On the Properties $A_{m,n}$ for Subspaces of $C^{kk}$

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1 Introduction

Consider the vector space $C^{kk}$ of all $k \times k$ matrices with entries from the complex numbers. For the standard basis of we will use matrices $E_{ij}$ with the value 1 in the $ij$ entry and zeros in all other entries. We are trying to determine which subspaces of $C^{kk}$ have certain properties that make them equivalent to rank one matrices.

Definition 1. Let $S$ be a subspace of $C^{kk}$. Then $S^\perp = \{ T \in C^{kk} \text{ such that } \forall M \in S, \text{tr}(MT) = 0 \}$ where $\text{tr}(\cdot)$ is the trace function.

Since the trace function is linear, it is easy to show that $S^\perp$ is a subspace of $C^{kk}$. It is important to note that $\dim(S) + \dim(S^\perp) = \dim(C^{kk}) = k^2$.

Definition 2. Let $S$ be a subspace of $C^{kk}$. Then the quotient space $Q_S = C^{kk}/S^\perp$.

We will denote elements of $Q_S$ as $[A]$ where $A$ is in $C^{kk}$. Then $[A] = [B]$ iff $A - B \in S^\perp$. Note that the $\dim(Q_S) = \dim(S)$ so to determine a basis for $Q_S$ we merely find $\dim(C^{kk}) - \dim(S^\perp)$ linearly independent vectors such that they are also linearly independent of the basis vectors for $S^\perp$ and let their equivalence classes form the basis.

Definition 3. Given two vectors $\overrightarrow{x} = \{x_i\}$ and $\overrightarrow{y} = \{y_j\}$ in $C^k$, we define their

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tensor product
\[
\vec{x} \otimes \vec{y} = \begin{pmatrix}
x_1 y_1 & x_1 y_2 & \cdots & x_1 y_k \\
x_2 y_1 & x_2 y_2 & \cdots & x_2 y_k \\
\vdots & \vdots & \ddots & \vdots \\
x_k y_1 & x_k y_2 & \cdots & x_k y_k
\end{pmatrix}
\]

Notice that \( \vec{x} \otimes \vec{y} \) is a rank one matrix, and for all \( u \in \mathbb{C}^k \), \( (\vec{x} \otimes \vec{y}) \vec{u} = \overline{(\vec{u}, \vec{y})} \vec{x} \) where \( (\vec{u}, \vec{y}) \) is the standard inner product on \( \mathbb{C}^k \).

**Definition 4.** We say that a subspace \( S \) has property \( A_{m,n} \) if for every array of equivalence classes \( [L_{ij}] \in Q_S \) with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), we can find vectors \( \vec{x}_i \) and \( \vec{y}_j \in \mathbb{C}^k \) such that \( [x_i \otimes y_j] = [L_{ij}] \).

Originally property \( A_{m,n} \) was defined for infinite dimensional vector spaces. See [2] for details. For finite dimensional vector spaces, property \( A_{1,1} \) was explored in [1], however different methods were used.

Note that by definition, if \( S \) has property \( A_{m,n} \), it must also have property \( A_{p,q} \), where \( p \leq m \) and \( q \leq n \).

**2 Property \( A_{1,1} \)**

**Example 1.**

Let \( S = \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \).

We will show that \( S \) has property \( A_{1,1} \).

A basis for \( S^\perp \) is

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.
\]

Therefore a basis for \( Q_S \) is

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.
\]

An arbitrary element of \( Q_S \) is:

\[
[L_{11}] = \begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}.
\]

If we let \( \vec{x}_1 = \begin{pmatrix} c_{11} \\ 0 \end{pmatrix} \) and \( \vec{y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), then \( [L_{11}] = [x_1 \otimes y_1] \) so \( S \) has property \( A_{1,1} \).
This subspace also has property $A_{m,1}$ for all $m \geq 1$. There are $m$ elements from $Q_S$, call them $[L_{11}], [L_{21}], \ldots, [L_{m1}]$ where each $[L_{11}] = \begin{pmatrix} c_{i1} & 0 \\ 0 & 0 \end{pmatrix}$. Let each $\vec{x}_i = \begin{pmatrix} c_{i1} \\ 0 \end{pmatrix}$ and let $\vec{y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and we are done.

In addition, this subspace has property $A_{1,n}$ for all $n \geq 1$. Now there are $n$ elements from $Q_S$, call them $[L_{11}], [L_{12}], \ldots, [L_{1n}]$ where each $[L_{11}] = \begin{pmatrix} c_{i1} & 0 \\ 0 & 0 \end{pmatrix}$. Let $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and each $\vec{y}_j = \begin{pmatrix} c_{ij} \\ 0 \end{pmatrix}$ and this shows that $S$ has property $A_{1,n}$.

In the previous example we saw that the subspace had both property $A_{m,1}$ and $A_{1,n}$. However, having one property does not necessarily imply the subspace also has the other property as the following example shows.

