$L(3,2,1)$-Labeling of Simple Graphs

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Abstract

An $L(3,2,1)$-labeling is a simplified model for the channel assignment problem. It is a natural generalization of the widely studied $L(2,1)$-labeling.

An $L(3,2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of positive integers such that for any two vertices x, y, if $d(x,y) = 1$, then $|f(x) - f(y)| \geq 3$; if $d(x,y) = 2$, then $|f(x) - f(y)| \geq 2$; and if $d(x,y) = 3$, then $|f(x) - f(y)| \geq 1$. The $L(3,2,1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3,2,1)$-labeling with k as the maximum label. In this paper we determine the $L(3,2,1)$-labeling number for paths, cycles, caterpillars, n-ary trees, complete graphs and complete bipartite graphs. We also present an upper bound for $k(G)$ in terms of the maximum degree of G.

1 Introduction

The assignment of FM frequencies to stations became a problem as technology advanced in the early 20th centuy. As more and more stations requested frequencies, it became difficult to assingn frequencies without having new stations

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interfere with the broadcast of other stations nearby. The channel assignment problem is an engineering problem in which the task is to assign a channel (nonnegative integer) to each FM radio station in a set of given stations such that there is no interference between stations and the span of the assigned channels is minimized. The level of interference between any two FM radio stations correlates with the geographic locations of the stations. Closer stations have a stronger interference, and thus there must be a greater difference between their assigned channels.

In 1980, Hale introduced a graph theory model of the channel assignment problem where it was represented as a vertex coloring problem [3]. Vertices on the graph correspond to the radio stations and the edges show the proximity of the stations.

In 1991, Roberts proposed a variation of the channel assignment problem in which the FM radio stations were considered either "close" or "very close." "Close" stations were vertices of distance two apart on the graph and were assigned channels that differed by two; stations that were considered "very close" were adjacent vertices on the graph and were assigned distinct channels [6].

More precisely, Griggs and Yeh defined the $L(2,1)$-labeling of a graph $G = (V, E)$ as a function f which assigns every x, y in V a label from the set of positive integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$ [2]. $L(2,1)$-labeling has been widely studied in recent years.

In 2001, Chartrand et al. introduced the radio-labeling of graphs; this was motivated by the regulations for the channel assignments in the channel assignment problem [1]. Radio-labeling takes into consideration the diameter of the graph, and as a result, every vertex is related.

Practically, interference among channels may go beyond two levels. $L(3,2,1)$-labeling naturally extends from $L(2,1)$-labeling, taking into consideration vertices which are within a distance of three apart; however, it remains less difficult than radio-labeling. An $L(3,2,1)$-labeling of a graph $G = (V, E)$ is a function f which assigns every x, y in V a label from the set of positive integers such that $|f(x) - f(y)| \geq 3$ if $d(x, y) = 1$, $|f(x) - f(y)| \geq 2$ if $d(x, y) = 2$, and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 3$. The $L(3,2,1)$-labeling number, $k(G)$, of G is the smallest number k such that G has an $L(3,2,1)$-labeling with k as the maximum label [4].

In this paper we determine the $L(3,2,1)$-labeling number for paths, cycles, caterpillars, n-ary trees, complete graphs and complete bipartite graphs. We also present an upper bound for $k(G)$ in terms of the maximum degree of G. The method for calculating the upper bound was introduced by Griggs and Yeh [2] and referenced by Jonas [5].

2 Definitions and Notation

Definition 1. Let $G = (V, E)$ be a graph and f be a mapping $f : V \longrightarrow \mathbb{N}$. f is an $L(3, 2, 1)$-*labeling* of G if, for all $x, y \in V$,

$$|f(x) - f(y)| \geq \begin{cases} 3, & \text{if } d(x, y) = 1; \\ 2, & \text{if } d(x, y) = 2; \\ 1, & \text{if } d(x, y) = 3. \end{cases}$$

Definition 2. The $L(3, 2, 1)$-*number*, $k(G)$, of a graph G is the smallest natural number k such that G has an $L(3, 2, 1)$-labeling with k as the maximum label. An $L(3, 2, 1)$-labeling of a graph G is called a *minimal* $L(3, 2, 1)$-*labeling* of G if, under the labeling, the highest label of any vertex is $k(G)$.

Note. If 1 is not used as a vertex label in an $L(3, 2, 1)$-labeling of a graph, then every vertex label can be decreased by one to obtain another $L(3, 2, 1)$-labeling of the graph. Therefore in a minimal $L(3, 2, 1)$-labeling 1 will necessarily appear as a vertex label.

Definition 3. Let $G = (V, E)$ be a graph.
G is called a *complete graph* on n vertices, K_n, if for all vertices $x, y \in V$, $(x, y) \in E$.
G is called a *complete bipartite graph*, $K_{m,n}$, if:

1. The set of vertices, V, can be partitioned into two disjoint sets of vertices, A and B, such that $|A| = m$, $|B| = n$, and $|V| = m + n$
2. For all $a_i, a_j \in A$, $(a_i, a_j) \notin E$ and for all $b_i, b_j \in B$, $(b_i, b_j) \notin E$
3. For all $a_i \in A$ and $b_j \in B$, $(a_i, b_j) \in E$.

