Self-Influencing Interpolation in Groundwater Flow

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Abstract

A technique for fitting $n+1$ data points is cubic spline interpolation; n cubic polynomials can be designed to fit any $n+1$ data points and interpolate between them. Traditionally, we use the assumption that the data points are independent of the cubic splines. We can relax this assumption to yield *self-referencing cubic splines*, which interpolate without complete knowledge of the location of the given data points. We explore this method through their use in modeling 2D groundwater flow with a circular heterogeneity using the method of analytic elements as a basis for the model.[1]

Self-Referencing Cubic Splines

A cubic spline interpolation involves fitting $n+1$ data points in such a way that the interpolating function is continuous and smooth. It is well known that using an interpolating polynomial to fit all $n+1$ data

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points at once leads to Runge's phenomenon – higher-order interpolating polynomials become less numerically stable. [1] One solution to this problem is to use cubic splines. Cubic splines are piecewise polynomials whose n parts are cubic functions restricted to the intervals between data points. Let $S(t)$ be a spline interpolating a set of values $\{(t_i, y_i)\}_{0\leq i\leq n}$, such that $S_i(t) = \sigma_{i0} + \sigma_{i1}t + \sigma_{i2}t^2 + \sigma_{i3}t^3$ where $t_i < t < t_{i+1}$. To maintain continuity and smoothness, the following conditions must be met:

Continuity: For $2 \leq i \leq n$, $S_{i-1}(t_i) = y_i = S_i(t_i)$. Also $S_1(t_1) = y_1$ and $S_n(t_{n+1}) = y_{n+1}$.

Smoothness: For $2 \leq i \leq n$, $S'_{i-1}(t_i) = S'_i(t_i)$ and $S''_{i-1}(t_i) = S''_i(t_i)$.

The above conditions are linear in the coefficients of the cubic polynomials, but where there are $4n$ unknown coefficients, there are only $4n-2$ equations. If we consider the spline as a model for a flexible rod (as it was historically [1]) then one further condition is required:

Naturality: $S''_1(t_1) = S''_n(t_{n+1}) = 0$.

One underlying assumption in cubic spline interpolation is that the value of the data points is completely known. In certain contexts, such as groundwater flow modeling, there can be a dependence between the data points and the coefficients of the spline. To account for this dependence, we work with a partial data set $\{(t_i, \tilde{u}_i)\}_{0\leq i\leq n}$ and an influence function, $F_i(t, \sigma_{i0}, \sigma_{i1}, \sigma_{i2}, \sigma_{i3})$. This is the effect due to i^{th} cubic polynomial on the point t. A spline which influences the data point it is supposed to fit is named a self-referencing cubic spline (SRCS). The revised continuity condition becomes

SRCS Continuity: For $2 \leq i \leq n$:

$$S_i(t_i) = \tilde{u}_i + \sum_j F_j(t, \sigma_{j0}, \sigma_{j1}, \sigma_{j2}, \sigma_{j3}) = S_{i+1}(t_i)$$

and

$$S_1(t_1) = \tilde{u}_1 + \sum_j F_j(t_1, \sigma_{j0}, \sigma_{j1}, \sigma_{j2}, \sigma_{j3})$$
$$S_n(t_{n+1}) = \tilde{u}_{n+1} + \sum_j F_j(t_1, \sigma_{j0}, \sigma_{j1}, \sigma_{j2}, \sigma_{j3})$$

In the context of groundwater flow, we will use self-referencing cubic splines to model the strength of a flow element meant to approximate the change in potential due to a heterogeneity in the hydraulic conductivity of the aquifer. Since the strength of such a flow feature influences the potential values over which the interpolation takes place, self-referencing cubic splines are useful in modeling this feature.

Groundwater Flow

Groundwater flows through underground aquifers comprised of porous material. The purpose of groundwater flow modeling is to predict the flow of water within this aquifer due to flow features such as rivers, lakes, and wells, along with the material properties of the aquifer. The classical assumption made in most groundwater models was independently formulated by Dupuit and Forchheimer [3]: *the variation of flow due to changes in depth is negligible.* Therefore, we consider only 2D models.

Specifically, we wish to determine the potential field $\Phi : \mathbb{R}^2 \to \mathbb{R}$ governing groundwater flow, from which other quantities can be derived. For example, we can estimate the velocity field $\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$ using Darcy's law [2]

$$\vec{v} = \frac{-\nabla\Phi}{n}.$$

While the potential at a particular point is not directly observable, we can relate it to the measurable hydraulic head ϕ by $\Phi = kH\phi$, where k is hydraulic conductivity and H is the width of the aquifer. Measuring hydraulic head involves drilling a monitor well into the aquifer and observing the height at which the water comes to rest. Because of this, it is not reasonable to assume a very large number of potential measurements. Thus, other means of modeling potentials are necessary.

