Mosaic Arithmetic

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Abstract

The mosaic of an integer n is the array of prime numbers resulting from iterating the Fundamental Theorem of Arithmetic on n and on any resulting composite exponents. Bildhauser, Erickson, Gillman and Tacoma generalized several arithmetic functions and attempted to define divisors on mosaics. We continue their work by investigating several possible definitions of multiplication for mosaics.

1 Introduction

Mullin first introduced the concept of a mosaic in 1964 and explored generalizations of common number theoretic ideas and functions on mosaics in subsequent papers [1964, 1965, 1967]. He offered the following definition of a *mosaic*.

Definition 1.1. *A* mosaic *is the array of prime numbers resulting from iterating the Fundamental Theorem of Arithmetic on n and on any resulting composite exponents.*

Constructing the mosaic of an integer is relatively easy as shown in the following example.

Example 1.1.

$$\begin{aligned} n &= 1,024,000,000. \\ &= 2^{16} \cdot 5^{6} \\ &= 2^{2^4} \cdot 5^{2\cdot 3} \\ &= 2^{2^{2^2}} \cdot 5^{2\cdot 3} = M(n). \end{aligned}$$

Gillman explored the consequences of partially expanding a mosaic through its first *i-levels* [1990]. He offered the following defintion of the levels of a mosaic.

Definition 1.2. Levels *of mosaics describe the different tiers of the mosaic.*

Not only do the levels refer to the location of individual primes in the mosaic but it also refers to the structure of the mosaic itself. We can consider the mosaic of n expanded through the i^{th} level, denoted by $M_i(n)$, as shown in the following example.

Example 1.2.

$$\begin{aligned} \textit{Let } n &= 1,024,000,000 \\ M_1(n) &= 2^{16} \cdot 5^{6} \\ M_2(n) &= 2^{2^4} \cdot 5^{2\cdot 3} \\ M_3(n) &= 2^{2^{2^2}} \cdot 5^{2\cdot 3}. \end{aligned}$$

Much of Mullin's and Gillman's work involved generalizing arithmetic functions. For example, Gillman defined the following generalization of the Möbius function.

$$\mu^*(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if the mosaic of } n > 1 \text{ has any prime number repeated} \\ (-1)^k & \text{if the mosaic of } n > 1 \text{ has no prime repeated, where } k \text{ is} \\ & \text{the number of } \textit{distinct} \text{ primes in the mosaic of } n. \end{cases}$$

Note that with this definition, μ_1 is the traditional μ function.

Bildhauser, Erickson, Gillman and Tacoma extended this this work by generalizing the Ω, ω, and λ to a mosaic context [2009]. They introduced the concept of a mosaic divisor in hopes of generalizing functions such as ϕ to mosaics. They offered a more complicated version of the following definition.

Definition 1.3. *A* prime i-mivisor *is a mosaic which has at most i levels and has a single prime at the first level. The set of all i-*mivisor *includes*

(a) 1,

(b) $P_{j,i}^{(\beta_{1,1},\beta_{1,2},\cdots,\beta_{1,s_1})}$ *where* $1 \leq \beta_j \leq \alpha_j$*, and*

(c) a product of 1-relatively prime i-mivisor from part (b).

The following example demonstrates the 3-mivisors of an integer, denoted by $M_3(n)$.

Example 1.3.

$$n = 2^{3^{5^3\cdot 7}\cdot 5} \cdot 3^{7^{11^2}}$$

The prime 3-mivisors of n are $2^{3^{5\cdot 7}\cdot 5}$ *and* $3^{7^{11}}$*, and the set* $M_3(n)$ *is*

$$\{1,\ 2^{3^{5\cdot 7}\cdot 5},\ 2^{3^{5^2\cdot 7}\cdot 5},\ 2^{3^{5^3\cdot 7}\cdot 5},\ 3^{7^{11}},\ 3^{7^{11^2}},\ 2^{3^{5\cdot 7}\cdot 5}\cdot 3^{7^{11}},\ 2^{3^{5^2\cdot 7}\cdot 5}\cdot 3^{7^{11}},$$
$$2^{3^{5^3\cdot 7}\cdot 5}\cdot 3^{7^{11}},\ 2^{3^{5\cdot 7}\cdot 5}\cdot 3^{7^{11^2}},\ 2^{3^{5^2\cdot 7}\cdot 5}\cdot 3^{7^{11^2}},\ 2^{3^{5^3\cdot 7}\cdot 5}\cdot 3^{7^{11^2}}\}$$

While this definition of a mivisor led to a generalization of ϕ, the generalization did not behave as the common ϕ function. As a result, it did not have the desired properties of the ϕ function since it was not sensitive to the levels of a mosaic nor was it i-multiplicative. Hence, the goal of this paper is to continue this work by first defining multiplication on mosaics in hopes that we can define divisor concepts that lead to a nicer generalization.

