^2))}{1!}(\sigma^2 - (h(t)(s^2))) + \frac{\Phi''(h(t)(s^2))}{2!}(\sigma^2 - (h(t)(s^2))^2)] \\
&= \Phi(h(t)(s^2)) + \frac{\Phi'(h(t)(s^2))}{1!}E[(\sigma^2 - (h(t)(s^2)))] + \frac{\Phi''(h(t)(s^2))}{2!}E[(\sigma^2 - (h(t)(s^2))^2)]
\end{aligned}$$

We can simplify this formula by solving for $E[(\sigma^2 - (h(t)(s^2)))^2]$,

$$\begin{aligned}
E[(\sigma^2 - (h(t)(s^2)))^2] &= E[\sigma^4 - 2\sigma^2(h(t)(s^2)) + (h(t)(s^2))^2] \\
&= E[\sigma^4] - 2(h(t)(s^2))E[\sigma^2] + (h(t)(s^2))^2
\end{aligned}$$

Thus,

$$E[\Phi(d_2)] = \Phi(h(t)(s^2)) + \Phi'(h(t)(s^2))[E[\sigma^2] - (h(t)(s^2))] + \frac{1}{2}\Phi''(h(t)(s^2))[E[\sigma^4] - 2(h(t)(s^2))E[\sigma^2] + (h(t)(s^2))^2]$$

where

$$\begin{aligned}
\Phi(d_2) &= \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_2^2/2} \, \mathrm{d}d_2 \\
\Phi'(d_2) &= \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \\
\Phi''(d_2) &= \frac{-d_2}{\sqrt{2\pi}} e^{-d_2^2/2} * d_2' \\
d_2 &= \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\
d_2' &= -\frac{2\ln(\frac{S}{K}) + 2rT + \sigma^2 T}{4\sigma^2\sqrt{\sigma^2 T}} \\
E[\sigma^2] &= \frac{t-1}{t-3} s^2 \\
E[\sigma^4] &= \frac{(t-1)^2 s^4}{(t-3)(t-5)}
\end{aligned}$$

Appendix 4

$E[\Phi(d_2)]$ with $h(t)(s^2) = s^2$

$$\begin{aligned}
E[\Phi(d_2)] &= E[\Phi(s^2) + \frac{\Phi'(s^2)}{1!}(\sigma^2 - s^2) + \frac{\Phi''(s^2)}{2!}(\sigma^2 - s^2)^2 + R_n] \\
&= \Phi(s^2) + \Phi'(s^2)E[\sigma^2 - s^2] + \frac{\Phi''(s^2)}{2!}E[(\sigma^2 - s^2)^2]
\end{aligned}$$

To simplify further we can solve for $E[\sigma^2 - s^2]$ and $E[(\sigma^2 - s^2)^2]$:

$$\begin{aligned}
E[\sigma^2 - s^2] &= E[\sigma^2] - s^2 \\
&= \frac{t-1}{t-3}s^2 - s^2 \\
&= \frac{2s^2}{t-3}
\end{aligned}$$

$$\begin{aligned}
E[(\sigma^2 - s^2)^2] &= E[\sigma^4] - 2s^2E[\sigma^2] + s^4 \\
&= E[\sigma^4] - (2s^2)(\frac{t-1}{t-3}s^2) + s^4 \\
&= E[\sigma^4] - \frac{t-1}{t-3}2s^4 + s^4 \\
&= E[\sigma^4] - \frac{t+1}{t-3}s^4 \\
&= \frac{(t-1)^2}{(t-3)(t-5)}s^4 - \frac{t+1}{t-3}s^4 \\
&= \frac{2s^4(t+3)}{(t-3)(t-5)}
\end{aligned}$$

Thus, we arrive at a simplified form for $E[\Phi(d_2)]$:

$$E[\Phi(d_2)] = \Phi(s^2) + \Phi'(s^2)(\frac{2s^2}{t-3}) + \Phi''(s^2)\frac{s^4(t+3)}{(t-3)(t-5)}$$

$E[\Phi(d_2)]$ with $h(t)(s^2) = E[\sigma^2]$

$$\begin{aligned}
E[\Phi(d_2)] &= E[\Phi(\frac{t-1}{t-3}s^2) + \frac{\Phi'(\frac{t-1}{t-3}s^2)}{1!}[\sigma^2 - (\frac{t-1}{t-3}s^2)] + \frac{\Phi''(\frac{t-1}{t-3}s^2)}{2!}[\sigma^2 - (\frac{t-1}{t-3}s^2)]^2] \\
&= \Phi(\frac{t-1}{t-3}s^2) + \Phi'(\frac{t-1}{t-3}s^2)E[\sigma^2 - (\frac{t-1}{t-3}s^2)] + \frac{\Phi''(\frac{t-1}{t-3}s^2)}{2!}E[(\sigma^2 - (\frac{t-1}{t-3}s^2))^2]
\end{aligned}$$

To simplify further we can solve for $E[\sigma^2 - (\frac{t-1}{t-3}s^2)]$ and $E[(\sigma^2 - (\frac{t-1}{t-3}s^2))^2]$:

$$\begin{aligned}
E[\sigma^2 - (\frac{t-1}{t-3}s^2)] &= E[\sigma^2] - (\frac{t-1}{t-3}s^2) \\
&= (\frac{t-1}{t-3}s^2) - (\frac{t-1}{t-3}s^2) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
E[(\sigma^2 - (\frac{t-1}{t-3}s^2))^2] &= E[(\sigma^2 - (\frac{t-1}{t-3}s^2))(\sigma^2 - (\frac{t-1}{t-3}s^2))] \\
&= E[\sigma^4 - 2(\frac{t-1}{t-3}s^2)E[\sigma^2] + \frac{(t-1)^2}{(t-3)^2}s^4] \\
&= E[\sigma^4 - 2(\frac{t-1}{t-3}s^2)(\frac{t-1}{t-3}s^2) + \frac{(t-1)^2}{(t-3)^2}s^4] \\
&= E[\sigma^4 - \frac{(t-1)^2s^4}{(t-3)^2}] \\
&= \frac{(t-1)^2s^4}{(t-3)(t-5)} - \frac{(t-1)^2s^4}{(t-3)^2} \\
&= \frac{2s^4(t-1)^2}{(t-3)^2(t-5)}
\end{aligned}$$

Thus, we arrive at a simplified form for $E[\Phi(d_2)]$:

$$E[\Phi(d_2)] = \Phi(\frac{t-1}{t-3}s^2) + \phi''(\frac{t-1}{t-3}s^2)\frac{s^4(t-1)^2}{(t-3)^2(t-5)}$$