Provided we assume no rainfall or other significant change in water volume within the aquifer, Darcy's law and the Dupuit-Forchheimer assumption imply that the potential field is harmonic, i.e., it is a solution to Laplace's equation

$$\nabla^2\Phi = 0$$

given the boundary conditions. One property of Laplace's equation we make frequent use of is the principle of superposition, which states that the sum of any two solutions to Laplace's equation is itself a solution to Laplace's equation. This follows directly from the linearity of ∇^2. Many traditional techniques for modeling groundwater flow approximate solutions to Laplace's equation on a grid using grid boundaries and flow features as boundary conditions. For our purposes, however, we use the analytic element method, which requires no grid and uses only flow features as boundary conditions.

Most flow features can be broken up into combinations of simple flow features: point sinks for wells, line sinks or sources for rivers, and so forth. Using the principle of superposition, we may approximate complex flow features by adding together the exact potential fields of several simple flow features. Since the potential functions for simple flow features may be calculated analytically, the method is called the analytic element method. [2]

In particular, we are interested in modeling an aquifer heterogeneity. In such a heterogeneity, the hydraulic conductivity is different than than that of the surrounding aquifer. Since hydraulic head is directly observable, we must assume it is continuous. However, if k_+ and k_- are the hydraulic conductivities of the two materials, then the potential has a jump discontinuity across the interface.

Third-Order Line Doublets

The simplest flow feature that models a jump in potential is the point dipole. The potential of a point dipole is found by taking the limit of a point sink and a point source as their strengths increase to infinity and the distance between them decreases to zero. To model a jump in potential along a line segment, we can integrate the potential of a point dipole over the segment. The resulting flow feature is called a line doublet. It can be shown that a point dipole has no net infiltration or exfiltration [2]. Second-order line doublets are already used by Haitjema's program GFLOW to model heterogeneities.

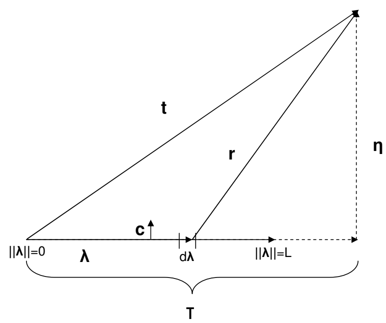
It is known that the potential of a point dipole in 2D Cartesian coordinates is

$$\Phi_{\text{dp}}(r) = \frac{-\sigma(\hat{c} \cdot \vec{r})}{2\pi\|\vec{r}\|^2}$$

where σ is the strength of the dipole and $\hat{c}$ is a unit vector pointing in the direction of the point source. From this we can write the potential of a third-order line doublet as the integral

$$\Phi_{\mathrm{d}}(\vec{x}) = \int_{\text{line}} \frac{-\sigma(\lambda)(\hat{c}\cdot\vec{r})}{2\pi\|\vec{r}\|^2}\, d\lambda$$

where now σ is a cubic polynomial $\sigma(\lambda) = \sigma_0 + \sigma_1\lambda + \sigma_2\lambda^2 + \sigma_3\lambda^3$. To evaluate this integral, we make use of the following diagram



Writing the above integral as four separate integrals and using the projection $\vec{t} = \vec{\eta} + \vec{\tau}$, essentially four integrals must be solved (as described in Appendix A) in terms of $\|\vec{\eta}\|$, $\|\vec{\tau}\|$ and L:

$$\begin{aligned}
I_0 &= \arctan\left(\frac{L - \|\vec{\tau}\|}{\|\vec{\eta}\|}\right) + \arctan\left(\frac{\|\vec{\tau}\|}{\|\vec{\eta}\|}\right)\\
I_1 &= \frac{1}{2}\ln\left(\frac{\|\vec{\tau}\|^2 + \|\vec{\eta}\|^2 + L^2 - 2\|\vec{\tau}\|L}{\|\vec{\tau}\|^2 + \|\vec{\eta}\|^2}\right)\\
I_2 &= \frac{L}{\|\vec{\eta}\|} - I_0\\
I_3 &= \frac{1}{2}\left(\frac{L^2 - 2L\|\vec{\tau}\|}{\|\vec{\eta}\|^2}\right) - I_1
\end{aligned}$$

The potential due to each term in the strength distribution can then be written as a linear combination of these terms.

$$\begin{aligned}
\Phi_0(\vec{x}) &= \sigma_0 I_0\\
\Phi_1(\vec{x}) &= \sigma_1(\|\vec{\tau}\|I_0 + \|\vec{\eta}\|I_1)\\
\Phi_2(\vec{x}) &= \sigma_2(\|\vec{\tau}\|^2 I_0 + 2\|\vec{\tau}\|\|\vec{\eta}\|I_1 + \|\vec{\eta}\|^2 I_2)\\
\Phi_3(\vec{x}) &= \sigma_3(\|\vec{\tau}\|^3 I_0 + 3\|\vec{\tau}\|^2\|\vec{\eta}\|I_1 + 3\|\vec{\tau}\|\|\vec{\eta}\|^2 I_2 + \|\vec{\eta}\|^3 I_3\\
\Phi_d &= \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3
\end{aligned}$$

In this derivation, we have made the tacit assumption that $\vec{\tau}$ and $\vec{\lambda}$ point in the same direction. Therefore, this solution is only accurate on the half-plane where this condition holds. However, we can exploit the symmetry about the axis $\vec{\tau} = \frac{1}{2}\vec{\lambda}$ to obtain a solution for the full plane.