A *star*, S_n, is a $K_{1,n}$ complete bipartite graph.

Definition 4. Let $G = (V, E)$ be a graph.
G is called a *path*, P_n, if $V = \{v_1, v_2, \ldots, v_n\}$ such that for $1 \leq i < n$, $(v_i, v_{i+1}) \in E$.
G is called a *cycle*, C_n, if $V = \{v_1, v_2, \ldots, v_n\}$ such that for $1 \leq i < n$ $(v_i, v_{i+1}) \in E$ and $(v_1, v_n) \in E$.

Definition 5. Let $G = (V, E)$ be a graph.
G is called a *tree* if G is connected and has no cycles.
G is called a *caterpillar* if G is a tree such that the removal of the degree one vertices produces a path called the spine of the caterpillar. A *uniform caterpillar* is a caterpillar with only degree one and degree Δ vertices. We denote a uniform caterpillar with n vertices on the spine by Cat_n.
G is called an *n-ary tree* if G is a rooted tree such that the root has degree n and all the other vertices have degree $n + 1$.

3 Theorems

3.1 Complete Graphs

Theorem 1. *For any complete graph on n vertices, $k(K_n) = 3n - 2$.*

Proof. Let $G = (V, E)$ be a complete graph with $V = \{v_1, v_2, \ldots, v_n\}$ and let f be a minimal $L(3,2,1)$-labeling of G with $f(v_i) < f(v_j)$ for all $i < j$. Then, for all $v_i, v_j \in V$ and $i \neq j$, $(v_i, v_j) \in E$, implying that $d(v_i, v_j) = 1$. Thus, $|f(v_i) - f(v_j)| \geq 3$ for all $v_i, v_j \in V$ and $i \neq j$. Since there exists v_i in V such that $f(v_i) = 1$, we have $f(v_1) = 1$. Also, since f is a minimal $L(3,2,1)$-labeling of G, we have that for all v_i with $1 < i \leq n$, $f(v_i) \geq f(v_{i-1}) + 3$. Recursively, we have

$$\begin{aligned}
f(v_n) &\geq f(v_{n-1}) + 3 \\
&\geq f(v_{n-2}) + 3(2) \\
&\geq f(v_{n-3}) + 3(3) \\
&\qquad \vdots \\
&\geq f(v_{n-i}) + 3(n-i) \\
&\qquad \vdots \\
&\geq f(v_1) + 3(n-1) = 1 + 3(n-1) = 3n - 2.
\end{aligned}$$

Therefore $k(K_n) = 3n - 2$. □

3.2 Complete Bipartite Graphs and Stars

Theorem 2. *For any complete bipartite graph, $k(K_{m,n}) = 2(m+n)$.*

Proof. Let $G = (V, E)$ be a complete bipartite graph, $K_{m,n}$, and let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be the two sets of vertices that partition V as per definition of a complete bipartite graph. Let f be a minimal $L(3,2,1)$-labeling of G with $f(a_1) \leq f(a_2) \leq \cdots \leq f(a_m)$ and $f(b_1) \leq f(b_2) \leq \cdots \leq f(b_n)$. Note that $d(a_i, a_j) = 2$ for all $a_i, a_j \in A$ with $i \neq j$; the same is true for pairs of vertices in B. Let $f(a_1) = 1$, then we need each $f(a_i)$ with $i \neq 1$ to be odd since f is minimal. Thus, $f(A) = \{1, 3, 5, \ldots, 1 + 2(m-1)\}$. Since each $a_i \in A$ is adjacent to every $b_i \in B$, we need $|f(a_i) - f(b_i)| \geq 3$. Note that we need $f(b_1) \geq f(a_m) + 3$. Then, since $f(a_m) = 2m - 1$, $f(b_1) \geq 3 + (2m-1)$. Since f is minimal we have $f(b_1) = 3 + (2m-1) = 2m + 2$. The labeling of vertices in B follows an argument similar to the labeling of vertices in A: since $f(b_1) = 2m - 2$ we need each $f(b_i)$ to be even. Thus, $f(B) = \{2m+2, 2m+4, \ldots, 2m+2+2(n-1)\}$. Then $f(b_n) = 2m + 2 + 2(n-1) = 2(m+n)$. Therefore $k(K_{m,n}) = 2(m+n)$. □

Corollary 3. *For a star, S_n, $k(S_n) = 2n + 2$.*

Proof. By definition, a star, S_n, is $K_{1,n}$. Therefore $k(S_n) = 2n + 2$.

□

3.3 Paths

Lemma 4. *For a path on n vertices, P_n, with $n \geq 8$, $k(P_n) \geq 8$.*

Proof. Let f be a minimal $L(3,2,1)$-labeling for a path on n vertices, P_n, and suppose $k(P_n) < 8$ for $n \geq 8$. Let v_1 be a vertex with label 1. There is an induced subpath of at least 5 vertices with v_1 as an end vertex. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ be this path. We will continue by considering the possibilities for $f(v_2)$.