2 Arithmetic Properties

Our approach to this problem, is to first define a system of arithmetic for mosaics that will hopefully lead to these nice generalizations. We begin by attempting to define a generalized addition and then move on to multiplication. Obviously any addition or multiplication system needs to be closed and associative. It would also be nice if they each had an identity and were commutative. Furthermore, we would like an addition or multiplication system which behaves like the respective integer operation when $i = 1$ and is sensitive to the levels of the mosaic much like the functions discussed by previous authors.

In order to investigate closure of operations, we require an alternative characterization of mosaics.

Definition 2.1. *Mosaics are arrays of primes with all primes connected by multiplication or exponentiation, and within a given level, the primes may appear only once as an exponent on any particular base one level lower.*

Theorem 2.1. Definition 1.1 *is equivalent to the characterization given in* Definition 2.1.

Proof. To do this proof, we are showing that a structure formed from Definition 2.1 can be constructed from the method described in Definition 1.1. Further the mosaic that results from the construction of Definition 1.1, also satisfies Definition 2.1. Both Definition 1.1 and Definition 2.1 follow from the Fundamental Theorem of Arithmetic which provides a unique decomposition for any composite number into primes, which leaves primes at all levels. It also results in a product of powers, which is to say all primes are connected by multiplication or exponentiation. Finally, the FTA results in distinct primes, in other words within a given level, each prime only appears once on any particular base. □

Since there is a bijection between the positive integers and their mosaics, the following two definitions are very natural first steps. We begin with *Box-Plus* addition where M(0)=0 and M(1)=1.

Definition 2.2. *The operation* $\boxplus$ *on the set of mosaics is given by* $M(a) \boxplus M(b) = M(a+b)$ *where* a *and* b *are non-negative integers with mosaics* $M(a)$ *and* $M(b)$ *respectively.*

Here is an example of Box-Plus addition.

Example 2.1.

$$\begin{aligned} 2^{3^2} \boxplus 2^{2^2} &= M(512) \boxplus M(16) \\ &= M(528) \\ &= 2^{2^2} \cdot 11 \cdot 3 \end{aligned}$$

Theorem 2.2. *Box-Plus is closed, associative, has an identity and is commutative.*

Proof. Box-Plus is closed since we always re-expand into a mosaic.
Box-Plus is associative since

$$\begin{aligned} (M(a) \boxplus M(b)) \boxplus M(c) &= M(a+b) \boxplus M(c) \\ &= M((a+b)+c) \\ &= M(a+(b+c)) \\ &= M(a) \boxplus M(b+c) \\ &= M(a) \boxplus (M(b) \boxplus M(c)). \end{aligned}$$

Box-Plus is commutative since

$$\begin{aligned} M(a) \boxplus M(b) &= M(a+b) \\ &= M(b+a) \\ &= M(b) \boxplus M(a). \end{aligned}$$

$M(0)$ is the additive identity since

$$\begin{aligned} M(0) \boxplus M(a) &= M(0+a) \\ &= M(a). \end{aligned}$$

□

We define a multiplication which operates in a similar fashion to Box-Plus addition, called *Box-Dot* multiplication.

Definition 2.3. *The operation* $\boxdot$ *on the set of mosaics is given by* $M(a) \boxdot M(b) = M(ab)$ *where* a *and* b *are integers with mosaics* $M(a)$ *and* $M(b)$ *respectively.*

Here is an example of Box-Dot multiplication.

Example 2.2.

$$2^{2^2} \boxdot 2 \cdot 3^2 = M(16) \boxdot M(18) = M(288) = 2^5 \cdot 3^2.$$

Theorem 2.3. *Box-Dot is closed, associative, has an identity and is commutative.*

Proof. Box-Dot is commutative since

$$\begin{aligned} M(a) \boxdot M(b) &= M(a \cdot b) \\ &= M(b \cdot a) \\ &= M(b) \boxdot M(a). \end{aligned}$$