**Example 2.** Let 

\[ S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \]

We will show that it has property $A_{m,1}$ for all $m \geq 1$, but property $A_{1,n}$ fails to hold for all $n > 1$.

A basis for $S^\perp$ is:

\[ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

A basis for $Q_S$ is:

\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]

Each $[L_{11}] \in Q_S = \begin{pmatrix} c_{i1} & 0 \\ 0 & 0 \end{pmatrix}$ so if we let each $\vec{x}_i = \begin{pmatrix} c_{i1} \\ 0 \end{pmatrix}$ and $\vec{y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it is clear that $S$ has property $A_{m,1}$.

To show property $A_{1,n}$ fails we let $L_{11} = \begin{pmatrix} j \\ j+1 \\ 0 \end{pmatrix}$ and we will show that we cannot find $\vec{x}_1$ and $\vec{y}_j$ such that $[\vec{x}_1 \otimes \vec{y}_j] = [L_{11}]$.

We must consider three cases. First suppose $\vec{x}_1 = \begin{pmatrix} p \\ 0 \end{pmatrix}$ where $p \in \mathbb{C}$ and $[\vec{x}_1 \otimes \vec{y}_1] = [L_{11}]$. Then we have $x_1 \otimes y_1 = \begin{pmatrix} p y_{11} \\ p y_{12} \\ 0 \end{pmatrix}$ so $x_1 \otimes y_1 - L_{11} = \begin{pmatrix} p y_{11} - 1 \\ p y_{12} \\ 0 \end{pmatrix} \notin S^\perp$ since the 2,1 entry is non-zero. This is a contradiction.

The second case, where $\vec{x}_1 = \begin{pmatrix} 0 \\ q \end{pmatrix}$, leads to a similar contradiction.

The third case supposes $\vec{x}_1 = \begin{pmatrix} p \\ q \end{pmatrix}$ with $p, q \neq 0$. Since $x_1 \otimes y_j = \begin{pmatrix} p y_{11} \\ p y_{12} \\ q y_{j1} \\ q y_{j2} \end{pmatrix}$ we have

\[ x_1 \otimes y_1 - L_{11} = \begin{pmatrix} p y_{11} - 1 \\ p y_{12} \\ q y_{j1} \\ q y_{j2} \end{pmatrix} \]

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and
\[ x_1 \otimes y_2 - L_{11} = \begin{pmatrix} p_{11} & 2 \\ q_{11} & 3 \end{pmatrix}. \]

In order for each difference to be in \( S^1 \) the 1,1 and 1,2 entries must equal zero. This leads to the following four equations:
\[ p_{11} = 1, \quad q_{11} = 2, \quad p_{22} = 2, \quad q_{22} = 3. \]

Therefore, \( \overline{p_{11}} = \frac{1}{p} = \frac{2}{q} \) and \( \overline{q_{11}} = \frac{2}{p} = \frac{3}{q} \). Simplifying we find that \( 2p = q \) and \( 3p = 2q \) or in other words, \( 3p = 4p \) which is true only when \( p = 0 \) so we have a contradiction. This shows that \( S \) does not necessarily have property \( A_{1,1} \) even though it has \( A_{m,1} \).

**Claim.** Every one-dimensional subspace of \( \mathbb{C}^{kk} \) has properties \( k_{1,1}, k_{m,1} \), and \( k_{1,1} \).

**Proof.** Since \( S \) is one-dimensional, so is \( Q_S \). Furthermore, \( S^\perp \) can not be all of \( \mathbb{C}^{kk} \) so there must exist a standard basis vector of \( \mathbb{C}^{kk} \) that is not in \( S^\perp \). The equivalence class of this basis vector forms a basis for \( Q_S \). We can then pick our vectors such that their tensor product and an arbitrary element of \( Q_S \) are equal. \( \square \)

### 3 Results

**Theorem 1.** Let \( S_1 \) be a subspace of \( \mathbb{C}^{kk} \) and \( P \in \mathbb{C}^{kk} \) be an invertible matrix. If \( S_2 = \{ P^{-1}AP \text{ such that } A \in S_1 \} \), then \( S_1 \) has property \( k_{m,n} \) iff \( S_2 \) has property \( k_{m,n} \).

**Proof.** Let \( T \) be in \( S_1^\perp \). Then for all \( L \in S_1 \), \( \text{tr}(LT) = 0 \). Since \( L \) is in \( S_1 \), \( P^{-1}LP \) is in \( S_2 \). Then \( \text{tr}(P^{-1}TP \cdot P^{-1}LP) = \text{tr}(LT) = 0 \) so \( P^{-1}TP \) is in \( S_2^\perp \).

Therefore \( S_2^\perp = \{ P^{-1}TP \text{ such that } T \in S_1^\perp \} \).