Case I: $f(v_2) = 4$:
Then we need $f(v_3) = 7$, $f(v_4) = 2$, $f(v_5) = 5$, and $f(v_6) \geq 8$, which contradicts our assumed $k(P_n)$.

Case II: $f(v_2) = 5$:
Then we need $f(v_3) \geq 8$, a contradiction to our assumed $k(P_n)$.

Case III: $f(v_2) = 6$:
Then we need $f(v_3) = 3$, forcing $f(v_4) \geq 8$, a contradiction.

Case IV: $f(v_2) = 7$:
Then $f(v_3) = 3$ or 4. Either possibility for $f(v_2)$ forces $f(v_4) \geq 9$, which is a contradiction.

Therefore we can conclude that $k(P_n) \geq 8$, when $n \geq 8$. □

Theorem 5. *For any path, P_n,*

$$k(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ 4, & \text{if } n = 2; \\ 6, & \text{if } n = 3, 4; \\ 7, & \text{if } n = 5, 6, 7; \\ 8, & \text{if } n \geq 8. \end{cases}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of P_n such that v_i is adjacent to v_{i+1} for $1 \leq i < n$. Define f such that $f(\{v_1, v_2, \ldots, v_8\}) = \{1, 4, 7, 2, 5, 8, 3, 6\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 8$. By definition of f we can conclude that $k(P_n) \leq 8$ for $n \geq 8$. We combine this result with that of Lemma 4 to get $k(P_n) = 8$ for $n \geq 8$.

For each P_n with $n < 8$, we proceed by cases using the labeling pattern defined by f and the observation that each $P_1, P_2, \ldots, P_7$ is an induced subpath of P_n with $n \geq 8$:

Case I: $n = 1$.
This is trivially true.

Case II: $n = 2$.
The labeling pattern $\{1, 4\}$ shows that $k(P_2) = 4$ because we cannot do any better.

Case III: $n = 3, 4$.

The labeling pattern $\{3, 6, 1, 4\}$ shows that $k(P_n) \leq 6$ for $n = 3, 4$. Suppose $k(P_n) < 6$. There is a vertex $v_i \in V$ such that $f(v_i) = 1$. If v_i has degree 2, then vertices v_{i-1} *and* v_{i+1} exist such that $f(v_{i-1}) \geq 4$ and $f(v_{i+1}) \geq 6$, a contradiction. If v_i has degree 1, let $v_i = v_1$, then the possibilities for $f(v_2)$ are 4 and 5. In either case, we would need $f(v_3) > 6$, a contradiction.

Case IV: $n = 5, 6, 7$.

The labeling pattern $\{3, 6, 1, 4, 7, 2, 5\}$ shows that $k(P_n) \leq 7$ for $n = 5, 6, 7$. Suppose $k < 7$. There is a vertex $v_i \in V$ such that $f(v_i) = 1$ and vertices v_{i+1} and v_{i+2} exist or v_{i-1} and v_{i-2} exist. Without loss of generality, suppose v_{i+1} and v_{i+2} exist. The possibilities for $f(v_{i+1})$ are 4, 5, and 6. If $f(v_{i+1}) = 4, 5$ we need $f(v_{i+2}) \geq 7$, a contradiction. If $f(v_{i+1}) = 6$ then $f(v_{i+2}) = 3$. Now if v_{i+2} has degree 2, then v_{i+3} exists and $f(v_{i+3}) > 7$. If v_{i+2} has degree 1 then vertices v_{i-1} and v_{i-2} exist. The only possibility for $f(v_{i-1})$ is 4. But this forces $f(v_{i-1}) \geq 7$, a contradiction. □

3.4 Cycles

Lemma 6. *Let n be an odd integer, $n > 3$, then $k(C_n) \neq 8$.*

Proof. Let f be a minimal $L(3, 2, 1)$-labeling of C_n where n is odd and $n > 3$. Suppose $k(C_n) = 8$. Then the only possible labelings with 8 as the maximum integer are as follows: $f_1(V) = \{8, 5, 2, 7, 4, 1, 6, 3, 8 \ldots\}$, $f_2(V) = \{8, 5, 2, 7, 4, 1, 8 \ldots\}$, $f_3(V) = \{8, 5, 1, 7, 4, X\}$, $f_4(V) = \{8, 5, 1, 7, 3, X\}$, $f_5(V) = \{8, 4, 1, 6, 3, 8, \ldots\}$, $f_6(V) = \{8, 4, 1, 7, 3, X\}$, $f_7(V) = \{8, 3, 6, 1, 8, \ldots\}$, $f_8(V) = \{8, 2, 5, X\}$, $f_9(V) = \{8, 2, 6, X\}$, and $f_{10}(V) = \{8, 1, 5, X\}$.

Observe that f_1, f_2, and f_7 are labelings for even cycles. Of the remaining labelings f_3, f_4, f_6, f_8, f_9, and f_{10} all fail as potential labelings for a cycle because the missing X labels are forced to be greater than the assumed $k(C_n)$. The only labeling left is f_5, which also fails to label a cycle because it does not provide a repeatable pattern. Therefore, no minimal $L(3, 2, 1)$-labeling exists for C_n with odd n and $n > 3$ such that $k(C_n) = 8$. □

Lemma 7. $k(C_4) = 8$.