Modeling Heterogeneities

With the third-order line doublet potential we can finally begin to model a heterogeneity. For our purposes we will consider a circular heterogeneity in which the hydraulic conductivity shifts from the external conductivity k to the internal conductivity k_1. Our notation in this respect follows the method used by [3] to model heterogeneities, though his modeling does not explicitly use splines.

In modeling a heterogeneity we first discretize its boundary with a set of control points, $\{(x_i, y_i)\}_{0\leq i\leq n}$. In what follows we will assume that the heterogeneity is closed, so that $(x_0, y_0) = (x_n, y_n)$. At each control point, we will calculate the potential Φ_i^+ due to all other flow features. If we assume that the control point is infinitesimally close to the exterior of the heterogeneity, again following Strack, then the jump in potential due to the heterogeneity is $\Delta\Phi_i = \bar{k}\Phi_i^+$, where $\bar{k} = \frac{k_+ - k_-}{k_+}$.

In order to use a self-referencing cubic spline to interpolate the expected potential jumps, we parameterize the boundary of the heterogeneity. We define the data points to be used in the self-referencing interpolation as $(t_i, \Delta\Phi_i)$ where

$$\begin{aligned} t_0 &= 0 \\ t_{i+1} &= t_i + \|(x_{i+1} - x_i, y_{i+1} - y_i)\| \end{aligned}$$

One further modification to the cubic spline conditions is necessary in order to model this situation. Since we have assumed that the heterogeneity is closed, the boundary parametrization is periodic. Therefore, it is reasonable to replace the naturality condition with a condition reflecting the perodicity of the spline:

Periodicity: $S_1'(t_0) = S_n'(t_n)$ and $S_1''(t_0) = S_n''(t_n)$.

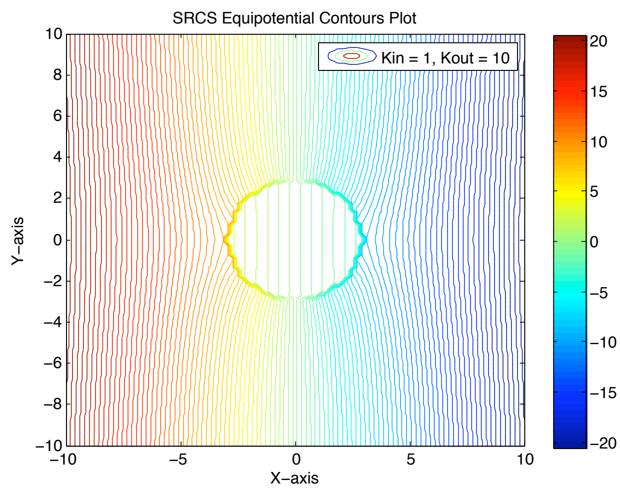
The SRCS continuity, smoothness, and periodicity conditions account for $4n$ equations linear in the $4n$ coefficients of the cubic spline, σ_{ij}. Using these condtions, we may now sketch out the steps necessary to construct a model using third-order line doublets whose strength distribution is a self-referencing cubic spline.

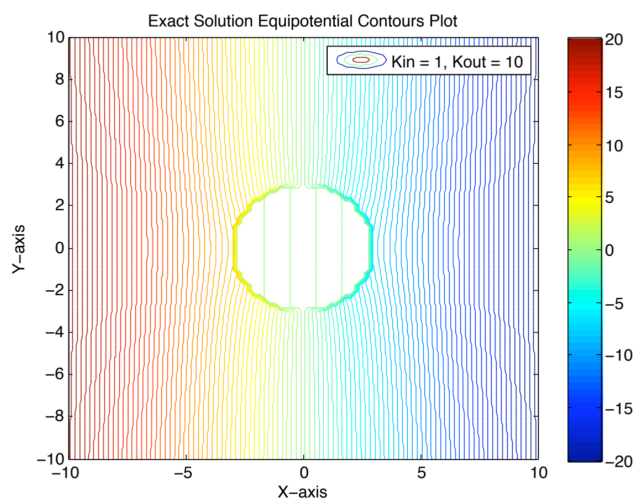
- Sum together the potentials of all flow features of known strength into a "background" potential field, Φ_B.
- Approximate the boundary of the heterogeneity with a closed loop of n line doublets with endpoints $\{(x_i, y_i)\}_{0 \le i \le n}$ where $(x_n, y_n) = (x_0, y_0)$. Parameterize this boundary so that t_i maps to (x_i, y_i) for $0 \le i \le n$.
- Using Φ_B, express the jump in potential $\Delta\Phi_i$ due to the heterogeneity at each of the endpoints using the formula $\Delta\Phi_i = \bar{k}\Phi_i$ as above.
- Using t_i and $\Delta\Phi_i$, solve the SRCS matrix equation.
- Use the resulting coefficients to determine the strength of each line doublet, and add these potential fields to the background potential field to model the total potential of the system.

