Mosaics: A Prime-al Art *

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July 31, 2007

Abstract

The mosaic of the integer n is the array of prime numbers resulting from iterating the Fundamental Theorem of Arithmetic on n and on any resulting composite exponents. In this paper, we generalize several number theoretic functions to the mosaic of n, first based on the primes of the mosaic, second by examining several possible definitions of a divisor in terms of mosaics. Having done so, we examine properties of these functions.

1 Introduction

1.1 Mosaics

In a series of papers [3], [4], [5], and [6] Mullin introduced the number theoretic concept of the mosaic of n and explored several ideas related to it. He defined the mosaic of an integer n as follows:

Definition 1. The mosaic of the integer n is the array of prime numbers resulting from iterating the Fundamental Theorem of Arithmetic (FTA) on n and on any resulting composite exponents.

*This material is based upon work supported by the National Science Foundation under Grant No. 033870. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Example.

$$\begin{aligned}
n &= 1,024,000,000 \dashleftarrow \text{Use the FTA to find the prime factorization of } n. \\
&= 2^{16} \cdot 5^{6} \qquad \dashleftarrow \text{Apply FTA to composite exponents 16 and 6.} \\
&= 2^{2^{4}} \cdot 5^{2\cdot 3} \qquad \dashleftarrow \text{Apply FTA again to composite number 4.} \\
&= 2^{2^{2^{2}}} \cdot 5^{2\cdot 3} \qquad \dashleftarrow \text{The mosaic of the integer; only primes remain.}
\end{aligned}$$

The first function he introduced was $\psi(n)$, the product of all of the primes in the mosaic of n. As an example, we have:

Example.

$$\psi(2^{17^{3}} \cdot 3^{5^{5}}) = 2 \cdot 17 \cdot 3 \cdot 3 \cdot 5 \cdot 5 = 7650$$

In classical number theory, a function is multiplicative if and only if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. Mullin defined f to be *generalized* multiplicative if and only if $f(mn) = f(m)f(n)$ whenever the mosaics of m and n have no primes in common. He showed that $\psi(n)$ is generalized multiplicative and that any multiplicative function is also generalized multiplicative. He also generalized the classical Möbius function, $\mu(n)$, defining the generalized Möbius function as:

$$\mu^*(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if the mosaic of } n > 1 \text{ has any prime number repeated} \\ (-1)^k & \text{if the mosaic of } n > 1 \text{ has no prime repeated, where } k \text{ is} \\ & \text{the number of \textit{distinct} primes in the mosaic of } n. \end{cases}$$

Examples.

$$\mu^*(5^{3^{2}}) = (-1)^3 = -1$$

$$\mu^*(2^{2^{3}}) = 0$$

Similarly, Mullin defined a function to be generalized additive if and only if $f(mn) = f(m) + f(n)$ whenever the mosaics of m and n have no primes in common. He defined $\psi^*(n)$ as the sum of the primes in the mosaic of n and showed that this function was generalized additive.

Example.

$$\psi^*(5^{2^{3}\cdot 7} \cdot 11^{13^{19}}) = 5 + 2 + 3 + 7 + 11 + 13 + 19 = \psi^*(5^{2^{3}\cdot 7}) + \psi^*(11^{13^{19}})$$

1.2 Levels of the mosaic of n

Following Mullin's work, Gillman, in his papers [1] and [2], defined new functions on the mosaic of n. He used the concept of *levels* of the mosaic to describe the different tiers of exponentation, an example of which follows:

Example. $2^{3\cdot 5^7}$ ←-- The two is in the first level, the three and five are in the second level, and the seven is in the third level of the mosaic.

Using this idea, Gillman generalized the classical Möbius function as follows:

$$\mu_i(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if the mosaic of } n \text{ has duplicate primes in the first } i \text{ levels} \\ & \text{(including multiplicities at the } i^{th} \text{ level)} \\ (-1)^k & \text{if the mosaic of } n \text{ consists of } k \text{ distinct primes in the first} \\ & i \text{ levels.} \end{cases}$$

Along with the Möbius function, Gillman generalized the multiplicative property of functions to the levels of mosaics.

Definition 2. A function is i-multiplicative if and only if $f(mn) = f(m)f(n)$ when m and n have no primes in common in the first i levels of their mosaic.

Combining these concepts, Gillman went on to prove μ_i is i-multiplicative for all i.

Example. $\mu_3(3^{5\cdot 11^{19}} \cdot 7^{13^2}) = (-1)^7 = (-1)^4 \cdot (-1)^3 = \mu_3(3^{5\cdot 11^{19}}) \cdot \mu_3(7^{13^2})$

He then furthered Mullin's work on $\psi(n)$ by generalizing it to depend on the levels of the mosaic. The function $\psi_{j,i}(n)$ for $j > i$ is computed as follows:

Definition 3. Expand n through the first j levels of its mosaic; for each prime p on the i^{th} level of this expansion, multiply p by the product of the primes in the $(i+1)^{st}$ through j^{th} levels above p, including multiplicities of the primes at the j^{th} level.

Examples.

$$\psi_{6,3}(2^{3^{5^{7^{11\cdot 13^2}}}} \cdot 3^{2\cdot 5^{3\cdot 7}}) = 2^{3^{5\cdot 7\cdot 11\cdot 13\cdot 2}} \cdot 3^{2\cdot 5^{3\cdot 7}}$$

$$\psi_{\infty,1}(n) = \textit{product of all primes in the mosaic}$$

Gillman also introduced the idea of i-relatively prime mosaics. That is, two integers, m and n, are i-relatively prime when they have no primes in common in the first i levels of their mosaics.

Example. The integers with mosaics 2^{3^5} and 7^{11^3} are 2-relatively prime, but *not* 3-relatively prime.

1.3 Our Motivation

In this paper, we will introduce new families of functions defined on the mosaic of n and determine which of these are i-multiplicative or i-additive. In section two, the functions will depend only on the primes present in the first i levels and their multiplicities. In sections three, four, five, and six, we define alternative generalizations of the concept of a divisor and explore functions that result from these definitions. Finally, in the last section, we present ideas for future consideration for study of the mosaic of n.

2 Mosaic Functions

The functions Ω, ω, λ, and ψ^* are among the functions that can be generalized to the mosaic of n. We discuss their generalizations specifically because we found them to either be i-multiplicative or i-additive.

2.1 The Functions Ω_i and ω_i

In classical number theory, $\Omega(n)$ is the total number of primes in the factorization of n, including repetitions. We generalize that in the following definition:

Definition 4. $\Omega_i(n)$ is the total number of primes in the first i levels of the mosaic of n, including multiplicities on the i^{th} level.

