On the Properties $\mathbb{A}_{m,n}$ for Subspaces of $\mathbb{C}^{kk}$

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1 Introduction

Consider the vector space $\mathbb{C}^{kk}$ of all $k \times k$ matrices with entries from the complex numbers. For the standard basis of we will use matrices E_{ij} with the value 1 in the ij entry and zeros in all other entries. We are trying to determine which subspaces of $\mathbb{C}^{kk}$ have certain properties that make them equivalent to rank one matrices.

Definition 1. *Let S be a subspace of $\mathbb{C}^{kk}$. Then $S^{\perp} = \{T \in \mathbb{C}^{kk}$ such that $\forall M \in S, tr\,(MT) = 0\}$ where $tr\,()$ is the trace function.*

Since the trace function is linear, it is easy to show that $S^{\perp}$ is a subspace of $\mathbb{C}^{kk}$. It is important to note that $dim\,(S) + dim\left(S^{\perp}\right) = dim\left(\mathbb{C}^{kk}\right) = k^2$.

Definition 2. *Let S be a subspace of $\mathbb{C}^{kk}$. Then the quotient space $Q_S = \mathbb{C}^{kk}/S^{\perp}$.*

We will denote elements of Q_S as $[A]$ where A is in $\mathbb{C}^{kk}$. Then $[A] = [B]$ iff $A - B \in S^{\perp}$. Note that the $dim\,(Q_S) = dim\,(S)$ so to determine a basis for Q_S we merely find $dim\left(\mathbb{C}^{kk}\right) - dim\left(S^{\perp}\right)$ linearly independent vectors such that they are also linearly independent of the basis vectors for $S^{\perp}$ and let their equivalence classes form the basis.

Definition 3. *Given two vectors $\overrightarrow{x} = \{x_i\}$ and $\overrightarrow{y} = \{y_j\}$ in $\mathbb{C}^k$, we define their*

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tensor product

$$\vec{x} \otimes \vec{y} = \begin{pmatrix} x_1\overline{y_1} & x_1\overline{y_2} & \cdots & x_1\overline{y_k} \\ x_2\overline{y_1} & x_2\overline{y_2} & \cdots & x_2\overline{y_k} \\ \vdots & \vdots & \ddots & \vdots \\ x_k\overline{y_1} & x_k\overline{y_2} & \cdots & x_k\overline{y_k} \end{pmatrix}$$

Notice that $\vec{x} \otimes \vec{y}$ *is a rank one matrix, and for all* $u \in \mathbb{C}^k$, $\left(\vec{x} \otimes \vec{y}\right) \vec{u} = \left(\vec{u}, \vec{y}\right) \vec{x}$ *where* $\left(\vec{u}, \vec{y}\right)$ *is the standard inner product on* C^k.

Definition 4. *We say that a subspace* S *has property* $\mathbb{A}_{m,n}$ *if for every array of equivalence classes* $[L_{ij}] \in Q_S$ *with* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*, we can find vectors* $\vec{x_i}$ *and* $\vec{y_j} \in \mathbb{C}^k$ *such that* $[x_i \otimes y_j] = [L_{ij}]$.

Originally property $\mathbb{A}_{m,n}$ was defined for infinite dimensional vector spaces. See [2] for details. For finite dimensional vector spaces, property $\mathbb{A}_{1,1}$ was explored in [1], however different methods were used.

Note that by definition, if S has property $\mathbb{A}_{m,n}$, it must also have property $\mathbb{A}_{p,q}$, where $p \leq m$ and $q \leq n$.

2 Property $\mathbb{A}_{1,1}$

Example 1.

$$\text{Let } S = span\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

We will show that S *has property* $\mathbb{A}_{1,1}$.