Comparison with Exact Solutions

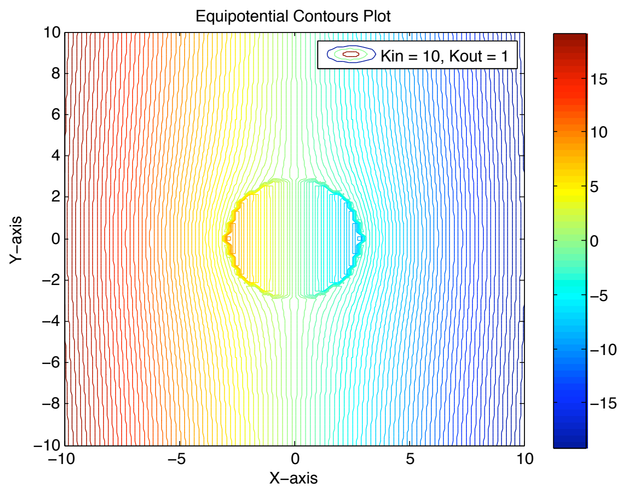
We implemented the algorithm described above in MATLAB 7.2, along with utilities for calculating the potential field of several simple flow features and displaying the combined total potential as a contour plot. For comparison, we also implemented the exact solution for a cylindrical heterogeneity with uniform flow along the x-axis, given in [3]. We found the same qualitative behaviors in the SRCS solution that appear in the exact solution. In the following contour plots, the exact solution is given on the bottom.

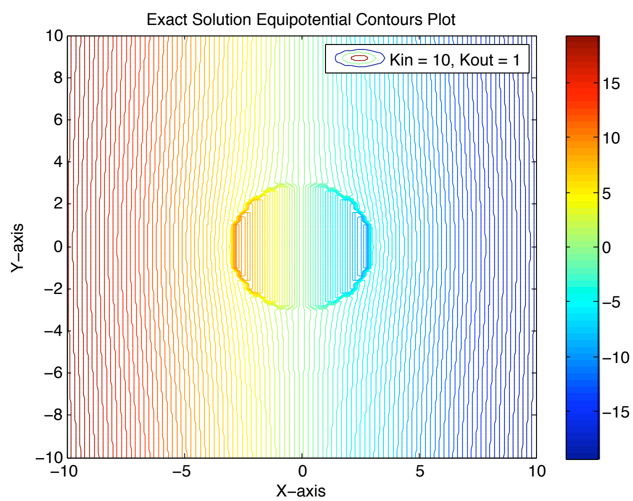
In this set, the external hydraulic conductivity is ten times the internal conductivity.





In this set, the internal hydraulic conductivity is ten times the external conductivity.

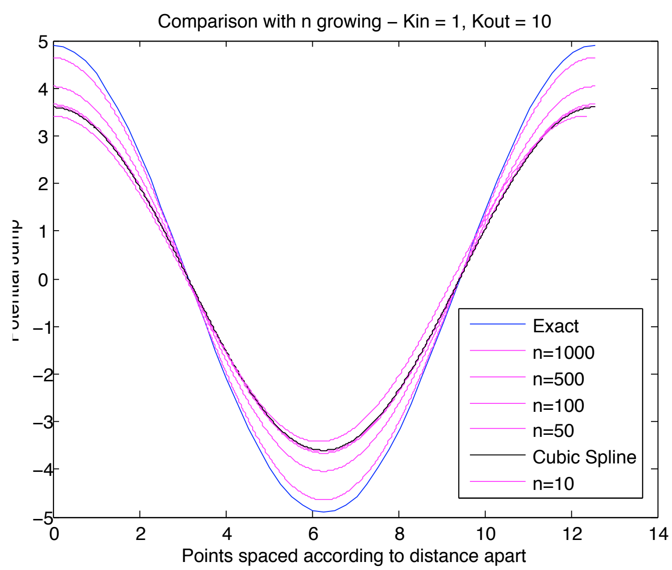




Conclusion and Future Research

Using self-referencing cubic splines as the strength distribution of a chain of third-order line doublets allows us to model reasonably well the qualitative effects of material heterogeneities within a groundwater aquifer. This technique is not specialized to the theory of groundwater flow; any model governed by potential theory could make use of self-referencing cubic splines.

As far as the general theory of SRCS is concerned, little is known. While we considered the existence and uniqueness properties of SRCS, we were unable to formulate reasonable limitations on influence functions, as in our specific examples we could not even expect continuity. Though we never encountered a singular SRCS matrix in our model, there are some obvious choices of influence functions that yield singular matrices. Specifically, one can consider the full SRCS matrix to be the sum of a normal cubic spline matrix and a matrix of influence functions. If the matrix of influence functions is chosen to be the negation of the normal cubic spline matrix, the full SRCS matrix is zero.