Examples.

$$\Omega_2(2^{3^7} \cdot 5^{11^6}) = \Omega_2(2^{3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3} \cdot 5^{11\cdot11\cdot11\cdot11\cdot11\cdot11}) = 15$$

$$\Omega_3(7^{5^{7^5}\cdot 19} \cdot 19^{13^{7\cdot 11^3}}) = \Omega_3(7^{5^{7\cdot7\cdot7\cdot7\cdot7}\cdot 19} \cdot 19^{13^{7\cdot 11\cdot11\cdot11}}) = 14$$

Noting that 1-additive implies j-additive for all j, we have the following theorem. However, i-additive implies j-additive for $j > i$, as we see in the theorem for ω_i.

Theorem 1. *Ω_i is j-additive for all i and j.*

Proof. Let m and n be 1-relatively prime. Then $\Omega_i(mn)$ is summing the number of prime divisors of the product mn and the number of primes in levels two though i of the mosaic of mn, including multiplicities at the i^{th} level. Since m and n are 1-relatively prime, the first term of this sum can be written as the number of prime divisors of m plus the number of prime divisors of n. Similarly, the second term can be written as the number of primes in levels two through i of the mosaic of m plus the number of primes in levels two through i of the mosaic of n (including multiplicities at the i^{th} level in each of these sums). Rearranging these sums results in $\Omega_i(m) + \Omega_i(n)$. Thus Ω_i is 1-additive and therefore j-additive for all j. $\square$

Examples.

$$\Omega_2(2^{3^7} \cdot 5^{11^6}) = 15 = 8 + 7 = \Omega_2(2^{3^7}) + \Omega_2(5^{11^6})$$

$$\Omega_3(7^{5^{3\cdot 2^4}} \cdot 13^{11^{17^8}}) = 17 = 7 + 10 = \Omega_3(7^{5^{3\cdot 2^4}}) + \Omega_3(13^{11^{17^8}})$$

Note: The result that Ω_i is i-additive depends only on the first level of the mosaics having *distinct* primes. That is, it is necessary and sufficient that the first level of the mosaics have distinct primes in order that Ω_i be i-additive. We illustrate this in the following examples.

Example.

$$\Omega_2(2^{3^5} \cdot 3^3) = 8 = 6 + 2 = \Omega_2(2^{3^5}) + \Omega_2(3^3)$$

$$\Omega_2(3^{3^5} \cdot 3^3) = \Omega_2(3^{2\cdot 3\cdot 41}) = 4 \neq 8 = 6 + 2 = \Omega_2(3^{3^5}) + \Omega_2(3^3)$$

In classical number theory, $\omega(n)$ is the number of distinct primes in the prime factorization of n. We generalize $\omega(n)$ to the mosaic of n as follows:

Definition 5. $\omega_i(n)$ is the number of *distinct* primes in the first i levels of the mosaic of n.

Examples.

$$\omega_2(2^{3^7} \cdot 5^{11^6}) = \omega_2(2^{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3} \cdot 5^{11\cdot 11\cdot 11\cdot 11\cdot 11\cdot 11}) = 4$$

$$\omega_3(19^{13^{7\cdot 11^3}} \cdot 7^{5^{7^5}} \cdot 19) = \omega_3(19^{13^{7\cdot 11\cdot 11\cdot 11}} \cdot 7^{5^{7\cdot 7\cdot 7\cdot 7\cdot 7}\cdot 19}) = 5$$

Unlike Ω_i, ω_i is not j-additive for all j. Rather, as the following theorem demonstrates, it is j-additive for $j \geq i$.

Theorem 2. *ω_i is i-additive.*

Proof. Let m and n be i-relatively prime. Then $\omega_i(mn)$ is summing the number of prime divisors of the product mn and the distinct primes in levels two though i of the mosaic of mn (which must also be distinct from the prime divisors). Since m and n are relatively prime, the first term of this sum can be written as the prime divisors of m plus the prime divisors of n. Similarly, the second term can be written as the number of distinct primes in levels two through i of the mosaic of m plus the number of distinct primes in levels two through i of the mosaic of n. Thus $\omega_i(mn) = \omega_i(m) + \omega_i(n)$ and therefore ω_i is i-additive. $\square$

Example.

$$\omega_4(7^{11\cdot 13^{5^{3^2}}} \cdot 29^{17^{89^2}}) = 9 = 5 + 4 = \omega_4(7^{11\cdot 13^{5^{3^2}}}) + \omega_4(29^{17^{89^2}})$$

$$\omega_4(11^{5\cdot 11} \cdot 7^{19^3\cdot 53^{5^7}}) = 6 \neq 2 + 5 = \omega_4(11^{5\cdot 11}) + \omega_4(7^{19^3\cdot 53^{5^7}})$$

2.2 The Function λ_i

Classical number theory defines the Liouville function as $\lambda(n) = (-1)^{\Omega(n)}$. We generalize this in the obvious way:

Definition 6. $\lambda_i(n) = (-1)^{\Omega_i(n)}$

Example.

$$\lambda_2(2^{3^7} \cdot 5^{11^6}) = (-1)^{\Omega_2(2^{3^7}\cdot 5^{11^6})} = (-1)^{15} = -1$$

This leads to the following theorem, again recalling that 1-multiplicative implies j-multiplicative for all j.

Theorem 3. *λ_i is j-multiplicative for all j.*

Proof. Assume m and n are 1-relatively prime. Therefore,

$$\Omega_i(mn) = \Omega_i(m) + \Omega_i(n)$$

Thus, it follows that

$$\begin{aligned}\lambda_i(mn) &= (-1)^{\Omega_i(mn)} \\ &= (-1)^{\Omega_i(m)+\Omega_i(n)} \\ &= (-1)^{\Omega_i(m)}(-1)^{\Omega_i(n)} \\ &= \lambda_i(m)\lambda_i(n)\end{aligned}$$

λ_i is 1-multiplicative and therefore j-multiplicative for all j. □

Example.

$$\lambda_2(2^{3^7} \cdot 5^{11^6}) = (-1)^{\Omega_2(2^{3^7} \cdot 5^{11^6})} = (-1)^{15} = (-1)^8(-1)^7 = \lambda_2(2^{3^7})\lambda_2(5^{11^6})$$

2.3 The Function $\psi^*_{j,i}$

Mullin defined the function ψ as the product of all primes in a mosaic. Gillman later extended this to the levels of the mosaic by introducing the function $\psi_{j,i}$. Mullin also defined the function ψ^* to be the sum of the primes in a mosaic. We wanted to generalize this idea to the levels of the mosaic as well, so we define the function $\psi^*_{j,i}$ as follows:

Definition 7. For $j > i$, compute $\psi^*_{j,i}(n)$ as follows: Expand n through the first j levels of its mosaic; for each prime p on the i^{th} level of this expansion, add p to the sum of the primes in the $(i+1)^{st}$ through j^{th} levels above p, including the multiplicities of the primes at the j^{th} level.

Examples.

$$\psi^*_{4,2}(17^{11^{3^7} \cdot 19} \cdot 23^{2^{3^5}}) = 17^{11+3+7+19} \cdot 23^{2+3+5}$$

$$\psi^*_{4,1}(3^{5^{2^3} \cdot 7}) = 3+5+2+3+7 = 20$$

Similar to the previous functions, we found that for $\psi^*_{j,1}$ 1-additive, and therefore k-additive for all k, as the following theorem shows.