Proof. The labeling pattern $\{1, 6, 3, 8\}$ shows that $k(C_4) \leq 8$. Now, let f be minimal $L(3, 2, 1)$-labeling of C_4 and suppose $k(C_4) < 8$. Let $f(v_1) = 1$ and v_2 and v_4 be the two vertices adjacent to v_1. Then the possibilities for $\{f(v_2), f(v_4)\}$ are $\{4, 6\}$, $\{4, 7\}$, and $\{5, 7\}$.

Case I: $\{f(v_2), f(v_4)\} = \{4, 6\}$:

Then we need $f(v_3) = 9$, a contradiction to our assumption that $k(C_4) < 8$.

Case II: $\{f(v_2), f(v_4)\} = \{4, 7\}$ or $\{5, 7\}$:

Then we need $f(v_3) = 10$, a contradiction to the assumed $k(C_4)$.

Therefore $k(C_4) = 8$. □

Lemma 8. $k(C_5) = 9$.

Proof. By Theorem 5 we know that $k(C_5) \geq 7$ since $k(P_5) = 7$. Let f be minimal $L(3,2,1)$-labeling of C_5 and suppose $k(C_5) = 7$. Let $f(v_1) = 1$ and let v_2 and v_5 be the two vertices adjacent to v_1. The possibilities for $\{f(v_2), f(v_5)\}$ are $\{4,6\}$, $\{4,7\}$, and $\{5,7\}$.

Case I: $\{f(v_2), f(v_5)\} = \{4,6\}$:
Then we need $f(v_3) = 8$, a contradiction to our assumption that $k(C_5) < 8$.

Case II: $\{f(v_2), f(v_5)\} = \{4,7\}$ or $\{5,7\}$:
Then we need $f(v_3) = 9$, a contradiction to the assumed $k(C_5)$.

Since $k(C_5) = 7$ is impossible, we have $k(C_5) \geq 8$. By Lemma 6 for odd cycles, we know that $k(C_5) \neq 8$. So we have $k(C_5) \geq 9$. The labeling pattern $\{5,1,7,3,9\}$ shows that $k(C_5) \leq 9$.

Therefore $k(C_5) = 9$. □

Lemma 9. $k(C_6) = 8$.

Proof. The labeling pattern $\{1,4,7,2,5,8\}$ shows that $k(C_6) \leq 8$. Now, let f be minimal $L(3,2,1)$-labeling of C_6 and suppose $k(C_4) < 8$. Let $f(v_1) = 1$ and let v_2 and v_6 be the two vertices adjacent to v_1. The possibilities for $\{f(v_2), f(v_6)\}$ are $\{4,6\}$, $\{4,7\}$, and $\{5,7\}$.

Case I: $\{f(v_2), f(v_6)\} = \{4,6\}$:
Then we need $f(v_3) = 7$ and $f(v_4) = 2$. For v_5, we need $f(v_5) = 9$, a contradiction to $k(C_6) < 8$.

Case II: $\{f(v_2), f(v_6)\} = \{4,7\}$ or $\{5,7\}$:
Then we need $f(v_3) = 8$, a contradiction.

Therefore $k(C_6) = 8$. □

Lemma 10. $k(C_7) = 10$.

Proof. Let $V = \{v_1, v_2, \ldots, v_7\}$ be the set the vertices of C_7. Suppose $k(C_7) < 10$ and let f be a minimal $L(3,2,1)$-labeling of C_7, then the possible values for $f(V)$ are in $S = \{1, 2, \ldots, 9\}$. Since the greatest distance between any two vertices in V is 3, none of the possible values for $f(V)$ can be repeated. Also, only up to two consecutive labels may be used. Using three consecutive labels is not possible due to the distance constraints of labeling V. Then at most 6 labels can be used from S, forcing the unlabeled vertex to be labeled with a value greater than 10, a contradiction to the assumed $k(C_7)$. Thus, $k(C_7) \geq 10$. The labeling pattern $\{1,5,8,2,10,7,4\}$ shows that $k(C_7) \leq 10$. Therefore, $k(C_7) = 10$. □

Lemma 11. *Let n be an even integer. If $n \geq 4$, then $n = 4a + 6b$; where a and b are non-negative integers.*

Proof. Let $S = \{4a + 6b \mid a, b \in \mathbb{Z};\ a, b \geq 0\}$ and let n be an even integer. Then $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. Suppose that $n \geq 4$.

Case I: $n \equiv 0 \pmod 4$:

Then there exists an integer q such that $n = 4q$. Since $n \geq 4$, $q \geq 1$. This implies $n = 4a + 6b$ where $a = q$ and $b = 0$. Then, $n \in S$.

Case II: $n \equiv 2 \pmod 4$:

Then there exists an integer q such that $n = 4q + 2$. This implies that $n = 4(q - 1) + 6$. Since $n \geq 4$ and $n \equiv 2 \pmod 4$, we get $n \geq 6$. Then $(q - 1) \geq 0$. From this, it follows that $n = 4a + 6b$ where $a = (q - 1)$ and $b = 1$. Then, $n \in S$.