The graph above summarizes the main result of our project. The black curve represents a normal cubic spline interpolation with 500 data points. The series of pink curves represent the SRCS interpolation with 10, 50, 100, 500, and 1000 data points, respectively. Finally, the blue curve represents the strength distribution of the exact solution. We found that the SRCS spline with 500 endpoints more closely approximated the exact solution than the usual cubic spline, and also that increasing the number of data points reduced the error in the approximation.

When implementing the SRCS matrix equation for the strength coefficients of the third-order line doublets, we neglected the influence of a line doublet on itself, as the potential is discontinuous along the line doublet. We found that the error between the SRCS and the exact solution decreased as the number of points used to interpolate in the model increased. Therefore, we suspect that a better result would come from an algorithm that properly handled the effect of a line doublet on itself, and that such an algorithm would yield more accurate results with fewer line doublets.

References

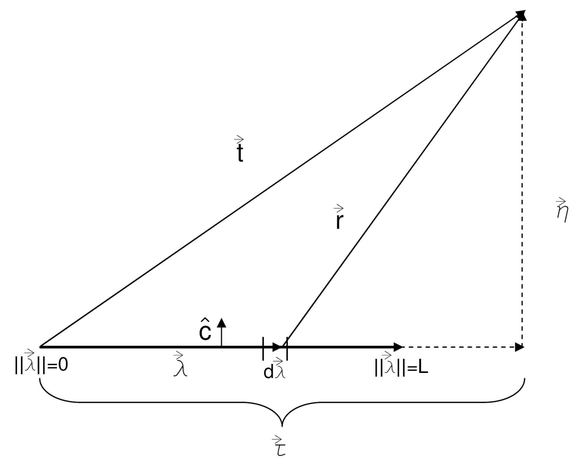
[1] Carl De Boor. *A Practical Guide to Splines*. Springer-Verlag, 1978.

[2] H.M. Haitjema. *Analytic Element Modeling of Groundwater Flow*. Academic Press, 1995.

[3] Otto D.L. Strack. *Groundwater Mechanics*. Prentice Hall, 1989.

Appendix

Apendix A: Geometry of a 2D Line Doublet



The line doublet is an analytic flow element created by either bringing a line source and a line sink of infinite strength infinitesimally close (in a way analogous with the point dipole) or by integrating a point dipole over a line. Since the latter route allows one to control the strength distribution of the line doublet more explicitly, we choose it. The above diagram details the geometry of the situation.

It is known that the potential of a 2D point dipole is

$$\Phi_{\mathrm{dp}}(r) = \frac{-\sigma(\hat{c} \cdot \vec{r})}{2\pi \|\vec{r}\|^2}$$

So one expects that the potential of a line doublet be

$$\Phi_{\mathrm{d}}(\vec{x}) = \int_{\mathrm{line}} \frac{-\sigma(\hat{c} \cdot \vec{r})}{2\pi \|\vec{r}\|^2}\, ds$$

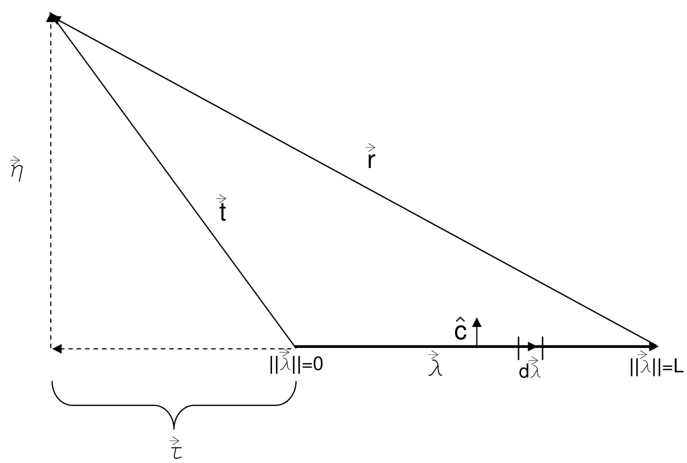
After parameterizing the line with $\lambda \in \left[0, L = \|\vec{L}\|\right]$, we can derive the following relations from the diagram

$$\begin{aligned} \|\vec{r}\|^2 &= \|\vec{\eta}\|^2 + (\lambda - \|\vec{\tau}\|)^2 \\ \hat{c} \cdot \vec{r} &= \hat{c} \cdot \vec{\eta} \end{aligned} \tag{1}$$

Note: These relations only hold if $\vec{\tau} = k\vec{\lambda}$ with $k \geq 0$. If $k < 0$, then we must redefine $\|\vec{\tau}\|$ appropiately as

$$\|\vec{\tau}\| := -\|\vec{\tau}\|$$

With this said, the derivations that follow can be considered to be a two pass process: once with $k > 0$ such as in the above diagram, and once with $k < 0$ as in the below diagram. Nicely, the derivations are the same using the redefinition of $\|\vec{\tau}\|$.