Theorem 4. *$\psi^*_{j,1}(n)$ is k-additive for all j and k.*

Proof. Let m and n be integers which are 1-relatively prime. $\psi^*_{j,1}(mn)$ is the sum of primes in the first j levels of the mosaic of mn, including multiplicities on the j^{th} level. This is equivalent to the sum of prime divisors of mn plus the sum of the primes in levels two through j of the mosaic of mn including multiplicities at the j^{th} level. Since m and n are relatively prime, the sum of prime divisors of mn can be written as the sum of prime divisors of m plus the sum of prime divisors of n. Similarly, the second term can be written as the sum of primes in levels two through j of m plus the number of primes in levels two through j of the mosaic of n including multiplicities at the j^{th} level in each. Rearranging these sums results in $\psi^*_{j,1}(m)+\psi^*_{j,1}(n)$. Thus $\psi^*_{j,1}$ is 1-additive and therefore k-additive for all k. $\square$

Example.

$$\begin{aligned}\psi^*_{3,1}(3^{7\cdot 11^{19}} \cdot 13^{2^{5^2}\cdot 13}) &= (3+7+11+19)+(13+2+5+5+13)\\ &= \psi^*_{3,1}(3^{7\cdot 11^{19}}) + \psi^*_{3,1}(13^{2^{5^2}\cdot 13})\end{aligned}$$

Interestingly, while $\psi^*_{j,i}$ is k-additive for all j and k when $i=1$, for any $i>1$, $\psi^*_{j,i}$ is 1-multiplicative and therefore k-multiplicative for all j and k.

Theorem 5. *For all $i>1$, $\psi^*_{j,i}(n)$ is k-multiplicative for all j and k.*

Proof. Let m and n be 1-relatively prime integers with prime factorizations $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ and $q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s}$, respectively. Because m and n are 1-relatively prime, $mn = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s}$. Further, because $i>1$, $\psi^*_{j,i}(mn)$ has the same first $(i-1)$ levels as the mosaic of mn and the i^{th} level is equal to the i^{th} level of $\psi_{j,i}(mn)$ with multiplication converted to addition. Thus the unchanged first level can be partitioned into the parts that have the same first $(i-1)$ levels as m and n and with the i^{th} levels equal to the i^{th} levels of $\psi_{j,i}(m)$ and $\psi_{j,i}(n)$ respectively with multiplication converted to addition.

$$\begin{aligned}\psi^*_{j,i}(mn) &= p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}q_1^{b_1}q_2^{b_2}\cdots q_s^{b_s}\\ &= (p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r})(q_1^{b_1}q_2^{b_2}\cdots q_s^{b_s})\\ &= \psi^*_{j,i}(m)\psi^*_{j,i}(n)\end{aligned}$$

where a_i and b_i are the 2^{nd} through i^{th} levels of the mosaic, with the $(i+1)^{st}$ to j^{th} levels brought down to the i^{th} level with multiplication converted to addition. Thus $\psi^*_{j,i}(mn)$ is 1-multiplicative and therefore k-multiplicative for all k. $\square$

Example.

$$\psi^*_{4,2}(17^{11^{3^7}\cdot 19} \cdot 23^{2^{3^5}}) = 17^{11+3+7+19}\cdot 23^{2+3+5} = \psi^*_{4,2}(17^{11^{3^7}\cdot 19})\cdot \psi^*_{4,2}(23^{2^{3^5}})$$

3 Submosaics

Many number theoretic functions are defined in terms of the divisors of n, so we must develop an analogous concept for the mosaic. In this and the following sections, we examine some possible definitions, and the resulting functions.

3.1 Definition of a Submosaic

A mosaic can be viewed as a connected graph where the primes in the mosaic are the vertices and there is an edge between vertices if one prime is multiplied by the other or one is an exponent of the other. In [4], Mullin introduced the concept of submosaics; therefore, submosaics seemed like a natural first step in our search for a divisor. Mullin defined a submosaic as the mosaic corresponding to a connected subgraph of the graph of the full mosaic. We found that this definition did not work like Mullin thought it would, and it did not help to create the i-multiplicative functions we were looking for. We decided to alter the definition to something more like divisors in the sense that removing a submosaic from a mosaic leaves another submosaic of the mosaic. This is similar to divisors because dividing a number by one of its divisors results in another divisor.

Definition 8. A restricted submosaic is a mosaic that corresponds to one of the connected subgraphs such that it leaves a connected subgraph when removed from the graph of the full mosaic. A restricted submosaic in the first i levels of n is found by expanding the mosaic of n through i levels and following the same process with multiplicities on the i^{th} level. We denote the set of all restricted submosaics of n in the first i levels by $S_i(n)$.

Example.

$$n = 7^{13} \cdot 11^{19}$$

$$S_2(7^{13} \cdot 11^{19}) = \{1,\ 13,\ 19,\ 7^{13},\ 11^{19},\ 7^{13} \cdot 11,\ 7 \cdot 11^{19},\ 7^{13} \cdot 11^{19}\}$$

3.2 The Function ${}_s\phi_i$

In classical number theory, $\phi(n)$ counts the number of integers less than or equal to n that are relatively prime to n. It is computed using the formula $\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$. Given this formula, we generalize ϕ using submosaics as follows:

Definition 9.

$${}_s\phi_i(n) = \sum_{d \in S_i(n)} \mu_i(d)(n \backslash d)$$

$(n \backslash d)$ is the integer resulting from the submosaic corresponding to the connected subgraph formed by removing the subgraph corresponding to the mosaic of d from the graph corresponding to the mosaic of n.

The pre-subscript s is used to denote the use of restricted submosaics. We need to use the pre-subscript because we will define other versions of the ϕ function later in the paper. The post-subscript, i, is used to determine which level of the mosaic is being examined.

Example.

$$\begin{aligned}
{}_s\phi_2(2^7 \cdot 13) &= (1)\big((2^7 \cdot 13)\backslash 1\big) \;+\; (-1)^1\big((2^7 \cdot 13)\backslash 7\big) + \;(-1)^1\big((2^7 \cdot 13)\backslash 13\big) \\
&\quad +\; (-1)^2\big((2^7 \cdot 13)\backslash 2^7\big) \;+\; (-1)^2\big((2^7 \cdot 13)\backslash (2 \cdot 13)\big) \\
&\quad +\; (-1)^3\big((2^7 \cdot 13)\backslash (2^7 \cdot 13)\big) \\
&= 1664 - 26 - 128 + 13 + 7 - 1 \\
&= 1529
\end{aligned}$$

3.3 The Functions ${}_s\tau_i$ and ${}_s\sigma_i$

The function $\tau(n)$ typically counts the number of divisors of an integer n. Our generalized τ function, ${}_s\tau_i(n)$, counts the number of restricted submosaics from the first i levels of n including multiplicities on the i^{th} level.