Box-Dot is closed since we always re-expand into a mosaic.
Box-Dot is associative since

$$\begin{aligned} (M(a) \boxdot M(b)) \boxdot M(c) &= M(a \cdot b) \boxdot M(c) \\ &= M((a \cdot b) \cdot c) \\ &= M(a \cdot (b \cdot c)) \\ &= M(a) \boxdot M(b \cdot c) \\ &= M(a) \boxdot (M(b) \boxdot M(c)). \end{aligned}$$

$M(1)$ is the identity since

$$\begin{aligned} M(1) \boxdot M(a) &= M(1 \cdot a) \\ &= M(a). \end{aligned}$$

□

Theorem 2.4. *Box-Dot multiplication is distributive over Box-Plus addition.*

Proof.

$$\begin{aligned} M(a) \boxdot (M(b) \boxplus M(c)) &= M(a) \boxdot (M(b + c)) \\ &= M(a \cdot b + a \cdot c) \\ &= M(a \cdot b) \boxplus M(a \cdot c) \\ &= (M(a) \boxdot M(b)) \boxplus (M(a) \boxdot M(c)) \end{aligned}$$

□

We can think of Box arithmetic as being sensitive to levels as we specified in our desired properties in *Section 2*. We can expand through any arbitrary i^{th} level and find the corresponding integer for that partially expanded mosaic. Iterate the Box operand on the two integers and then re-expand to the i^{th} level.

Theorem 2.5. *$M(a)|M(b)$ if and only if $a|b$.*

Proof. $M(a)|M(b)$
$\Leftrightarrow$ There exists $M(c)$ such that $M(a) \boxdot M(c) = M(b)$
$\Leftrightarrow M(a \cdot c) = M(b)$
$\Leftrightarrow$ There exists c such that $a \cdot c = b$
$\Leftrightarrow a|b$ □

Corollary 2.1. *$\boxdot$ is equivalent to regular integer multiplication.*

Corollary 2.2. *Any operation $*$ on mosaics that has the property $M(a) * M(b) = M(a \cdot b)$ is equivalent to $\boxdot$.*

Therefore, anything with this property is going to be equivalent to *Box-Dot* multiplication and will therefore not be interesting in terms of having different properties.

Corollary 2.2 shows us that *Box-Dot* multiplication is the only arithmetic operation on mosaics that behaves like integer arithmetic. Since we know we cannot achieve these properties, we can now move on to other multiplication systems that have different properties.

3 Mosaic Arithmetic

As we have seen in *Corollary 2.2*, we should abandon any attempts at defining an arithmetic that works similarly to integer arithmetic since it will always be equivalent to box-arithmetic. Therefore, we consider a multiplication system that is consistent with regular arithmetic when $i = 1$, but works differently otherwise.

3.1 Circle-Star Multiplication

Circle-Star multiplication mirrors integer multiplication in the sense that you add powers on a common base; however, in circle-star multiplication, the base can be more than just one level. The divisors that subsequently fall out of Circle-Star multiplication appear to be similar to the *prime i-mivisors* defined by Bildhauser, Erickson, Gillman, and Tacoma [2009], which motivated us to consider this multiplication system.

Definition 3.1. *Letting*

$$a = \prod p_i^{\alpha_i},$$

we have

$$M(a) = \prod p_i^{M(\alpha_i)},$$

and we can define

$$M_{p_i}(a) = p_i^{M(\alpha_i)}.$$

Thus,

$$M(a) = \prod M_{p_i}(a).$$

With this representation of a mosaic, we define the following multiplication.

Definition 3.2. *Circle-Star is denoted*

$$M(a) \circledast M(b) = \prod M_{p_i}(a) \circledast \prod M_{q_i}(b)$$

and is found by iterating the following processes.

1. *Take all pieces of the mosaic such that $p_m = q_n$ and expand through j-levels where j is the highest level for which $M_{p_m}(a)$ and $M_{q_n}(b)$ are identical. Add vector exponents at the $(j+1)^{st}$ level, put each element of the vector into mosaic form and append at the $(j+1)^{st}$ level of $M_{p_m}(a)$.*
2. *Append any mosaic left over to the end of the product by multiplication.*

As shown below, up through a common base between the factors nothing will change.

Example 3.1.

$$2^{3^3} \circledast 2^{3^5} = 2^{3^{M(3+5)}} = 2^{3^{2^3}}$$

In the case that there is a difference in the number of levels between the factors, the mosaics will need to be collapsed down so the they both have any remaining composits at the $(j+1)^{st}$ level.

Example 3.2.

$$2^{3^{2^3}} \circledast 2^{3^3} = 2^{3^{M(8)}} \circledast 2^{3^3} = 2^{3^{11}}$$

At the j^{th} level, should there be multiple bases, treat the exponents at the $(j+1)^{st}$ level as vectors and add accordingly.

