VIRTUALLY CYCLIC SUBGROUPS OF THREE-DIMENSIONAL CRYSTALLOGRAPHIC GROUPS

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Abstract. An enumeration of the virtually cyclic subgroups of the three-dimensional crystallographic groups ("space groups") is given. Additionally, we offer explanations of the underlying group theory and develop several exclusion theorems which simplify our calculations.

1. Introduction

1.1. **Crystallographic Groups.** The crystallographic groups may be understood geometrically, linear-algebraically, or abstract-algebraically. All three approaches will play a role in the theorems that follow, so we will outline all three here. The geometric approach is the most easily understood by intuition alone, so we will explore it first.

1.1.1. *Geometric Approach.* We first consider an infinite regular tiling of a Euclidean plane; similar efforts may be made on spherical or hyperbolic geometries, but this is beyond the scope of our research. We require that there be two linearly independent translations which each preserve the pattern's symmetry— that is, that we can move the plane by specific finite distances in two distinct directions and return the pattern to its original state. Beyond these two required translations, we may be able to reflect the plane across some line or rotate it about the origin and preserve its symmetry; in some cases, it may be possible to translate and then reflect the plane such that neither the translation nor the reflection alone preserve symmetry, but their combination does— this will be termed a "glide-reflection." In all there are seventeen possible combinations of translations, reflections, rotations, and glide-reflections which uniquely describe ways to preserve symmetries of plane patterns. For further discussion of the geometry of these groups, see the excellent chapter in Gallian [2].

The three– and n-dimensional crystallographic cases represent the same fundamental concept extended into a higher-dimensional space. For three dimensions this may be understood as the symmetries of a three-dimensional object like a chemical crystal; in this case, operations called inversion, axial screwing, and rotoinversion are added. For higher dimensions than this it is necessary to call on more abstract geometries or to shift to the linear– or abstract-algebraic model of crystallographic groups.

1.1.2. *Linear–Algebraic Approach.* Now, instead of rigid motions in n-dimensional Euclidean space, we consider algebraic isometries of an n-dimensional vector space V A symmetry operator may be represented as (v, ϕ), denoting a map $x \to v + \phi x$ with $v, x \in V$ and ϕ an $n \times n$ orthogonal matrix. In this notation, v represents any translation like those described in 1.1.1, ϕ represents a reflection, rotation, or other transform which fixes the origin, and combinations with nontrivial v and ϕ components denote the glide-reflections and other compound operations. This is the approach most frequently used by physical chemsists in their analysis of crystal structures; a mathematical discussion, including a simple but thorough proof of the completeness of the list of 17 plane groups and some ties to topology, is found in Schwarzenberger [6].

1.1.3. *Abstract–Algebraic Approach.* We now forgo direct consideration of the n-space entirely and consider only the structure of its isometries. Since the symmetry operations from 1.1.2 under composition satisfy requirements of associativity, identity, and invertibility, but do not necessarily commute, they may be considered as elements of non-Abelian groups. The translations described in 1.1.1 and represented as vector addition in 1.1.2 are a subgroup and provide the basis for the definition of an n-dimensional crystallographic group developed by Bieberbach.

Definition 1.1. A group G is n-dimensional crystallographic if it admits a subgroup T such that:

(1) $T \cong (\mathbb{Z})^n$
(2) $T \triangleleft G$
(3) T is maximal abelian in G

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(4) $[G : T] < \infty$

This subgroup T is the "Bieberbach lattice" of G.

The matrix operations described in 1.1.2 also form a subgroup, although it is not normal. The more complicated compound operations in the space are parts of the action of these matrix operations on T. Together, these two types of operations form the so-called "point group" F — a misleading name, since it is not necessarily a subgroup of G.

Several properties are of interest here:

- All origin-preserving elements of the point group F have finite order.
- All elements of the lattice T have infinite order.
- For each compound element $z \in F$, z has infinite order, and there is some finite power n such that $z^n \in T$.

1.2. **Infinite virtually cyclic subgroups.** Ultimately we intend to compute the infinite virtually cyclic subgroups of the 3-dimensional crystallographic groups. To do so, we must explain what these subgroups are.

1.2.1. *Definitions.*

Definition 1.2. A group G is cyclic if and only if it is generated by a single element.

All cyclic groups G are isomorphic either to $\mathbb{Z}_n, n \in \mathbb{Z}^+$ or to $\mathbb{Z}$ according as there is or is not some finite integer i such that, for each element $g \in G$, $g^i = e$.

Definition 1.3. The index of a subgroup H in its supergroup G (written $[G : H]$) is the order of the factor group G/H; if the factor group is finite, this is the same as the number of cosets of H in G. Equivalently, it is the number of elements $g \in G$ such that gH— the coset of H by g— is a distinct set.

A group G is virtually cyclic if it has a subgroup H of finite index such that H is cyclic.