Definition 10.

$$ {}_s\tau_i(n) = \sum_{d \in S_i(n)} 1 $$

Example.

$$ n = 5^{19^7} \cdot 23 $$

$$ S_3(5^{19^7} \cdot 23) = \{1,\ 7,\ 23,\ 5^{19^7},\ 19^7,\ 5^{19} \cdot 23,\ 5 \cdot 23,\ 5^{19^7} \cdot 23\} $$

$$ {}_s\tau_3(5^{19^7} \cdot 23) = |S_3(5^{19^7} \cdot 23)| = 8 $$

Similarly, $\sigma(n)$ sums the divisors of an integer n, so ${}_s\sigma_i(n)$ is the sum of restricted submosaics from the first i levels of n including multiplicities on the i^{th} level.

Definition 11.

$$ {}_s\sigma_i(n) = \sum_{d \in S_i(n)} d $$

Example.

$$ {}_s\sigma_3(5^{19^7} \cdot 23) = 1 + 7 + 23 + 5^{19^7} + 19^7 + (5^{19} \cdot 23) + (5 \cdot 23) + (5^{19^7} \cdot 23) $$

3.4 Mullin's problem

We discovered that none of these functions are i-multiplicative. The problem is that if $d \in S_i(mn)$, then $d_1 \in S_i(m)$ and $d_2 \in S_i(n)$ for some d_1 and d_2 such that $d_1 d_2 = d$. However, $d_1 \in S_i(m)$ and $d_2 \in S_i(n)$ does not necessarily imply that $d_1 d_2 \in S_i(mn)$.

Example.

$$ 3 \in S_2(2^5 \cdot 7^3) \qquad 5 \in S_2(2^5 \cdot 7^3) $$

$$ 3 \cdot 5 \notin S_2(2^5 \cdot 7^3) $$

After discovering this, we looked again at Mullin's work with submosaics. If we let $C(n)$ be the set of all submosaics of n, he made the assumption that $C(mn) = C(m) \times C(n)$, but we discovered that this is not true. As we found with restricted submosaics, we also discovered that $d_1 \in C(m)$ and $d_2 \in C(n)$ does not necessarily imply that $d_1 d_2 \in C(mn)$.

Example.

$$2 \in C(13^2 \cdot 17^5) \quad 17 \in C(13^2 \cdot 17^5)$$
$$2 \cdot 17 \notin C(13^2 \cdot 17^5)$$

4 Mivisors

Since the assumptions made by Mullin about functions with submosaics were false, that is, they did not act as a divisor of the full mosaic, our quest for a divisor continued.

4.1 Definition of a Mivisor

Mivisor, from mosaic divisor, was our first generalization of a divisor for mosaics, which we examine in this section.

Definition 12. d is a mivisor of n if and only if its base primes are a subset of the base primes of n and the exponents are the same as for each base prime. We denote the set of all mivisors of n by $M(n)$.

Example. Let $n = 2^{3^5} \cdot 3^{5^{17}} \cdot 5$. The mivisors of n are:

$$M(2^{3^5} \cdot 3^{5^{17}} \cdot 5) = \{1,\ 2^{3^5},\ 3^{5^{17}},\ 5,\ 2^{3^5} \cdot 3^{5^{17}},\ 2^{3^5} \cdot 5,\ 3^{5^{17}} \cdot 5,\ 2^{3^5} \cdot 3^{5^{17}} \cdot 5\}$$

We chose this structure because the mosaic above each prime in the first level is fixed and we did not want to change the entire mosaic when dividing by the mivisor.

Lemma. *$M(mn) = M(m) \times M(n)$ when m and n are i-relatively prime.*

Proof. Let the prime-power factorizations of m and n be $p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ and $q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$, respectively. Since m and n are i-relatively prime, the set of primes in the first level of m and the set of primes in the first level of n have no common elements. Therefore, the prime-power factorization of mn is

$$mn = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}.$$

If $d \in M(mn)$, then

$$d = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} q_1^{f_1} q_2^{f_2} \cdots q_t^{f_t}$$

where e_i is either 0 or a_i for $i = 1, 2, \ldots, s$ and f_j is either 0 or b_i for $j = 1, 2, \ldots, t$. Now let $d_1 = gcd(d, m)$ and $d_2 = gcd(d, n)$. Then

$$d_1 = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \text{ and } d_2 = q_1^{f_1} q_2^{f_2} \cdots q_t^{f_t}.$$

It follows that $d_1 \in M(m)$ and $d_2 \in M(n)$. Since $d = d_1 d_2$, $d \in M(m) \times M(n)$

Similarily, if $d_1 \in M(m)$ and $d_2 \in M(n)$, then $d_1 d_2 \in M(m) \times M(n)$ and $d_1 d_2 \in M(mn)$. Thus the sets are the same. □

We are able to prove the following theorem using this lemma.

Theorem 6. *If f is an i-multiplicative function, then $F(n) = \sum_{d \in M(n)} f(d)$, is also i-multiplicative.*

Proof. To show that F is an i-multiplicative function, we must show that when m and n are i-relatively prime, $F(mn) = F(m)F(n)$. So assume that m and n are i-relatively prime. We know

$$F(mn) = \sum_{d \in M(mn)} f(d).$$

By the lemma, because m and n are i-relatively prime, each mivisor of mn can be written as the product $d = d_1 d_2$ where $d_1 \in M(m)$ and $d_2 \in M(n)$, and d_1 and d_2 are i-relatively prime.

$$F(mn) = \sum_{d_1 \in M(m), d_2 \in M(n)} f(d_1 d_2)$$

Because f is i-multiplicative, and d_1 and d_2 are i-relatively prime, we see that

$$\begin{aligned} F(mn) &= \sum_{d_1 \in M(m)} \sum_{d_2 \in M(n)} f(d_1) f(d_2) \\ &= \sum_{d_1 \in M(m)} f(d_1) \sum_{d_2 \in M(n)} f(d_2) \\ &= F(m)F(n). \end{aligned}$$

□

4.2 The Function ${}_{m_1}\phi_i$

Using the classical formula for ϕ and the mivisors, we are able to redefine ϕ for mosaics. Unfortunately, we lose the conceptual definition of ϕ counting something.

Definition 13.

$${}_{m_1}\phi_i(n) = \sum_{d \in M(n)} \mu_i(d) \frac{n}{d}$$

Example.

$${}_{m_1}\phi_2(2^3) = (-1)^2 \frac{2^3}{2^3} + (-1)^0 \frac{2^3}{1} = 9$$

Having redefined our functions in terms of mivisors, we wanted to know if the typical number theoretic properties of these functions still held. We found our function ${}_{m_1}\phi_i$ to, in fact, be i-multiplicative.