Therefore, if n is an even integer and $n \geq 4$ then $n = 4a + 6b$, where a and b are non-negative integers.

□

Lemma 12. *Let n be an odd integer. If $n \geq 9$ and $n \neq 11$, then $n = 4a + 5b$; where a and b are non-negative integers.*

Proof. Let $S = \{4a + 5b \mid a, b \in \mathbb{Z};\ a, b \geq 0\}$ and let n be an odd integer. Then $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$. Suppose $n \geq 9$ and $n \neq 11$.

Case I: $n \equiv 1 \pmod 4$:

Then there exists an integer q such that $n = 4q + 1$. This implies that $n = 4(q - 1) + 5$. Since $n \geq 9$, we need $(q - 1) \geq 1$. This implies that $n = 4a + 5b$ where $a = (q - 1)$ and $b = 1$. Then, $n \in S$.

Case II: $n \equiv 3 \pmod 4$:

Then there exists an integer q such that $n = 4q + 3$. This implies that $n = 4(q - 3) + 5(3)$. Since $n \geq 9$, $n \neq 11$, and $n \equiv 3 \pmod 4$, we get $n \geq 15$. Then we need $(q - 3) \geq 0$. Since $n = 4a + 5b$ where $a = (q - 3)$ and $b = 3$, we get $n \in S$.

Therefore, if n is an odd integer such that $n \geq 9$ and $n \neq 11$ then $n = 4a+5b$ for non-negative integers a and b. □

Theorem 13. *For any cycle, C_n with $n \geq 3$,*

$$k(C_n) = \begin{cases} 7, & \text{if } n = 3; \\ 8, & \text{if } n \text{ is even;} \\ 9, & \text{if } n \text{ is odd and } n \neq 3, 7; \\ 10, & \text{if } n = 7. \end{cases}$$

Proof. Let $n \geq 3$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices of C_n such that for $1 \leq i < n$, (v_i, v_{i+1}) and (v_1, v_n) are edges in C_n. The following cases describe all the possibilities for $k(C_n)$.

Case I: $n = 3$:

Observe that C_3 is a complete graph. Therefore, by Theorem 1 for complete graphs, $k(C_3) = 7$.

Case II: n is even:

We have $k(C_4) = 8$ by Lemma 7 and we have $k(C_6) = 8$ by Lemma 9. We also know from Theorem 5 for paths that $k(P_n) = 8$ for $n \geq 8$. Then for all even cycles, $k(C_n) \geq 8$. Recall the labelings used for C_4 and C_6 in Lemmas 7 and 9, respectively: for C_4 we used $f(V) = \{1, 6, 3, 8\}$ and for C_6 we used $f(V) = \{1, 4, 7, 2, 5, 8\}$. Observe that the labeling used for C_4 can be repeated infinitely for all C_n with n a multiple of 4. Likewise, the labeling used for C_6 can be repeated infinitely for all C_n with n a multiple of 6. Moreover, the labelings of C_4 and C_6 can be joined together to label C_{10} as such: $f(V) = \{1, 6, 3, 8, 1, 4, 7, 2, 5, 8\}$. From Lemma 11, we know that every even integer greater than or equal to 4 can be expressed as some combination of non-negative multiples of 4 and 6. From this it becomes obvious that the labeling of every even C_n can be composed of combinations of non-negative multiples of the C_4 and C_6 labeling patterns. Therefore $k(C_n) = 8$ for all even n.

Case III: n is odd and $n \neq 3$ or 7:

By Lemma 8 we have $k(C_5) = 9$, by Theorem 5 for paths we know that $k(C_n) \geq 8$ for $n \geq 8$ since $k(P_n) = 8$, and by Lemma 6 we know that $k(C_n) \neq 8$ for odd cycles. This implies that for all odd cycles $k(C_n) \geq 9$. In Lemma 8 we labeled C_5 with $\{5, 1, 7, 3, 9\}$. Note that labeling of C_5 can be repeated infinitely for all C_n with n a multiple of 5. Also, we can combine the labeling for C_5 with the labeling for C_4 used in Lemma 7 to label C_9 as such: $\{5, 1, 7, 3, 9, 1, 6, 3, 8\}$. By Lemma 8 we have that every odd integer greater than or equal to 9, with the exception of 11, can be expressed as some combination of non-negative multiples of 4 and 5. This implies that every C_n, with $n \geq 9$ and $n \neq 11$, is composed of combinations of non-negative multiples of C_4 and C_5. Therefore, since $k(C_5) = 9$, we have $k(C_n) = 9$ for all C_n, where $n \geq 9$ and $n \neq 11$. For C_{11} we know that $k(C_{11}) \geq 9$ (from Lemma 2 and Theorem 3). Define f for C_{11} such that $f(V) = \{1, 6, 3, 8, 5, 1, 9, 6, 2, 8, 4\}$. Since $\max(f(V)) = 9$, $k(C_{11}) = 9$.