Example 3.3.

$$2^{3^2 \cdot 5^3} \circledast 2^{3^5 \cdot 5^7} = 2^{3^7 \cdot 5^{M(10)}} = 2^{3^7 \cdot 5^{2 \cdot 5}}$$

In the event that there are multiple primes appearing at the first level, locate any common bases between $M(a)$ and $M(b)$ and combine accordingly.

Example 3.4.

$$\begin{aligned} 2^{3^2 \cdot 7} \cdot 3^{7^5 \cdot 5} \cdot 5^7 \circledast 2^{5^2 \cdot 7} \cdot 3^{7^3 \cdot 5} &= 2^{M(3 \cdot 3 \cdot 7)} \cdot 3^{7^5 \cdot 5} \cdot 5^7 \circledast 2^{M(5 \cdot 5 \cdot 7)} \cdot 3^{7^3 \cdot 5} \\ &= 2^{M(63+175)} \cdot 3^{7^{M(5+3)} \cdot 5^{M(1+1)}} \cdot 5^7 \\ &= 2^{2 \cdot 7 \cdot 17} \cdot 3^{7^{2^3} \cdot 5^2} \cdot 5^7 \end{aligned}$$

Theorem 3.1. *Circle-Star is closed, has an identity and is commutative.*

Proof. Circle-Star is closed since we always re-expand into a mosaic.

$M(1)$ is the multiplicative identity, since

$$\begin{aligned} M(1) \circledast M(a) &= M(a) \circledast M(1) \\ &= M(1) \circledast M(a) \\ &= M(1 \cdot a) \\ &= M(a). \end{aligned}$$

Circle-Star is commutative, since we maintain a common base and then add on the $(j+1)^{st}$ level, integer addition is associative. For cases in which there is no common base prime combination we offer the following proof of commutativity.

$$\begin{aligned} M(a) \circledast M(b) &= \prod M_{p_i}(a) \circledast M_{q_j}(b) \\ &= \prod_{p_i = q_j} M_{p_i}(a) \prod_{p_i \neq q_j} M_{p_i}(a) \circledast \prod_{q_j = p_i} M_{q_j}(b) \prod_{q_j \neq p_i} M_{q_j}(b) \end{aligned}$$

Letting $M_{p_i}(\cdot)$ be the product of $M_{p_i}(a) \circledast M_{q_j}(b)$. When $p_i = q_j$ we show commutativity as follows.

$$\begin{aligned} &= \prod_{p_i = q_j} M_{p_i}(\cdot) \prod_{p_i \neq q_j} M_{p_i}(a) \prod_{q_j \neq p_i} M_{q_j}(b) \\ &= \prod_{q_j \neq p_i} M_{q_j}(b) \prod_{p_i = q_j} M_{p_i}(\cdot) \prod_{p_i \neq q_j} M_{p_i}(a) \\ &= \prod_{q_j = p_i} M_{q_j}(b) \prod_{q_j \neq p_i} M_{q_j}(b) \circledast \prod_{p_i = q_j} M_{p_i}(a) \prod_{p_i \neq q_j} M_{p_i}(a) \end{aligned}$$

□

Unfortunately, Circle-Star is not associative, as demonstrated by the following counterexample.

Example 3.5.

$$\begin{aligned}(2^{3^2} \circledast 2^{3^3}) \circledast 2^{5^2} &= 2^{3^5} \circledast 2^{5^2} \\ &= 2^{M(243)} \circledast 2^{M(25)} \\ &= 2^{M(268)}. \\ 2^{3^2} \circledast (2^{3^3} \circledast 2^{5^2}) &= 2^{3^2} \circledast (2^{M(27)} \circledast 2^{M(25)}) \\ &= 2^{M(9)} \circledast 2^{M(52)} \\ &= 2^{M(61)}. \\ \text{Clearly, } 2^{M(268)} &\neq 2^{M(61)}.\end{aligned}$$

Because Circle-Star is not associative, it fails to be a useful or interesting operation, even though when the first level is common between two primes it does act like regular arithmetic. Unfortunately, we cannot control on what level we multiply with Circle-Star multiplication, so it also fails to meet our goals for mosaic arithmetic as layed out in our opening paragraph of *Section 2*. Therefore, we continue to explore alternative systems of arithmetic.