A basis for $S^{\perp}$ *is*

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}.$$

Therefore a basis for Q_S *is*

$$\left\{\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]\right\}.$$

An arbitrary element of Q_S *is:*

$$[L_{11}] = \left[\begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}\right].$$

If we let $\vec{x_1} = \begin{pmatrix} c_{11} \\ 0 \end{pmatrix}$ *and* $\vec{y_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$*, then* $[L_{11}] = [x_1 \otimes y_1]$ *so* S *has property* $\mathbb{A}_{1,1}$.

This subspace also has property $\mathbb{A}_{m,1}$ for all $m \geq 1$. There are m elements from Q_S, call them $[L_{11}], [L_{21}], \ldots, [L_{m1}]$ where each $[L_{i1}] = \left[\begin{pmatrix} c_{i1} & 0 \\ 0 & 0 \end{pmatrix}\right]$. Let each $\vec{x_i} = \begin{pmatrix} c_{i1} \\ 0 \end{pmatrix}$ and let $\vec{y_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and we are done.

In addition, this subspace has property $\mathbb{A}_{1,n}$ for all $n \geq 1$. Now there are n elements from Q_S, call them $[L_{11}], [L_{12}], \ldots, [L_{1n}]$ where each $[L_{1j}] = \left[\begin{pmatrix} c_{1j} & 0 \\ 0 & 0 \end{pmatrix}\right]$. Let $\vec{x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and each $\vec{y_j} = \begin{pmatrix} \overline{c_{1j}} \\ 0 \end{pmatrix}$ and this shows that S has property $\mathbb{A}_{1,n}$.

In the previous example we saw that the suspace had both property $\mathbb{A}_{m,1}$ and $\mathbb{A}_{1,n}$. However, having one property does not necessarily imply the subspace also has the other property as the following example shows.

Example 2. *Let*

$$S = span\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$$

We will show that it has property $\mathbb{A}_{m,1}$ for all $m \geq 1$, but property $\mathbb{A}_{1,n}$ fails to hold for all $n > 1$.

A basis for $S^\perp$ is:

$$\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

A basis for Q_S is:

$$\left\{\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]\right\}.$$

Each $[L_{i1}] \in Q_S = \left[\begin{pmatrix} c_{i1} & 0 \\ d_{i1} & 0 \end{pmatrix}\right]$ so if we let each $\vec{x_i} = \begin{pmatrix} c_{i1} \\ d_{i1} \end{pmatrix}$ and $\vec{y_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it is clear that S has property $\mathbb{A}_{m,1}$.

To show property $\mathbb{A}_{1,n}$ fails we let $L_{1j} = \begin{pmatrix} j & 0 \\ j+1 & 0 \end{pmatrix}$ and we will show that we cannot find $\vec{x_1}$ and $\vec{y_j}$ such that $\left[\vec{x_1} \otimes \vec{y_j}\right] = [L_{1j}]$.

We must consider three cases. First suppose $\vec{x_1} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ where $p \in \mathbb{C}$ and $\left[\vec{x_1} \otimes \vec{y_1}\right] = [L_{11}]$. Then we have $x_1 \otimes y_1 = \begin{pmatrix} p\overline{y_{11}} & p\overline{y_{12}} \\ 0 & 0 \end{pmatrix}$ so $x_1 \otimes y_1 - L_{11} = \begin{pmatrix} p\overline{y_{11}} - 1 & p\overline{y_{12}} \\ -2 & 0 \end{pmatrix} \notin S^\perp$ since the $2,1$ entry is non-zero. This is a contradiction.

The second case, where $\vec{x_1} = \begin{pmatrix} 0 \\ q \end{pmatrix}$, leads to a similar contradiction.