Also, as we want the potential for a third-order line doublet, we let the strength coefficient σ become cubic in the parameter λ, so that

$$\sigma(\lambda) = \sigma_0 + \sigma_1\lambda + \sigma_2\lambda^2 + \sigma_3\lambda^3$$

where $\sigma_0, \sigma_1, \sigma_2$, and σ_3 are real coefficients. We can then write the potential as

$$\Phi_{\mathrm{d}}(\vec{x}) = \sum_{i=0}^{3} \frac{-\sigma_i}{2\pi} \int_0^L \frac{\lambda^i(\hat{c}\cdot\vec{r})}{\|\vec{r}\|^2}\, d\lambda \tag{2}$$

Applying the two relations (1) to (2), the potential can be rewritten as

$$\Phi_{\mathrm{d}}(\vec{x}) = \sum_{i=0}^{3} \frac{-\sigma_i}{2\pi} \int_0^L \frac{\lambda^i(\hat{c}\cdot\vec{\eta})}{\|\vec{\eta}\|^2 + (\lambda - \|\vec{\tau}\|)^2}\, d\lambda \tag{3}$$

Factoring out $\|\vec{\eta}\|^2$ from the bottom of (3), we may rewrite this as

$$\Phi_{\mathrm{d}}(\vec{x}) = \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|^2}\sum_{i=0}^{3}\sigma_i\int_0^L \frac{\lambda^i\,d\lambda}{1+\left(\frac{\lambda-\|\vec{\tau}\|}{\|\vec{\eta}\|}\right)^2} \tag{4}$$

The following substitution is the initial step in solving each I_i, and for that reason we only apply it explicitly to I_0

$$\begin{aligned} u &= \frac{(\lambda-\|\vec{\tau}\|)}{\|\vec{\eta}\|} \\ \lambda &= \|\vec{\eta}\|u+\|\vec{\tau}\| \\ d\lambda &= \|\vec{\eta}\|du \end{aligned} \tag{5}$$

$$\begin{aligned} \Phi_{\mathrm{d}}(\vec{x}) &= \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|^2}\sum_{i=0}^{3}\sigma_i\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{\|\vec{\eta}\|(u\|\vec{\eta}\|+\|\vec{\tau}\|)^i}{1+u^2}du \\ &= \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\sum_{i=0}^{3}\sigma_i\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{(u\|\vec{\eta}\|+\|\vec{\tau}\|)^i}{1+u^2}du \end{aligned}$$

Define Φ_i with $i=0,1,2,$ and 3, as

$$\Phi_i = \sigma_i\frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{(u\|\vec{\eta}\|+\|\vec{\tau}\|)^i}{1+u^2}du \tag{6}$$

The following is a list of integrals which are computed in Appendix A to make the rest of our work easier.

$$\begin{aligned} I_0 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{1}{1+u^2}du = \arctan\left(\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}\right)+\arctan\left(\frac{\|\vec{\tau}\|}{\|\vec{\eta}\|}\right) \\ I_1 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{u}{1+u^2}du = \frac{1}{2}\ln\left(\frac{\|\vec{t}\|^2+L^2-2\|\vec{\tau}\|L}{\|\vec{t}\|^2}\right) \\ I_2 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{u^2}{1+u^2}du = \frac{L}{\|\vec{\eta}\|}-I_0 \\ I_3 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{u^3}{1+u^2}du = \frac{1}{2}\left(\frac{L^2-2L\|\vec{\tau}\|}{\|\vec{\eta}\|^2}\right)-I_1 \end{aligned}$$

We then solve each integral separately and write the full potential as a linear combination of σ_i and the integral terms, which we relabel as $I_i(\vec{x})$.

Solving Φ_0

$$\begin{aligned}\Phi_0 &= \sigma_0 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{1}{1+u^2}du \\ &= \sigma_0 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|} I_0\end{aligned}$$

Solving Φ_1

$$\begin{aligned}\Phi_1 &= \sigma_1 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{\|\vec{\eta}\|u+\|\vec{\tau}\|}{1+u^2}du \\ &= \sigma_1 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\left(\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{\|\vec{\tau}\|}{1+u^2}du + \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{\|\vec{\eta}\|u}{1+u^2}du\right) \\ &= \sigma_1 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\left(\|\vec{\tau}\|I_0 + \|\vec{\eta}\|I_1\right)\end{aligned}$$

Solving Φ_2

$$\begin{aligned}\Phi_2 &= \sigma_2 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{(\|\vec{\eta}\|u+\|\vec{\tau}\|)^2}{1+u^2}du \\ &= \sigma_2 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{(\|\vec{\tau}\|^2+2\|\vec{\tau}\|\|\vec{\eta}\|u+\|\vec{\eta}\|^2u^2)}{1+u^2}du \\ &= \sigma_2 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\left(\|\vec{\tau}\|^2 I_0 + 2\|\vec{\tau}\|\|\vec{\eta}\|I_1 + \|\vec{\eta}\|^2 I_2\right)\end{aligned}$$