3.2 P-Arithmetic

We would like to generalize addition and multiplication operations on mosaics that do not refer to the integer associated with the mosaic, as Box-Arithmetic does. Ideally, we would like these operations to have an identity, as well as be associative and commutative. Therefore we turn our attention to an ordered system of primes in order to deal with mosaics. Since mosaics consist only of arrays of primes, it is logical to define a context in which the only numbers that exist are 0, 1, and all the primes. So let $P = \{p_0 = 0, p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5, p_5 = 7, etc.\}$.

We can add and multiply in P by operating on the indices and then finding the prime at that particular index. We have defined these operations below.

Definition 3.3. *Circle-Plus on the set P is given by*

$$p_i \oplus p_j = p_{i+j}$$

Example 3.6.

$$2 \oplus 11 = p_2 \oplus p_6 = p_{2+6} = p_8 = 17$$

Definition 3.4. *Circle-Times on the set P is given by*

$$p_i \otimes p_j = p_{i \times j}$$

Example 3.7.

$$3 \otimes 17 = p_3 \times p_8 = p_{24} = 83$$

Theorem 3.2. *Circle-Plus is closed, associative, has an identity and is commutative.*

Proof. Circle-Plus is trivially closed on P.
p_0 is the additive identity on P since

$$\begin{aligned}p_0 \oplus p_i &= p_{0+i} \\ &= p_i.\end{aligned}$$

Circle-Plus is associative on P since

$$\begin{aligned}(p_i \oplus p_j) \oplus p_k &= p_{i+j} \oplus p_k \\ &= p_{(i+j)+k} \\ &= p_{i+(j+k)} \\ &= p_i \oplus p_{j+k} \\ &= p_i \oplus (p_j \oplus p_k).\end{aligned}$$

Circle-Plus is commutative on P since

$$\begin{aligned} p_i \oplus p_j &= p_{i+j} \\ &= p_{j+i} \\ &= p_j \oplus p_i. \end{aligned}$$

□

Theorem 3.3. *Circle-Times is closed, associative, has an identity and is commutative.*

Proof. Circle-Times is trivially closed in P.

p_1 is the multiplicative identity on P since

$$\begin{aligned} p_1 \otimes p_i &= p_{1\times i} \\ &= p_i \end{aligned}$$

Circle-Times is commutative on P since

$$\begin{aligned} p_i \otimes p_j &= p_{i\times j} \\ &= p_{j\times i} \\ &= p_j \otimes p_i. \end{aligned}$$

Circle-Times is associative on P since

$$\begin{aligned} (p_i \otimes p_j) \otimes p_k &= p_{i\times j} \otimes p_k \\ &= p_{(i\times j)\times k} \\ &= p_{i\times (j\times k)} \\ &= p_i \otimes p_{j\times k} \\ &= p_i \otimes (p_j \otimes p_k). \end{aligned}$$

□

Theorem 3.4. *Circle-Times distributes over Circle-Plus.*

Proof.

$$\begin{aligned} p_a \otimes (p_b \oplus p_c) &= p_a \otimes p_{b+c} \\ &= p_{a\times(b+c)} \\ &= p_{(a\times b)+(a\times c)} \\ &= p_{a\times b} \oplus p_{a\times c} \\ &= (p_a \otimes p_b) \oplus (p_a \otimes p_c) \end{aligned}$$

□

If we let $-p_i = p_{-i} \in -P$ and consider the set $-P \cup P$, we can expand this arithmetic to form a unit ring.

Theorem 3.5. *There exists an additive inverse for each p in $-P \cup P$.*

Proof. For any prime p_i, there exists a prime denoted by p_{-i} such that

$$\begin{aligned} p_i \oplus p_{-i} &= p_{i+(-i)} \\ &= p_0 \\ &= 0 \end{aligned}$$

□

Theorem 3.6. *$(-P \cup P, \otimes, \oplus)$ forms a unit ring.*

Proof. *Theorems 3.2, 3.3, 3.4* and *3.5* together show that $(-P \cup P, \otimes, \oplus)$ is a unit ring. □

In P, the primes would be any p_a, such that a is prime in on the set of integers. These numbers are prime since we can only obtain these numbers via multipying $p_a \otimes p_1$.

3.3 P-Arithmetic Applied to Mosaics

Now that we have established a unit ring on the set $-P \cup P$ we are able to generalize the Circle-Plus and Circle-Times operations to mosaics. This is desirable because these operations always result in primes and have interesting properties in terms of composites. These operations require us to use the alternative characterization of mosaics.

We can add on mosaics by iterating the Circle-Plus operation on each level as defined below.