The third case supposes $\vec{x_1} = \begin{pmatrix} p \\ q \end{pmatrix}$ with $p, q \neq 0$. Since $x_1 \otimes y_j = \begin{pmatrix} p\overline{y_{j1}} & p\overline{y_{j2}} \\ q\overline{y_{j1}} & q\overline{y_{j2}} \end{pmatrix}$ we have

$$x_1 \otimes y_1 - L_{11} = \begin{pmatrix} p\overline{y_{11}} - 1 & p\overline{y_{12}} \\ q\overline{y_{11}} - 2 & q\overline{y_{12}} \end{pmatrix}$$

and

$$x_1 \otimes y_2 - L_{11} = \begin{pmatrix} p\overline{y_{21}} - 2 & p\overline{y_{22}} \\ q\overline{y_{21}} - 3 & q\overline{y_{22}} \end{pmatrix}.$$

In order for each difference to be in $S^\perp$ the $1,1$ and $1,2$ entries must equal zero. This leads to the following four equations:

$$p\overline{y_{11}} = 1, \quad q\overline{y_{11}} = 2, \quad p\overline{y_{21}} = 2, \quad q\overline{y_{21}} = 3$$

Therefore, $\overline{y_{11}} = \frac{1}{p} = \frac{2}{q}$ and $\overline{y_{21}} = \frac{2}{p} = \frac{3}{q}$. Simplifying we find that $2p = q$ and $3p = 2q$ or in other words, $3p = 4p$ which is true only when $p = 0$ so we have a contradiction. This shows that S does not neccessarily have property $\mathbb{A}_{1,n}$ even though it has $\mathbb{A}_{m,1}$.

Claim. *Every one-dimensional subspace of $\mathbb{C}^{kk}$ has properties $\mathbb{A}_{1,1}$, $\mathbb{A}_{m,1}$, and $\mathbb{A}_{1,n}$.*

Proof. Since S is one-dimensional, so is Q_S. Furthermore, $S^\perp$ can not be all of $\mathbb{C}^{kk}$ so there must exist a standard basis vector of $\mathbb{C}^{kk}$ that is not in $S^\perp$. The equivalence class of this basis vector forms a basis for Q_S. We can then pick our vectors such that their tensor product and an arbitrary element of Q_S are equal. □

3 Results

Theorem 1. *Let S_1 be a subspace of $\mathbb{C}^{kk}$ and $P \in \mathbb{C}^{kk}$ be an invertible matrix. If $S_2 = \left\{P^{-1}AP \text{ such that } A \in S_1\right\}$ then S_1 has property $\mathbb{A}_{m,n}$ iff S_2 has property $\mathbb{A}_{m,n}$.*

Proof. Let T be in $S_1^\perp$. Then for all $L \in S_1$, $tr\,(LT) = 0$. Since L is in S_1, $P^{-1}LP$ is in S_2. Then $tr\left(P^{-1}TP \cdot P^{-1}LP\right) = tr\,(LT) = 0$ so $P^{-1}TP$ is in $S_2^\perp$. Therefore $S_2^\perp = \left\{P^{-1}TP \text{ such that } T \in S_1^\perp\right\}$.

Suppose S_1 has property $\mathbb{A}_{m,n}$. If $L_{ij} \in \mathbb{C}^{kk}$ for $1 \le i \le m$ and $1 \le j \le n$ let $M_{ij} = PL_{ij}P^{-1}$. Since S_1 has property $\mathbb{A}_{m,n}$ we can find $\vec{x_i}$ and $\vec{y_j}$ such that $M_{ij} - \vec{x_i} \otimes \vec{y_j} \in S_1^\perp$. So $P^{-1}\left(M_{ij} - \vec{x_i} \otimes \vec{y_j}\right)P = L_{ij} - P^{-1}\left(\vec{x_i} \otimes \vec{y_j}\right)P = L_{ij} - P^{-1}\vec{x_i} \otimes P^* \vec{y_j}$ where P^* is the conjugate transpose of P. Therefore S_2 has property $\mathbb{A}_{m,n}$. By symmetry the converse is also true. □

Corollary. *If A and B are similar $k \times k$ matrices and $S_1 = span\left\{I, A, A^2, \ldots, A^{n-1}\right\}$ and $S_2 = span\left\{I, B, B^2, \ldots, B^{n-1}\right\}$ then S_1 has property $\mathbb{A}_{m,n}$ iff S_2 has property $\mathbb{A}_{m,n}$.*