Theorem 7. *${}_{m_1}\phi_i$ is i-multiplicative.*

Proof.

$$\begin{aligned}
{}_{m_1}\phi_i(mn) &= \sum_{d\in M(mn)} \mu_i(d)\frac{mn}{d} \\
&= \sum_{d\in M(m)\times M(n)} \mu_i(d)\frac{mn}{d} \\
&= \sum_{d_1\in M(m)} \sum_{d_2\in M(n)} \mu_i(d_1)\mu_i(d_2)\frac{m}{d_1}\frac{n}{d_2} \\
&= \sum_{d_1\in M(m)} \mu_i(d_1)\frac{m}{d_1} \sum_{d_2\in M(n)} \mu_i(d_2)\frac{n}{d_2} \\
&= {}_{m_1}\phi_i(m) {}_{m_1}\phi_i(n)
\end{aligned}$$

□

4.3 The Functions ${}_{m_1}\tau$ and ${}_{m_1}\sigma$

The function ${}_{m_1}\tau(n)$ counts the number of mivisors of n.

Definition 14.

$${}_{m_1}\tau(n) = \sum_{d\in M(n)} 1$$

Example.

$$n = 11^7 \cdot 17^3$$
$$M(11^7 \cdot 17^3) = \{1,\ 11^7,\ 17^3,\ 11^7 \cdot 17^3\}$$
$${}_{m_1}\tau(11^7 \cdot 17^3) = |M(11^7 \cdot 17^3)| = 2^2$$

Note: The function ${}_{m_1}\tau$ will always result in a power of two with the exponent equal to the number of primes in the first level.

Using Theorem 6 with $f(d) = 1$, which is obviously i-multiplicative, we obtain

Corollary. *${}_{m_1}\tau$ is i-multiplicative for all i.*

Example.

$${}_{m_1}\tau(11^7 \cdot 17^3) = |M(11^7 \cdot 17^3)| = 4 = 2 \cdot 2 = {}_{m_1}\tau(11^7) \cdot {}_{m_1}\tau(17^3)$$

Our generalized σ function, ${}_{m_1}\sigma_i(n)$, sums the mivisors of n. That is

Definition 15.

$${}_{m_1}\sigma(n) = \sum_{d\in M(n)} d$$

Example.

$${}_{m_1}\sigma(11^7 \cdot 17^3) = 1 + 11^7 + 17^3 + 11^7 \cdot 17^3$$

Using Theorem 6 with $f(d) = d$, which is i-multiplicative, we have the following

Corollary. *${}_{m_1}\sigma$ is i-multiplicative for all i.*

Example.

$${}_{m_1}\sigma(11^7 \cdot 17^3) = 1 + 11^7 + 17^3 + 11^7 \cdot 17^3 = (1 + 11^7)(1 + 17^3) = {}_{m_1}\sigma(11^7) \cdot {}_{m_1}\sigma(17^3)$$

We noticed that for all three of these functions we were using the same set of mivisors for all i given any n. The values of ${}_{m_1}\tau$ and ${}_{m_1}\sigma$ did not change as i varied, and ${}_{m_1}\phi_i$ only changed with i because μ_i changed as i varied. It is because of these problems that we changed our functions to use $\psi_{i,1}$.

4.4 The Functions ${}_{m_2}\phi_i$, ${}_{m_2}\tau_i$, and ${}_{m_2}\sigma_i$

We tried to see if our function would make more sense if the set we took the sum over changed with i as well, so we included $\psi_{i,1}$ in our formulas for ${}_{m_2}\phi_i$, ${}_{m_2}\tau_i$, and ${}_{m_2}\sigma_i$. To make our function different for different values of i, we consider ${}_{m_2}f_i(n) = {}_{m_1}f_i \circ \psi_{i,1}(n)$. In particular,

Definition 16.

$${}_{m_2}\phi_i(n) = {}_{m_1}\phi_i(\psi_{i,1}(n)) = \sum_{d \in M(\psi_{i,1}(n))} \mu_i(d) \frac{\psi_{i,1}(n)}{d}$$

Example.

$${}_{m_2}\phi_2(2^3) = (-1)^2 \frac{2 \cdot 3}{2 \cdot 3} + (-1)^1 \frac{2 \cdot 3}{2} + (-1)^1 \frac{2 \cdot 3}{3} + (-1)^0 \frac{2 \cdot 3}{1} = 2$$

Fortunately, ${}_{m_2}\phi_i$ behaves nicely, as we see in the following.

Theorem 8. *${}_{m_2}\phi_i$ is i-multiplicative.*

Proof. Let m and n be i-relatively prime integers.

$$\begin{aligned} {}_{m_2}\phi_i(mn) &= {}_{m_1}\phi_i(\psi_{i,1}(mn)) \\ &= {}_{m_1}\phi_i(\psi_{i,1}(m) \cdot \psi_{i,1}(n)) \\ &= {}_{m_1}\phi_i(\psi_{i,1}(m)) \cdot {}_{m_1}\phi_i(\psi_{i,1}(n)) \\ &= {}_{m_2}\phi_i(m) \cdot {}_{m_2}\phi_i(n) \end{aligned}$$

□

Clearly, if the integer n is squarefree, then ${}_{m_1}\phi_i(n) = {}_{m_2}\phi_i(n) = \phi(n)$.

Again, we use $\psi_{i,1}$ to fix the problem with ${}_{m_1}\tau_i$.

Definition 17.

$${}_{m_2}\tau_i(n) = {}_{m_1}\tau(\psi_{i,1}(n)) = \sum_{d \in M(\psi_{i,1}(n))} 1$$

Example.

$$\begin{aligned}{}_{m_2}\tau_2(2^{3^5}) &= \text{number of mivisors of } \psi_{2,1}(2^{3^5}) \\ &= \text{number of mivisors of } 2\cdot 3^5 \\ &= 2^2\end{aligned}$$

The function ${}_{m_2}\tau_i$ will also always result in a power of two. In this case the exponent is equal to the number of *distinct* primes in the first i levels of n. Another way to describe the results of ${}_{m_2}\tau_i$ is in terms of ω_i, which was defined earlier. Hence, another formula is ${}_{m_2}\tau_i(n) = 2^{\omega_i(n)}$.

Example.

$$\begin{aligned}{}_{m_2}\tau_3(3^{7^2}) &= 2^{\omega_3(3^{7^2})} \\ &= 2^3\end{aligned}$$

where 2^3 is the total number of mivisors of $\psi_{3,1}(3^{7^2})$ and the exponent, 3, is the number of distinct primes of 3^{7^2}.