Definition 3.5. *Circle-Plus is found by applying Circle-Times to each level of each addend separately. Then Circle-Plus across levels.*

Example 3.8.

$$\begin{aligned} 2^{3\cdot 7} \oplus 3^5 \cdot 7 &= {p_2}^{p_3 \otimes p_5} \oplus {p_3}^{p_4} \otimes p_5 \\ &= {p_2}^{p_{3\times 5}} \oplus {p_{3\times 5}}^{p_4} \\ &= {p_{(2+15)}}^{p_{(15+4)}} \\ &= {p_{17}}^{p_{19}} \\ &= 53^{67} \end{aligned}$$

Theorem 3.7. *Circle-Plus is closed, associative, and commutative on the set of mosaics.*

Proof. Circle-Plus is trivially closed on the set of mosaics.

Circle-Plus is associative on the set of mosaics since Circle-Plus on P is associative and we can think of operating on mosaics as associating across each level independently.

Circle-Plus is commutative on the set of mosaics since Circle-Plus on P is commutative and we can think of operating on mosaics as commuting across each level independently. □

We can multiply in P on mosaics iterating the Circle-Times operation on each level as defined below.

Definition 3.6. *$M_1 \otimes M_2$ equals the mosaic resulting by applying $\otimes$ to all of the primes at each level.*

Example 3.9.

$$\begin{aligned} 2^{3\cdot 5} \otimes 2 \cdot 3^5 &= {p_2}^{p_3 \otimes p_4} \otimes p_2 \otimes {p_3}^{p_4} \\ &= {p_2}^{p_{3\cdot 4}} \otimes {p_{2\cdot 3}}^{p_4} \\ &= {p_{2\cdot 6}}^{p_{4\cdot 12}} \\ &= {p_{12}}^{p_{48}} \\ &= 31^{211} \end{aligned}$$

Theorem 3.8. *Circle-Times is closed, associative and commutative on the set of mosaics.*

Proof. Circle-Times is trivially closed on the set of mosaics.

Circle-Times is associative on the set of mosaics since Circle-Times on P is associative and we can think of operating on mosaics as associating across each level independently.

Circle-Times is commutative on the set of mosaics since Circle-Times on P is commutative and we can think of operating on mosaics as commuting across each level independently. □

Because of the nuances of our definitions for Circle-Plus and Circle-Times operations on mosaics, identities for these operations must be separately defined. To find the additive and multiplicative identities, you do not simplify across levels, you rather only Circle-Plus or Circle-Times on the first level (without simplifying across the first level).

Theorem 3.9. *Circle-Plus has an additive identity on the set of mosaics.*

Proof. p_0 is an additive identity on the set of mosaics since

$$\begin{aligned} p_0 \oplus p_i^{M(\alpha_i)} &= {p_{0+i}}^{M(\alpha_i)} \\ &= {p_i}^{M(\alpha_i)}. \end{aligned}$$

□

Theorem 3.10. *Circle-Times has a multiplicative identity on the set of mosaics.*

Proof. p_1 is a multiplicative identity on the set of mosaics since

$$\begin{aligned} p_1 \otimes {p_i}^{M(\alpha_i)} &= {p_{1\times i}}^{M(\alpha_i)} \\ &= {p_i}^{M(\alpha_i)} \end{aligned}$$

□

The product of mosaic multiplication will always consist of a single number at each level, due to the process of simplifying across levels prior to multiplying the factors. Therefore, any mosaic that has more than one number appearing at any level is a prime mosaic.

4 Further Work

Many open questions remain on the topic of mosaics and mosaic arithmetic. More exploration on P aritmetic on mosaics needs to be completed. We have offered a definition of additive and multiplicative identities, but we are not quite satisfied with the way it weakens our operation (since it deviates from our definition. We are unsure if there is another definition to better define an identity, or if the problem lies within the definition of our P-Arithmetic. However, we have the following conjecture.

Conjecture 4.1. *P-Arithmetic on mosaics has an additive and multiplicative identity.*

Eventually, if P arithmetic on mosaics has nice properties, we would like to generalize more number theoretic functions to mosaics.

We would also like to develop an aritmetic that encompasses more of our three desired properties. We offer the following conjecture about a multiplication which has this property.

Conjecture 4.2. *Let $M_{i,j}$ be a mosaic expanded through i levels then truncated at the k^{th} level. Let $*_i$ be a multiplication on the i^{th} level such that $(M_i(a) *_i M_i(b))_{i,j} = M_{i,j}(a) *_j M_{i,j}(b)$.*

5 References

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