Theorem 2. *Suppose S is a subspace of $\mathbb{C}^{kk}$ and $\exists\, i,j$ such that $\forall T \in S^\perp, T_{ij} = 0$. Then S fails to have property $\mathbb{A}_{2,2}$.*

Proof. For S to have property $\mathbb{A}_{2,2}$ we must have $\forall\, k,l,\ 1 \le k,l \le 2$, $[x_k \otimes y_l] = [L_{kl}]$. Since $\forall T \in S^\perp, T_{ij} = 0$ the i,j^{th} entry of $x_k \otimes y_l - L_{kl} = 0$. This leads to the following system of equations (where each c_{kl} is the i,j entry of L_{kl}):

$x_{1,i} \cdot y_{1,j} - c_{11} = 0$
$x_{1,i} \cdot y_{2,j} - c_{12} = 0$
$x_{2,i} \cdot y_{1,j} - c_{21} = 0$
$x_{2,i} \cdot y_{2,j} - c_{22} = 0.$

Let $c_{11} = 0$ and the remaining $c_{k,l} \neq 0$. Then we have $x_{1,i} \cdot y_{1,j} = 0$ which implies that either $x_{1,i} = 0$ or $y_{1,j} = 0$. Suppose first that $x_{1,i} = 0$. The second equation implies that $x_{1,i} \cdot y_{2,j} = c_{12} \neq 0$ which is a contradiction. If we let $y_{1,j} = 0$ then we get a similar contradiction with the third equation. Therefore S cannot have property $\mathbb{A}_{2,2}$. □

Corollary. *Suppose $A \in \mathbb{C}^{kk}$ is a diagonal matrix having at least one eigenvalue of multiplicity 1 and let $S = span\left\{I, A, A^2, \ldots, A^{c-1}\right\}$ where c is the degree of the minimal polynomial of A. Then S fails to have property $\mathbb{A}_{2,2}$.*

Proof. Clearly $dim\,(S) = c$ so $dim\left(S^{\perp}\right) = k^2 - c$. There are $k^2 - k$ basis vectors for $S^{\perp}$ that are standard basis vectors in $\mathbb{C}^{kk}$. These correspond to the entries in A that are zero. Any remaining basis vectors for $S^{\perp}$ can be constructed as follows. If the same eigenvalue appears in the i, i and j, j positions of A, where $i < j$, then we place a 1 in the i, i position and a -1 in the j, j position. These remaining vectors fill out the basis for $S^{\perp}$.

Now let the eigenvalue of multiplicity 1 be in the n, n position in A. Clearly every basis vector in $S^{\perp}$ will have a zero in the n, n position so by theorem 2, S fails to have property $\mathbb{A}_{2,2}$. □

4 Future Work

We are close to proving the following conjecture.

Conjecture 1. *Let A be a $k \times k$ Jordan block having one eigenvalue. Define S as the span of $\left\{I, A, A^2, \ldots, A^{e-1}\right\}$, where e is the degree of the minimum polynomial in A. Then S does not have property $\mathbb{A}_{m,n}$ for $m, n > k$.*

A proof of this conjecture may help lead to a proof of the following, more general statement.

Conjecture 2. *If S is a subspace of $\mathbb{C}^{kk}$ then S does not have property $\mathbb{A}_{m,n}$ for $m, n > k$.*

References

[1] E.A. Azoff, *On Finite Rank Operators and Preannihilators*, Memoirs of the American Mathematical Society, vol. 64, no. 357, American Mathematical Society, Providence, RI, 1986.

[2] H. Bercovici, C. Foias, and C. Pearcy, *Dual Algebras with Applications to Invariant Subspaces and Dilation Theory* CBMS Regional Conference Series in Mathematics, no. 56, American Mathematical Society, Providence, RI, 1985.