Case IV: $n = 7$:

By Lemma 10, we have $k(C_7) = 10$. □

3.5 Caterpillars

Lemma 14. *For a uniform caterpillar, Cat_n with $n > 2$, $k(Cat_n) > 2\Delta + 2$.*

Proof. Suppose $k(Cat_n) \leq 2\Delta + 2$. Begin with any vertex, v on the spine, label it i where $1 \leq i \leq 2\Delta + 2$. Let m be the number of available labels for the vertices adjacent to v. We will show that $m \leq \Delta$ if $i = 1$ or $2\Delta + 2$ and $m \leq \Delta - 1$ otherwise. To determine m, we count the possible labels, j, for vertices adjacent to v such that $|i - j| \geq 3$. As 1 and $2\Delta + 2$ are the respective minimum and maximum possibilities for any label, we know that $1 \leq j \leq i - 3$

or $i+3 \leq j \leq 2\Delta+2$. Since any pair of vertices adjacent to v has a distance of 2, no two consecutive numbers may be used as labels. Then, we can compute m by:

$$m \leq \left(\frac{(i-3)-1}{2}+1\right)+\left(\frac{(2\Delta+2)-(i+3)}{2}+1\right) \tag{1}$$

Case I: $i=1$:

As 1 is the lowest possible label for v, all available labels would be greater than 1. As such, equation 1 becomes:

$$m \leq \left(\frac{(2\Delta+2)-(1+3)}{2}+1\right) \leq \left(\frac{2\Delta-2}{2}+1\right) \leq \Delta \tag{2}$$

Thus, when v is labeled with with 1, there are Δ labels available for the Δ adjacent vertices.

Case II: $i=2\Delta+2$:

Then, since $2\Delta+2$ is the highest label possible for v, equation 1 becomes:

$$m \leq \left(\frac{((2\Delta+2)-3)-1}{2}+1\right) \leq \left(\frac{2\Delta-2}{2}+1\right) \leq \Delta \tag{3}$$

Thus, when v is labeled with $2\Delta+2$, there are Δ labels available for the Δ adjacent vertices.

Case III: $1 < i < 2\Delta+2$:

As $n > 2$, at least one label must be used on the spine aside from 1 and $2\Delta+2$. For this spinal vertex, equation 1 is used and yields:

$$\begin{aligned} m &\leq \frac{(i-3)-1}{2}+\frac{(2\Delta+2)-(i+3)}{2}+2 \\ &\leq \frac{i-3-1+2\Delta+2-i-3}{2}+2 \\ &\leq \frac{2\Delta-5}{2}+2 \leq \Delta-\frac{1}{2}. \end{aligned} \tag{4}$$

As m must be an integer (there can be no partial labels), for any v labeled with $1 < i < 2\Delta+2$, there are at most $\Delta-1$ labels available for Δ vertices. Therefore, by the Pigeon-Hole Principle, any vertex labeled $1 < i < 2\Delta+2$ on Cat_n cannot have $k(Cat_n) \leq 2\Delta+2$. Thus, $k(Cat_n) > 2\Delta+2$. □

Theorem 15. *For a uniform caterpillar, Cat_n, with $\Delta > 2$ and $n > 2$, $k(Cat_n) = 2\Delta+3$.*

Proof. The labeling pattern below illustrates the pattern of an $L(3,2,1)$-labeling with $k(Cat_n) = 2\Delta+3$.

To label the spine, we use the pattern 1, 2Δ, $2\Delta+3$, $2\Delta-2$ repeatedly. To label the legs adjacent to a vertex labeled i on the spine use positive values from the set $\{i \pm (3+2p) :\ 0 \leq p \leq 2\Delta\}$. As $k(Cat_n) > 2\Delta+2$, $k(Cat_n) = 2\Delta+3$ is the lowest possible k. □

Corollary 16. *For any caterpillar Cat_n with $\Delta > 2$ and n vertices on the spine, either $k(Cat_n) = 2\Delta + 2$ or $k(Cat_n) = 2\Delta + 3$. When there are no more than two vertices of degree Δ within any subgraph of four spinal vertices on Cat_n, $k(Cat_n) = 2\Delta + 2$.*

Proof. This follows from the proof of the uniform caterpillar: only vertices labeled 1 and $2\Delta + 2$ had Δ label options for Δ vertices. All other spinal labels yielded at most $\Delta - 1$ possible labels. Thus, all spinal vertices labeled with $1 < i < 2\Delta + 2$ must have a degree of at most $\Delta - 1$, in order to keep the largest label to $2\Delta + 2$ or less (case III, Theorem 14). Also, cases I and II of Theorem 14 show that a non-uniform caterpillar with $k(Cat_n) = 2\Delta + 2$ may only have a degree of Δ at vertices labeled with 1 and $2\Delta + 2$. Notice that the labels 1 and $2\Delta + 2$ can be repeated every four labels and that using this pattern maximizes the total number of vertices on the caterpillar. The pattern below illustrates one labeling of the vertices that fulfills this requirement and maximizes the quantity of vertices.