Suppose \( S_1 \) has property \( k_{m,n} \). If \( L_{ij} \in \mathbb{C}^{kk} \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), let \( M_{ij} = PL_{ij}P^{-1} \). Since \( S_1 \) has property \( k_{m,n} \), we can find \( \overline{x_i} \) and \( \overline{y_j} \) such that \( M_{ij} - \overline{x_i} \otimes \overline{y_j} \in S_1^\perp \). So \( \overline{P^{-1}

\begin{pmatrix} M_{ij} - \overline{x_i} \otimes \overline{y_j} \end{pmatrix} = L_{ij} - L_{ij} - \overline{x_i} \otimes \overline{y_j} \) where \( P^* \) is the conjugate transpose of \( P \). Therefore \( S_2 \) has property \( k_{m,n} \). By symmetry the converse is also true. \( \square \)

**Corollary.** If \( A \) and \( B \) are similar \( k \times k \) matrices and \( S_1 = \text{span} \{ I, A, A^2, \ldots, A^{n-1} \} \) and \( S_2 = \text{span} \{ I, B, B^2, \ldots, B^{n-1} \} \) then \( S_1 \) has property \( k_{m,n} \) iff \( S_2 \) has property \( k_{m,n} \).

**Theorem 2.** Suppose \( S \) is a subspace of \( \mathbb{C}^{kk} \) and \( \exists i, j \) such that \( \forall T \in S^\perp, T_{ij} = 0 \). Then \( S \) fails to have property \( k_{2,2} \).

**Proof.** For \( S \) to have property \( k_{2,2} \) we must have \( \forall k, l, 1 \leq k, l \leq 2, [x_k \otimes y_l] = [L_{kl}] \). Since \( \forall T \in S^\perp, T_{ij} = 0 \) the \( i,j \)th entry of \( x_k \otimes y_l - L_{kl} = 0 \). This leads to the following system of equations (where each \( c_{kl} \) is the \( i,j \)th entry of \( L_{kl} \)):
\[ x_{1,i} \cdot y_{1,j} - c_{11} = 0 \]
\[ x_{1,i} \cdot y_{2,j} - c_{12} = 0 \]
\[ x_{2,i} \cdot y_{1,j} - c_{21} = 0 \]
\[ x_{2,i} \cdot y_{2,j} - c_{22} = 0. \]

Let \( c_{11} = 0 \) and the remaining \( c_{k,l} \neq 0 \). Then we have \( x_{1,i} \cdot y_{1,j} = 0 \) which implies that either \( x_{1,i} = 0 \) or \( y_{1,j} = 0 \). Suppose first that \( x_{1,i} = 0 \). The second equation implies that \( x_{1,i} \cdot y_{2,j} = c_{12} \neq 0 \) which is a contradiction. If we let \( y_{1,j} = 0 \) then we get a similar contradiction with the third equation. Therefore \( S \) cannot have property \( \mathbb{A}_{2,2} \).

**Corollary.** Suppose \( A \in \mathbb{C}^{kk} \) is a diagonal matrix having at least one eigenvalue of multiplicity 1 and let \( S = \text{span} \{I, A, A^2, \ldots , A^{c-1}\} \) where \( c \) is the degree of the minimal polynomial of \( A \). Then \( S \) fails to have property \( \mathbb{A}_{2,2} \).  

**Proof.** Clearly \( \dim(S) = c \) so \( \dim(S^\perp) = k^2 - c \). There are \( k^2 - k \) basis vectors for \( S^\perp \) that are standard basis vectors in \( \mathbb{C}^{kk} \). These correspond to the entries in \( A \) that are zero. Any remaining basis vectors for \( S^\perp \) can be constructed as follows. If the same eigenvalue appears in the \( i,i \) and \( j,j \) positions of \( A \), where \( i < j \), then we place a 1 in the \( i,i \) position and a \(-1\) in the \( j,j \) position. These remaining vectors fill out the basis for \( S^\perp \).

Now let the eigenvalue of multiplicity 1 be in the \( n,n \) position in \( A \). Clearly every basis vector in \( S^\perp \) will have a zero in the \( n,n \) position so by theorem 2, \( S \) fails to have property \( \mathbb{A}_{2,2} \).

4 Future Work

We are close to proving the following conjecture.

**Conjecture 1.** Let \( A \) be a \( k \times k \) Jordan block having one eigenvalue. Define \( S \) as the span of \( \{I, A, A^2, \ldots , A^{e-1}\} \), where \( e \) is the degree of the minimum polynomial in \( A \). Then \( S \) does not have property \( \mathbb{A}_{m,n} \) for \( m,n > k \).

A proof of this conjecture may help lead to a proof of the following, more general statement.

**Conjecture 2.** If \( S \) is a subspace of \( \mathbb{C}^{kk} \) then \( S \) does not have property \( \mathbb{A}_{m,n} \) for \( m,n > k \).

References