Solving Φ_3

$$\begin{aligned}\Phi_3 &= \sigma_3 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{(\|\vec{\eta}\|u+\|\vec{\tau}\|)^3}{1+u^2}du \\ &= \sigma_3 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{\|\vec{\tau}\|^3+3\|\vec{\tau}\|^2\|\vec{\eta}\|u+3\|\vec{\tau}\|\|\vec{\eta}\|^2u^2+\|\vec{\eta}\|^3u^3}{1+u^2}du \\ &= \sigma_3 \frac{-(\hat{c}\cdot\vec{\eta})}{2\pi\|\vec{\eta}\|}\left(\|\vec{\tau}\|^3 I_0 + 3\|\vec{\tau}\|^2\|\vec{\eta}\|I_1 + 3\|\vec{\tau}\|\|\vec{\eta}\|^2 I_2 + \|\vec{\eta}\|^3 I_3\right)\end{aligned}$$

Result

Finally, we can state the full potential due to the line doublet as

$$\Phi_{\mathrm{d}}(\vec{x}) = \sum_{i=0}^{3} \Phi_i. \tag{7}$$

Appendix AA

$$
\begin{aligned}
I_0 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{1}{1+u^2} du \\
&= \arctan\left(\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}\right) - \arctan\left(\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}\right) \\
&= \arctan\left(\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}\right) + \arctan\left(\frac{\|\vec{\tau}\|}{\|\vec{\eta}\|}\right) \\
I_1 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{u}{1+u^2} du \\
&= \frac{1}{2}\ln(u^2+1)\Big|_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \\
&= \frac{1}{2}\ln\left(1+\left(\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}\right)^2\right) - \ln\left(1+\left(\frac{\|\vec{\tau}\|}{\|\vec{\eta}\|}\right)^2\right) \\
&= \frac{1}{2}\ln\left(\frac{\|\vec{\eta}\|^2+L^2+\|\vec{\tau}\|^2-2\|\vec{\tau}\|L}{\|\vec{\eta}\|^2}\right) - \ln\left(\frac{\|\vec{\eta}\|^2+{\|\vec{\tau}\|^2}^2}{\|\vec{\eta}\|}\right) \\
&= \frac{1}{2}\ln\left((\|\vec{\eta}\|^2+\|\vec{\tau}\|^2)+L^2-2\|\vec{\tau}\|L\right) - \ln\left(\|\vec{\eta}\|^2+\|\vec{\tau}\|^2\right) \\
&= \frac{1}{2}\ln\left(\|\vec{t}\|^2+L^2-2\|\vec{\tau}\|L\right) - \ln(\|\vec{t}\|^2) \\
&= \frac{1}{2}\ln\left(\frac{\|\vec{t}\|^2+L^2-2\|\vec{\tau}\|L}{\|\vec{t}\|^2}\right) \\
I_2 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{u^2}{1+u^2} du \\
&= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} 1 - \frac{1}{1+u^2} du \\
&= \frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|} - \frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|} - I_0 \\
&= \frac{L}{\|\vec{\eta}\|} - I_0
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} \frac{u^3}{1+u^2} du \\
&= \int_{\frac{-\|\vec{\tau}\|}{\|\vec{\eta}\|}}^{\frac{L-\|\vec{\tau}\|}{\|\vec{\eta}\|}} u^2 - \frac{u}{1+u^2} du \\
&= \frac{1}{2}\left(\frac{L^2 - 2L\|\vec{\tau}\|}{\|\vec{\eta}\|^2}\right) - I_1
\end{aligned}
$$

Appendix B: Cubic Spline Matrix Implementation

When we take a closer look at the mechanics used for solving this problem, we find there is a slightly shorter path to the solution which is still routed in the foundation of cubic splines. This method begins with decomposing a normal cubic spline coefficient matrix into two pieces. For this example we shall assume even spacing across the x-axis of the data points being used for interpolation. Solving a typical spline problem, we must solve the matrix equation

$$A\vec{x} = \vec{b}$$

Now we will decompose A into two pieces, L and T, both having the same size as A, and regroup these pieces

$$
\begin{aligned}
\vec{b} &= A\vec{x} \\
&= DT\vec{x} \\
&= D(T\vec{x}) \\
&= D\vec{t}
\end{aligned}
$$

D is a regular cubic spline matrix which interpolates on the data normally, but it has the change that every spline ranges from $x = 0$ to

$x = L$. Here is a D matrix set up to interpolate on four data points.

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & L & L^2 & L^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L & L^2 & L^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & L & L^2 & L^3 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2L & -3L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2L & -3L^2 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & -6L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & -6L \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6L \end{pmatrix}$$

T is an upper triangular matrix with 1's on the diagonal which was derived from shifting a polynomial piece of a spline interpolation alter its range from $[x_i, x_{i+1}]$ to $[0, L = x_{i+1} - x_i]$.

The next step is to combine this decomposition technique with the influence function matrix B that contains all the information about how the line doublets will affect each other. For a small example where we use three data points to approximate a circle. Since we have neglected the influence of any line doublet connected to a control point, there is only one line doublet affecting each control point. Conveniently, since the distances are all the same, the flow functions are the same save for the strength coefficients.