□

3.6 N-ary Tree

Lemma 17. *For any n-ary tree, G, $k(G) \leq 2n + 6$.*

Proof. This proof is by construction. Let $G = (V, E)$ be an n-ary tree. First, observe that the induced subgraph of each vertex and all of the Δ vertices adjacent to it form a star. With this observation, we propose the following $L(3, 2, 1)$-labeling of each star: let f be a minimal $L(3, 2, 1)$-labeling of a star with central vertex x and define f such that the adjacent vertices of x are labeled with the lowest positive integers from the set $\{f(x) \pm (3 + 2i) : i \in \mathbb{Z}^+ \cup \{0\}\}$. In other words, adjacent vertices of x are first labeled with as many positive elements as possible from the set $\{f(x) - 3, f(x) - 5, f(x) - 7 \ldots\}$ and the remaining adjacent vertices are labeled with the lowest positive elements from the set $\{f(x) + 3, f(x) + 5, f(x) + 7, \ldots\}$.

We will show that these stars can be joined via overlapping edges to create an n-ary tree such that f is an $L(3, 2, 1)$-labeling. It follows from construction that for any two adjacent vertices x and y, $|f(x) - f(y)| \geq 3$. It also follows from construction that for any two vertices x and y that are mutually adjacent to a vertex w, $|f(x) - f(y)| \geq 2$. As for vertices x, y with $d(x, y) = 3$, we observe that for each star (induced subgraph of G), with central vertex x: if $f(x)$ is even then for all vertices y adjacent to x, $f(y)$ is odd; and if $f(x)$ is odd then for all vertices y adjacent to x, $f(y)$ is even. Hence for vertices x, y with $d(x, y) = 3$, x and y have different parity. Thus, $d(x, y) = 3$ implies $|f(x) - f(y)| \geq 1$. Thus, f is an $L(3, 2, 1)$-labeling for any n-ary tree.

Claim: The highest value of f used in the proposed $L(3, 2, 1)$-labeling is $2\Delta + 4$. Note that the worst case for the proposed $L(3, 2, 1)$-labeling is when the central vertex, x, of a star has $f(x) = 3$. This is because none of the vertices adjacent to x can be labeled using elements of the set $\{f(x) - 3, f(x) - 5, f(x) - 7, \ldots\}$.

Hence, since all the labels for the vertices adjacent to x must come from the set $\{f(x)+3, f(x)+5, f(x)+7, \ldots\}$, the highest label needed is $f(x)+3+2(\Delta-1) = 2\Delta+4$.

Therefore, since $\Delta = n+1$, $k(G) \leq 2n+6$.

□

Lemma 18. *Let $G=(V,E)$ be an n-ary tree. Suppose $k(G) < 2n+6$. Let f be a minimal $L(3,2,1)$-labeling of G, and let x be a vertex with $f(x)=1$. Let $A=\{a_1, a_2, \ldots, a_\Delta\}$, be the set of vertices adjacent to x. Neither 6 nor 7 can appear as a label for the elements of A.*

Proof. We consider the cases when 6 or 7 is used as a label for elements of A.

Case I: Suppose there is a vertex a_1 in A, such that $f(a_1)=6$.

Let $B=\{b_1, b_2, \ldots, b_{\Delta-1}\}$ be the set of vertices adjacent to a_1, not including x. (Recall that $f(x)=1$). By the definition of an $L(3,2,1)$-labeling, $f(b_i)$ differs by at least 3 from $f(a_1)=6$ for all i. This implies that the minimum label used in B is 3 or 9. If the minimum label is 3, then there is a vertex, b_1 in B such that $f(b_1)=3$. Let $C=\{c_1, c_2, \ldots, c_\Delta\}$ be the set of vertices adjacent to b_1. By the definition of an $L(3,2,1)$-labeling, $f(c_i) \geq f(b_1)+3=6$ for all i. Then there exists c_i in C such that $f(c_i) \geq 6+2(\Delta-1) = 2\Delta+4 = 2n+6$, a contradiction to our assumption that $k(G) < 2n+6$. If the minimum label used in B is 9, then for all b_i in B, $f(b_i) \geq 9$. This implies that there is b_j in B such that $f(b_j) \geq 9+2(\Delta-2) = 2\Delta+5 = 2n+7$. This contradicts our assumption that $k(G) < 2n+6$. Therefore, 6 cannot appear as a vertex label in A.

Case II: Suppose there is a vertex in A labeled 7, say $f(a_1)=7$:

Let $B=\{b_1, b_2, \ldots, b_{\Delta-1}\}$ be the set of vertices adjacent to a_1, not including x. The elements of B are distance 2 away from x and they are distance 3 away from any non-adjacent element of A. By the definition of an $L(3,2,1)$-labeling the smallest vertex label we could use for elements in B is 3. However, using 3 as a vertex label in B leads to a contradiction as we have shown in case I. The same argument also shows that the smallest label in B cannot be more than 3. Therefore, 7 cannot appear as a vertex label in A.