Using the corollary that states that ${}_{m_1}\tau$ is i-multiplicative for all i, we can prove the following

Corollary. *${}_{m_2}\tau_i$ is i-multiplicative.*

Proof. Let m and n be i-relatively prime integers.

$$\begin{aligned}{}_{m_2}\tau_i(mn) &= {}_{m_1}\tau_i(\psi_{i,1}(mn)) \\ &= {}_{m_1}\tau(\psi_{i,1}(m)\cdot\psi_{i,1}(n)) \\ &= {}_{m_1}\tau(\psi_{i,1}(m))\cdot {}_{m_1}\tau(\psi_{i,1}(n)) \\ &= {}_{m_2}\tau_i(m)\cdot {}_{m_2}\tau_i(n)\end{aligned}$$

□

Example.

$$\begin{aligned}{}_{m_2}\tau_2(2^{3^2\cdot 5}\cdot 7^{11^7}\cdot 13^{17^{13}}\cdot 31) &= 2^8 \\ &= 2^5\cdot 2^3 \\ &= {}_{m_2}\tau_2(2^{3^2\cdot 5}\cdot 7^{11^7})\cdot {}_{m_2}\tau_2(13^{17^{13}}\cdot 31)\end{aligned}$$

We re-defined ${}_m\sigma_i$ using $\psi_{j,i}$ as follows:

Definition 18.

$${}_{m_2}\sigma_i(n) = {}_{m_1}\sigma_i(\psi_{i,1}(n)) = \sum_{d\in M(\psi_{i,1}(n))} d$$

Example.

$${}_{m_2}\sigma_3(3^{7^2}) = 1+3+7+2+(3\cdot 7)+(3\cdot 2)+(7\cdot 2)+(3\cdot 7\cdot 2) = 96$$

Using the corollary that states that ${}_{m_1}\sigma$ is i-multiplicative for all i, we can prove the following

Corollary. *${}_{m_2}\sigma_i$ is i-multiplicative.*

Proof. Let m and n be relatively prime integers.

$$\begin{aligned}
{}_{m_2}\sigma_i(mn) &= {}_{m_1}\sigma_i(\psi_{i,1}(mn)) \\
&= {}_{m_1}\sigma(\psi_{i,1}(m)\psi_{i,1}(n)) \\
&= {}_{m_1}\sigma(\psi_{i,1}(m)) \cdot {}_{m_1}\sigma(\psi_{i,1}(n)) \\
&= {}_{m_2}\sigma_i(m) \cdot {}_{m_2}\sigma_i(n)
\end{aligned}$$

□

Example.

$$\begin{aligned}
&{}_{m_2}\sigma_2(7^{11^3} \cdot 5^{19^2}) \\
&= 1 + 7 + 11^3 + 5 + 19^2 + (7 \cdot 11^3) + (7 \cdot 5) + (7 \cdot 19^2) + (11^3 \cdot 5) + (11^3 \cdot 19^2) \\
&\quad + (5 \cdot 19^2) + (7 \cdot 11^3 \cdot 5) + (7 \cdot 5 \cdot 19^2) + (7 \cdot 11^3 \cdot 19^2) + (11^3 \cdot 5 \cdot 19^2) \\
&\quad + (7 \cdot 11^3 \cdot 5 \cdot 19^2) \\
&= (1 + 5 + 19^2 + 5 \cdot 19^2) + 7(1 + 5 + 19^2 + 5 \cdot 19^2) + 11^3(1 + 5 + 19^2 + 5 \cdot 19^2) \\
&\quad + 7 \cdot 11^3(1 + 5 + 19^2 + 5 \cdot 19^2) \\
&= (1 + 7 + 11^3 + 7 \cdot 11^3)(1 + 5 + 19^2 + 5 \cdot 19^2) \\
&= {}_{m_2}\sigma_2(7^{11^3}) \cdot {}_{m_2}\sigma_2(5^{19^2})
\end{aligned}$$

5 New-visors

While mivisors worked as a divisor of mosaics, we were displeased. We wanted to find a divisor that did not rely on $\psi_{i,1}$ in order for functions to change as i varied. Consequently, we re-examined the functions ϕ, τ, and σ with a differently structured divisor.

5.1 Definition of a New-Visor

New-visors, from new types of divisors, are the next concept analogous to divisors which we examine.

Definition 19. A number d is a new-visor of n if d and n have the same mosaic through the first i levels and when a_j is an element of the unfactored $(i+1)$ row of n and b_j is the corresponding element of the unfactored $(i+1)$ row of d, $0 \le b_j \le a_j$ for each a_j and b_j. We denote the set of i-level new-visors of n by $N_i(n)$.

Note: When $i = 1$, new-visors are equivalent to normal divisors, unlike mivisors which do not correspond.

Examples.

$$n = 2^{2\cdot 3^3} \cdot 5$$

$$N_2(2^{2\cdot 3^3} \cdot 5) = \{2^{2^0 \cdot 3^0} \cdot 5,\ 2^{2^0 \cdot 3^1} \cdot 5,\ ...,\ 2^{2^1 \cdot 3^2} \cdot 5,\ 2^{2^1 \cdot 3^3} \cdot 5\}$$

$$N_3(2^{2\cdot 3^3} \cdot 5) = \{2^{2\cdot 3^{3^0}} \cdot 5,\ 2^{2\cdot 3^{3^1}} \cdot 5\}$$

$$N_4(2^{2\cdot 3^3} \cdot 5) = \{2^{2\cdot 3^3} \cdot 5\}$$

Lemma. *$N_i(mn) = N_i(m) \times N_i(n)$ when m and n are i-relatively prime.*

Proof. After applying the FTA to generate i levels of the mosaics of m and n, let the unfactored powers in the $(i+1)$ level of m be $a_1, a_2, ..., a_s$ and let the unfactored powers in the $(i+1)$ level of n be $b_1, b_2, ..., b_t$. Since m and n are i-relatively prime, applying the FTA to mn would also give $a_1, a_2, ..., a_s, b_1, b_2, ..., b_t$ as the unfactored powers in the $(i+1)$ level. If d is a new-visor of mn, then d has $e_1, e_2, ..., e_s, f_1, f_2, ..., f_t$ as the unfactored powers in the $(i+1)$ level with the same first i levels as mn where $0 \leq e_j \leq a_j$ for $j = 1, 2, ..., s$ and $0 \leq f_k \leq b_k$ for $k = 1, 2, ..., t$. Now let d_1 be a new-visor of m such that d_1 has the same first i levels as m and the unfactored powers in the $(i+1)$ level are $e_1, e_2, ..., e_s$. Let d_2 be a new-visor of n such that d_2 has the same first i levels as n and the unfactored powers in the $(i+1)$ level are $f_1, f_2, ..., f_s$. It follows that $d_1 \in N_i(m)$ and $d_2 \in N_i(n)$. Thus d_1 and d_2 are i-relatively prime and $d = d_1 d_2$, so $d \in N_i(m) \times N_i(n)$.

Similarily, if $d_1 \in M(m)$ and $d_2 \in M(n)$, then $d_1 d_2 \in M(m) \times M(n)$ and $d_1 d_2 \in M(mn)$. Then the sets are the same. □

Theorem 9. *If f is an i-multiplicative function, then $F(n) = \sum_{d \in N_i(n)} f(d)$, is also i-multiplicative.*

Proof. The proof follows from Theorem 6 using the set of new-visors in place of the set of mivisors. □

5.2 The Function ${}_n\phi_i$

We now revisit the functions explored in the previous section. We begin with the function ${}_n\phi_i$.