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_1 & F_L & F_{L^2} & F_{L^3} \\ F_1 & F_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & F_1 & F_L & F_{L^2} & F_{L^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_1 & F_L & F_{L^2} & F_{L^3} \\ F_1 & F_L & F_{L^2} & F_{L^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_1 & F_L & F_{L^2} & F_{L^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since this information is given with the notion that each line doublet ranges from $t = 0$ to $t = L$, we must also multiply this matrix by the shifting matrix T. Combining this with the work above, we have

$$\begin{aligned} A\vec{x} &= \vec{b} + BT\vec{x} \\ DT\vec{x} - BT\vec{x} &= \vec{b} \\ (D - B)(T\vec{x}) &= \vec{b} \\ (D - B)(\vec{t}) &= \vec{b} \end{aligned}$$

Solving the matrix equation produces the information to plot the line doublets with strength distributions ranging from $t = 0$ to $t = L$. If we would like to connect the polynomials to produce a strength distribution graph (as you would typically see for a regular cubic spline interpolation), we must only multiply our vector of coeffiencits by T^{-1}, that is $T^{-1}\vec{t} = \vec{x}$.

Appendix BB: Shifting Matrix T

When interpolating with cubic splines, we form a sequence of cubic polynomials that connect the the various knots or control points which are at t_i with $i = 1, 2, ..., n$. Let these polynomials be denoted as $\hat{S}_i(t)$. These polynomials have all the "nice" features of continuity, smoothness with first and second derivatives, and periodicity. Each interval $[t_i, t_{i+1}]$ has a corresponding spline $\hat{S}_i$ which we want to map to the contour plot appropriately. If we choose $S_i(t)$ to be the cubic polynomial that describes the strength distribution across the line doublet corresponding to the interval $[t_i, t_{i+1}]$, then since the line doublets start at $t = 0$, we should see

$$S_i(t) = \hat{S}_i(t + t_i).$$

This satisfies the property that we should see $S_i(0) = \hat{S}_i(t_i)$. If we already have completed the cubic spline interpolation, then we know the values of the coefficients for each $\hat{S}_i(t)$ and for the control points t_i. Let $S_i(t) = \sigma_{i0} + \sigma_{i1}t + \sigma_{i2}t^2 + \sigma_{i3}t^3$ and $\hat{S}_i(t) = \hat{\sigma}_{i0} + \hat{\sigma}_{i1}t + \hat{\sigma}_{i2}t^2 + \hat{\sigma}_{i3}t^3$.

$$\begin{aligned} S_i(t) &= \hat{S}_i(t + t_i) \\ &= \hat{\sigma}_{i0} + \hat{\sigma}_{i1}(t + t_i) + \hat{\sigma}_{i2}(t + t_i)^2 + \hat{\sigma}_{i3}(t + t_i)^3 \\ &= \hat{\sigma}_{i0} + \hat{\sigma}_{i1}t + \hat{\sigma}_{i1}t_i + \hat{\sigma}_{i2}t^2 + 2\hat{\sigma}_{i2}tt_i + \hat{\sigma}_{i2}t_i^2 + \hat{\sigma}_{i3}t^3 + 3\hat{\sigma}_{i3}t^2t_i + 3\hat{\sigma}_{i3}tt_i^2 + \hat{\sigma}_{i3}t_i^3 \\ &= \left(\hat{\sigma}_{i0} + \hat{\sigma}_{i1}t_i + \hat{\sigma}_{i2}t_i^2 + \hat{\sigma}_{i3}t_i^3\right) + t\left(\hat{\sigma}_{i1} + 2\hat{\sigma}_{i2}t_i + 3\hat{\sigma}_{i3}t_i^2\right) + t^2\left(\hat{\sigma}_{i2} + 3\hat{\sigma}_{i3}t_i\right) + t^3\left(\hat{\sigma}_{i3}\right) \end{aligned}$$

To wrap this information up, from the above relation on $S_i(t)$ and $\hat{S}_i(t)$ we can infer that

$$\begin{bmatrix} \sigma_{i0} \\ \sigma_{i1} \\ \sigma_{i2} \\ \sigma_{i3} \end{bmatrix} = \begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 \\ 0 & 1 & 2t_i & 3t_i^2 \\ 0 & 0 & 1 & 3t_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\sigma}_{i0} \\ \hat{\sigma}_{i1} \\ \hat{\sigma}_{i2} \\ \hat{\sigma}_{i3} \end{bmatrix}$$

or more simply we let $\vec{\sigma} = [\sigma_{i0}\ \sigma_{i1}\ \sigma_{i2}\ \sigma_{i3}]^t$ and $\vec{\hat{\sigma}} = [\hat{\sigma}_{i0}\ \hat{\sigma}_{i1}\ \hat{\sigma}_{i2}\ \hat{\sigma}_{i3}]^t$. Then

$$\vec{\sigma} = T(t_i)\vec{\hat{\sigma}}$$

where $T(t_i)$ is called the *shifting matrix block* for the point t_i. The shifting matrix T is a square matrix of size $4n$ with n shifting matrix blocks on the diagonal. For example, a shifting matrix for $n = 3$ would look like

$$T = \begin{pmatrix} 1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2t_1 & 3t_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2t_3 & 3t_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2t_3 & 3t_3^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3t_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$