□

Lemma 19. *Let $G=(V,E)$ be an n-ary tree. Suppose $k(G) < 2n+6$. Let f be a minimal $L(3,2,1)$-labeling of G, and let x be a vertex with $f(x)=1$. Let $A=\{a_1, a_2, \ldots, a_\Delta\}$ be the set of vertices adjacent to x. The only possible sets of labels for the vertices of A are $\{4, 6, 8, \ldots, 2\Delta+2\}$, $\{5, 7, 9, \ldots, 2\Delta+3\}$ and a set of the form $\{4, \ldots, 2\Delta+3\}$. The last set contains even numbers up to a certain point then odd numbers afterwards.*

Proof. Let x and A be defined as above. Then by definition of $L(3,2,1)$-labeling, the smallest label of any vertex in A is 4. Also, by our assumption on $k(G)$ no label of any vertex of A can be bigger than $2\Delta+3$. Since any two elements of A are distance 2 apart in G, their labels must be at least 2 apart. In the first set there are Δ even labels that we can use for labeling the vertices in A. Likewise, there are enough labels in the sets $\{4, \ldots, 2\Delta + 3\}$ and $\{5, 7, 9, , 2\Delta + 3\}$ for labeling each element of A, so we can use these sets in an $L(3, 2, 1)$-labeling.

Now we show that no other sets are possible. Suppose the minimum label used for elements of A was at least 6. Then in order to have enough labels, a label at least $6 + 2(\Delta - 1) = 2\Delta + 4 = 2n + 6$ would need to be used, which contradicts the assumption $k(G) < 2n + 6$. Therefore, the smallest label must be less than 6. If the smallest label is 5, or if we use only even labels, the labels are $\{5, 7, 9, \ldots, 2\Delta + 3\}$, and $\{4, 6, 8, \ldots, 2\Delta + 2\}$, respectively. Otherwise the labels must come from the set $\{4, \ldots, 2\Delta + 3\}$.

□

Theorem 20. *For any n-ary tree, G, $k(G) = 2n + 6$.*

Proof. Let G be an n-ary tree. By Lemma 17 we know that $k(G) \leq 2n + 6$. If $k(G) < 2n + 6$ then, by Lemma 18, we know that neither 6 nor 7 can appear as a label in A. By Lemma 19 we know that the labels of A must come from one of the following sets: $\{4, 6, 8, \ldots, 2\Delta + 2\}$, $\{5, 7, 9, \ldots, 2\Delta + 3\}$ or one of the sets $\{4, \ldots, 2\Delta + 3\}$. Notice that 6 is an element of the set $\{4, 6, 8, \ldots, 2\Delta + 2\}$, 7 is an element of the set $\{5, 7, 9, \ldots, 2\Delta + 3\}$, and either 6 or 7 is an element of any set of the form $\{4, \ldots, 2\Delta + 3\}$. Therefore none of these sets can provide the labels for A.

Therefore, $k(G) = 2n + 6$.

□

3.7 The Upper Bound in Terms of Maximum Degree, Δ

Theorem 21. *If G is a graph with maximum degree Δ, $k(G) \leq \Delta^3 + \Delta^2 + 3\Delta$.*

Proof. Let $N_i(v)$ be the set of vertices of distance i from a vertex v; the members of the sets of $1 \leq i \leq 3$ shall be referred to as neighbors of v. When constructing a greedy $L(3, 2, 1)$-labeling f on a graph G, each vertex surrounding v is given the lowest available label such that no two labels are repeated within distance 3 of v. Furthermore, each neighbor of v blocks a certain number of labels from use: $N_1(v)$ blocks up to 5 labels, $N_2(v)$ up to 3, and $N_3(v)$ prohibits the use of the label given to itself. The maximum label needed for v is calculated through a sum of the quantity of vertices within distance three of v while weighting each set of neighbors according to the quantity of labels each blocks. As the maximum degree of G is at most Δ, $|N_1| \leq \Delta$, $|N_2| \leq \Delta(\Delta - 1)$, and $|N_3| \leq \Delta(\Delta - 1)^2$. Thus, $k(G) \leq 5\Delta + 3\Delta(\Delta - 1) + \Delta(\Delta - 1)^2$, or $k(G) \leq \Delta^3 + \Delta^2 + 3\Delta$. □

References

[1] G. Chartrand, D. Erwin, F. Harary, and P. Zhang, *Radio labeling of graphs*, Bull. Inst. Combin. Appl., 33 (2001), 77-85.

[2] J. R. Griggs and R. K. Yeh, *Labeling graphs with a condition at distance two*, SIAM J. Discrete Math., 5 (1992) 586-5995.

[3] W. K. Hale, *Frequency assignment: theory and application*, Proc. IEEE, 68 (1980), 1497-1514.

[4] L. Jia-zhuang and S. Zhen-dong, *The $L(3, 2, 1)$-labeling problem on graphs*, Mathematica Applicata, 17 (4) (2004), 596-602.

[5] K. Jonas, *Graph coloring analogues with a condition at distance two: L(2,1)-labelling and list λ-labellings*, Ph.D. thesis, University of South Carolina (1993), 8-9.

[6] F. S. Roberts, *T-colorings of graphs: recent results and open problems*, Discrete Math., 93 (1991), 229-245.