Definition 20.

$$_n\phi_i(n) = \sum_{d \in N_i(n)} \mu_i(d) \frac{n}{d}$$

Example.

$$_n\phi_2(13^{5^2}) = (-1)^3 \frac{13^{5^2}}{13^{5^2}} + (-1)^2 \frac{13^{5^2}}{13^{5^1}} + (-1)^1 \frac{13^{5^2}}{13^{5^0}}$$

We found our function ${}_n\phi_i$ to be i-multiplicative.

Theorem 10. *${}_n\phi_i$ is i-multiplicative.*

Proof. Similar to proof that ${}_{m_1}\phi_i$ is i-multiplicative. $\square$

If the mosaic of n has i levels of primes, then $\mu_j(n) = {}_n\phi_j(n)$ for $j > i$ because the only new-visor of n will be n itself. If the mosaic of n has a prime repeated in the first i levels, then ${}_n\phi_j(n) = 0$ for $j > i$ because $\mu_j(d) = 0$ for all new-visors of n.

5.3 The Functions ${}_n\tau_i$ and ${}_n\sigma_i$

The generalized τ function, ${}_n\tau_i(n)$, counts the number of new-visors of n. Thus, we have:

Definition 21.

$$ {}_n\tau_i(n) = \sum_{d \in N_i(n)} 1 $$

It can be easily shown that

$$ {}_n\tau_i(n) = \prod (a_j + 1) $$

where a_j is an element of the unfactored $(i+1)$ row of n.

Examples.

$$ {}_n\tau_2(2^{3^5}) = 5 + 1 = 6 $$

$$ {}_n\tau_2(7^{11^{23}}) = 23 + 1 = 24 $$

$$ {}_n\tau_2(2^{3^5} \cdot 7^{11}) = (5+1)(1+1) = 12 $$

By Theorem 9, we find

Corollary. *${}_n\tau_i$ is i-multiplicative for all i.*

Example.

$$ \begin{aligned} {}_n\tau_2(3^{5^2} \cdot 7^{11 \cdot 13^{19}}) &= (2+1)(1+1)(19+1) \\ &= 120 \\ &= 3 \cdot 40 \\ &= {}_n\tau_2(3^{5^2}) \cdot {}_n\tau_2(7^{11 \cdot 13^{19}}) \end{aligned} $$

The generalized σ function, ${}_n\sigma_i$, sums the new-visors of n, therefore,

Definition 22.

$$ {}_n\sigma_i(n) = \sum_{d \in N_i(n)} d $$

Examples.

$$ {}_n\sigma_2(2^{3^5}) = 2^{3^0} + 2^{3^1} + 2^{3^2} + ... + 2^{3^5} $$

$$ {}_n\sigma_2(7^{11}) = 7^{11^0} + 7^{11^1} $$

Again, by Theorem 9 we see

Corollary. ${}_n\sigma_i$ *is i-multiplicative for all i.*

Example.

$$\begin{aligned} {}_n\sigma_2(19^{5^3} \cdot 7^{2\cdot 13}) &= 19^{5^0} \cdot 7^{2^0\cdot 13^0} + 19^{5^1} \cdot 7^{2^0\cdot 13^0} + ... + 19^{5^3} \cdot 7^{2^1\cdot 13^1} \\ &= {}_n\sigma_2(19^{5^3}) \cdot {}_n\sigma_2(7^{2\cdot 13}) \end{aligned}$$

6 Trivisors

Neither the mivisors nor the new-visors had all of the properties of divisors that we wanted. In particular, we want a type of divisor that will allow us to use the notion of a greatest common divisor of two mosaics, always have 1 as a divisor of the integer, and change as i varies.

6.1 Definition of a Trivisor

We define trivisors, from third divisor, as follows.

Definition 23. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

For each p_j, a *prime* i-trivisor of n is $p_j^{\alpha_j}$ expanded through i levels with multiplicities above the i^{th} level truncated, denoted P_j.

$P_j^{(a_{j,1},a_{j,2},\cdots,a_{j,s_j})}$ denotes $p_j^{\alpha_j}$ expanded through i levels including the multiplicities $a_{j,1}, a_{j,2}, \cdots, a_{j,s_j}$ on the i^{th} level.

If $n = \prod_{j=1}^{k} P_j^{(a_{j,1},a_{j,2},\cdots,a_{j,s_j})}$, then an i-trivisor of n is

(a) 1,

(b) $P_j^{(b_{j,1},b_{j,2},\cdots,b_{j,s_j})}$ where $1 \le b_j \le a_j$, or

(c) a product of 1-relatively prime i-trivisors from part (b).

Example.

$$n = 2^{3^{5\cdot 7}\cdot 5} \cdot 3^{7^{11^2}}$$

$$T_3(2^{3^{5\cdot 7}\cdot 5} \cdot 3^{7^{11^2}}) = \{1,\ 2^{3^{5\cdot 7}\cdot 5},\ 3^{7^{11}},\ 3^{7^{11^2}},\ 2^{3^{5\cdot 7}\cdot 5} \cdot 3^{7^{11}},\ 2^{3^{5\cdot 7}\cdot 5} \cdot 3^{7^{11^2}}\}$$

Note: Of these 3-trivisors, $2^{3^{5\cdot 7}\cdot 5}$ and $3^{7^{11}}$ are *prime* 3-trivisors.

Lemma. $T_i(mn) = T_i(m) \times T_i(n)$

Proof. Let m and n be i-relatively prime integers such that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_u^{\beta_u}$. After applying the FTA to generate i levels of the mosaics of m and n, let

$$m = \prod_{j=1}^{t} P_j^{(a_{j,1},a_{j,2},\cdots,a_{j,r_j})} \text{ and } n = \prod_{j=1}^{u} Q_j^{(b_{j,1},b_{j,2},\cdots,b_{j,s_j})}.$$

Since m and n are i-relatively prime, $mn = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} q_1^{\beta_1} q_2^{\beta_2} \cdots q_u^{\beta_u}$ and

$$mn = \prod_{j=1}^{t} P_j^{(a_{j,1}, a_{j,2}, \cdots, a_{j,r_j})} \prod_{j=1}^{u} Q_j^{(b_{j,1}, b_{j,2}, \cdots, b_{j,s_j})}.$$

If d is an i-trivisor of mn, then

$$d = \prod_{k=1}^{v} P_k^{'(c_{k,1}, c_{k,2}, \cdots, c_{k,r_k})} \prod_{k=1}^{w} Q_k^{'(e_{k,1}, e_{k,2}, \cdots, e_{k,s_k})}$$

where for each k, $P_k^{'} = P_j$ for some j and $1 \leq c_{k,\ell} \leq a_{j,\ell}$. Let d_1 be an i-trivisor of m such that $d_1 = \prod_{k=1}^{v} P_k^{'(c_{k,1}, c_{k,2}, \cdots, c_{k,r_k})}$. Let d_2 be and i-trivisor of n such that $d_2 = \prod_{k=1}^{w} Q_k^{'(e_{k,1}, e_{k,2}, \cdots, e_{k,s_k})}$. It follows that $d_1 \in T_i(m)$ and $d_2 \in T_i(n)$. Then d_1 and d_2 are i-relatively prime and $d = d_1 d_2$, so $d \in T_i(m) \times T_i(n)$.

Similarly, if $d_1 \in T(m)$ and $d_2 \in T(n)$, then $d_1 d_2 \in T_i(m) \times T_i(n)$ and $d_1 d_2 \in T_i(mn)$. Therefore the sets are the same. □

We need this lemma for the proof of the following theorem.

Theorem 11. *If f is an i-multiplicative function, then $F(n) = \sum_{d \in T_i(n)} f(d)$ is also i-multiplicative.*

Proof. Follows from Theorem 6. □

Definition 24. The greatest common i-trivisor of m and n, $GCT_i(m, n)$, is the greatest common product of prime i-trivisors and exponents.

Examples.

$$GCT_3(2^{3^5},\ 2 \cdot 5^{3^2}) = 1$$

$$GCT_3(2^{3^{5^{20}}} \cdot 3^{7^{11}},\ 2^{3^{5^{10}}} \cdot 7) = 2^{3^{5^{10}}}$$

Definition 25. Two integers m and n are GCT$_i$ relatively prime if and only if $GCT_i(m, n) = 1$.

6.2 The Function ${}_t\phi_i$

Using the idea of GCT$_i$ relatively prime allows us to define a new generalized ϕ function that is based on the original definition of the ϕ function rather than just the formula. That is, ${}_t\phi_i(n)$ is the number of integers GCT$_i$ relatively prime to n that are less than or equal to n. Unfortunately, ${}_t\phi_i(n)$ is NOT an i-multiplicative function.

Example.

$${}_t\phi_2(2) = 1$$

$${}_t\phi_2(3) = 2$$

$${}_t\phi_2(2 \cdot 3) = 3 \neq_t \phi_2(2) \cdot_t \phi_2(3)$$

However, it is still of interest to us because by counting something it serves a purpose that the other ϕ functions do not.

6.3 The Functions ${}_t\tau_i$ and ${}_t\sigma_i$

Similar to previous sections, we let ${}_t\tau_i$ count the number of i-trivisors of n. Thus,

Definition 26.

$$ {}_t\tau_i(n) = \sum_{d \in T_i(n)} 1 $$

where $T_i(n)$ is the set of i-trivisors of n.

Example.

$$ \begin{aligned} n &= 7^{11 \cdot 13^{5^2}} \\ T_3(7^{11 \cdot 13^{5^2}}) &= \{1,\ 7^{11 \cdot 13^{5^1}},\ 7^{11 \cdot 13^{5^2}}\} \\ {}_t\tau_3(7^{11 \cdot 13^{5^2}}) &= |T_3(7^{11 \cdot 13^{5^2}})| = 3 \end{aligned} $$

By Theorem 11, we find

Corollary. *${}_t\tau_i$ is i-multiplicative.*

Example.

$$ \begin{aligned} {}_t\tau_2(5^{3 \cdot 5^7} \cdot 37^{17^7}) &= 64 \\ &= 8 \cdot 8 \\ &= {}_t\tau_2(5^{3 \cdot 5^7}) \cdot {}_t\tau_2(37^{17^7}) \end{aligned} $$

We were able to find and prove another formula to compute ${}_t\tau_i$ using the preceeding corollary and the following lemma.

Lemma. *Let p be a prime and α be a positive integer, then ${}_t\tau_i(p^\alpha) = (\prod a_j) + 1$ where a_j is an element of the unfactored $(i+1)$ level of $p^\alpha = P^{(a_1, a_2, \ldots, a_k)}$.*

Proof. 1 is an i-trivisor of p^α and so is $P^{(b_1, b_2, \ldots, b_k)}$ where $1 \le b_j \le a_j$ for all j. Since there are $\prod a_j$ ways to select the set $(b_1, b_2, \ldots, b_j)$, there are $(\prod a_j) + 1$ i-trivisors of p^α. $\square$

Theorem 12. *Let n have the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. Then*

$$ {}_t\tau_i(n) = \prod_{k=1}^{s} \left(\left(\prod a_j \right) + 1 \right) $$

where a_j is an element of the unfactored $(i+1)$ level of $p_k^{\alpha_k}$.

Proof. Because ${}_t\tau_i$ is i-multiplicative for all i, we see that

$$ {}_t\tau_i(n) = {}_t\tau_i(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = {}_t\tau_i(p_1^{\alpha_1}) {}_t\tau_i(p_2^{\alpha_2}) \cdots {}_t\tau_i(p_s^{\alpha_s}). $$

Inserting the values from the lemma, we see that

$$ {}_t\tau_i(n) = \prod_{k=1}^{s} \left(\left(\prod a_j \right) + 1 \right) $$

$\square$

Examples.

$$ {}_t\tau_3(5 \cdot 7^{11^{7^3 \cdot 13}}) = (1+1)((3 \cdot 1)+1) = 8 $$
$$ {}_t\tau_2(17^{5 \cdot 7^3 \cdot 19}) = (1 \cdot 3 \cdot 1)+1 = 4 $$

The function ${}_t\sigma_i(n)$ sums the i-trivisors of n.

Definition 27.

$$ {}_t\sigma_i(n) = \sum_{d \in T_i(n)} d $$

Examples.

$$ {}_t\sigma_3(7^{5^{11^2}}) = 1 + 7^{5^{11}} + 7^{5^{11^2}} $$
$$ {}_t\sigma_4(2^{3^{2^{3^2}}} \cdot 3) = 1 + 2^{3^{2^3}} + 2^{3^{2^{3^2}}} + 3 + (2^{3^{2^3}} \cdot 3) + (2^{3^{2^{3^2}}} \cdot 3) $$

Again, using Theorem 11 we obtain

Corollary. ${}_t\sigma_i$ *is* i*-multiplicative.*

Example.

$$ \begin{aligned} {}_t\sigma_2(2^{3^2} \cdot 13^{7^2}) &= 1 + 2^3 + 2^{3^2} + 13^7 + 13^{7^2} + (2^3 \cdot 13^7) \\ &\quad + (2^{3^2} \cdot 13^7) + (2^3 \cdot 13^{7^2}) + (2^{3^2} \cdot 13^{7^2}) \\ &= (1 + 2^3 + 2^{3^2})(1 + 13^7 + 13^{7^2}) \\ &= {}_t\sigma_2(2^{3^2}) \cdot {}_t\sigma_2(13^{7^2}) \end{aligned} $$

7 Future Work

In conclusion, we generalized several number theoretic functions in terms of the levels of the mosaic and explored their properties, building on the work of Mullin and Gillman. Further, we re-defined the notion of a divisor for mosaics in three ways so we could generalize the functions ϕ, τ, and σ, all of which have divisors in their classical definitions. Although our concepts of a divisor were helpful, none were perfectly analogous to the normal divisor of an integer. In the future, we could generalize more number theoretic functions and properties to the mosaic of n, as well as perfecting our notion of a divisor in terms of mosaics.

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