1.2.2. *Virtually cyclic groups.* All finite groups are virtually cyclic because they admit the trivial group as a subgroup of finite index, and the trivial group is cyclic. Hence, to compute the finite virtually cyclic subgroups of the 3-dimensional crystallographic groups is simply to compute the finite subgroups.

Infinite virtually cyclic subgroups are significantly less trivial to construct. There are two distinct types — semidirect products and amalgamated free products.

1.3. **Semidirect products.** The semidirect product of two groups is a generalization of the direct product through an automorphism. We will use the notation $A \rtimes_\alpha B$, where A, B are groups such that $A \triangleleft A \rtimes_\alpha B$ and α denotes a homomorphism $\alpha : B \to \text{Aut}\, A$. Given the group structures of A, B and the homomorphism α, the group structure of the semidirect product is determined as follows: for elements $a_i \in A, b_i \in B$,

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot \alpha(b_1)a_2, b_1 \cdot b_2)$$

$A \rtimes_\alpha B$ reduces to $A \times B$ when $\alpha(b) : a \to a$ for all $a \in A, b \in B$.

Example 1.4. Consider the groups $\mathbb{Z}_3 \cong \langle x | x^3 = e \rangle$, $\mathbb{Z}_2 \cong \langle y | y^2 = e \rangle$. Let $\alpha(y) : x \to x^2$. Then

$$\mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2 \cong \langle x, y | x^3 = y^2 = e, yx = x^2 y \rangle.$$

The calculation of these semidirect product groups is simplified dramatically when groups A, B are restricted, as they will be in 2.1.3. Specifically, A will be a finite subgroup of a crystallographic group, and $B \cong \mathbb{Z}$.

1.4. **Amalgamated Free Products.** Let A and B be groups and $B \triangleleft A$, $B \triangleleft C$. Then the amalgamated free product $A \underset{B}{*} C$ is generated by the elements of A and of C with the common elements from B identified.

Example 1.5. Consider the groups $D_3 \cong \langle r, f | r^3 = f^2 = e, fr = r^2 f \rangle$ and $\mathbb{Z}_6 \cong \langle x | x^6 = e \rangle$ with subgroup $\mathbb{Z}_3$. Then

$$D_3 \underset{\mathbb{Z}_3}{*} \mathbb{Z}_6 \cong \langle r, f, x | r^3 = f^2 = x^6 = e, x^2 = r, fr = r^2 f \rangle.$$

1.5. **Completeness.** According to the work of Scott and Wall [7], given a group G, the only infinite virtually cyclic subgroups of G will be semidirect products $N \rtimes_\alpha \mathbb{Z}$ and amalgamated free products $A \underset{B}{*} C$ for $N, A, B, C < G$. Combining this result with a list of finite virtually cyclic subgroups of G then then yields the complete list of virtually cyclic subroups of G.

2. Computational technique for semidirect products

2.1. **Methods.** To compute the semidirect products which will give us candidate infinite virtually cyclic subgroups, we first must know the finite subgroups of the 3-dimensional crystallographic groups and the automorphisms of those groups.

2.1.1. *Finite Subgroups.* The finite subgroups of the crystallographic groups may be derived exclusively from their point groups, because any finite subgroup of a 3-dimensional crystallographic group G is isomorphic to a subgroup of the point group of G. Information about the group structure of these point groups is found in the Tables [3]; a group-theoretical enumeration of them follows easily. Categorized by complexity and structural class, they are:

- e
- $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$
- D_2, D_3, D_4, D_6
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_6$
- $D_3 \times \mathbb{Z}_2, D_4 \times \mathbb{Z}_2, D_6 \times \mathbb{Z}_2$
- $A_4, S_4, A_4 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2$

2.1.2. *Automorphisms.* We next must know the automorphisms of each of these subgroups in order to compute their semidirect products. The exclusion theorems in Section 3 reduce dramatically the list of subgroups that need be considered, so the automorphisms will not be fully enumerated here. They are listed in their entirety in Table 1.

2.1.3. *Computing the Semidirect Product.* We are concerned with semidirect products of the form $N \rtimes_\alpha \mathbb{Z}$, where N is a finite subgroup of a crystallographic group. As explained in 1.3, the semidirect product is computed (in this case) using $\alpha : \mathbb{Z} \to \operatorname{Aut} N$; since $\mathbb{Z}$ is cyclic, the behavior of α is completely determined by the choice of some $\phi \in \operatorname{Aut} N$ such that $\alpha(1) = \phi$. In fact, given a presentation $N \cong \langle n_1, n_2, \ldots, n_i | \ldots \rangle$, we may obtain a presentation

$$N \rtimes_\alpha \mathbb{Z} \cong \langle n_1, n_2, \ldots, n_i, x | ..., x \cdot n_j = \phi(n_j) \cdot x \rangle$$

2.2. **Identifying Semidirect Products in the International Tables.** We begin by choosing a finite group N from the initial list of candidates in 2.1.1. We then first consider whether it satisfies the requirement of Theorem 3.2. If it does not, we may exclude $N \rtimes_\alpha \mathbb{Z}$ for all α by Lemma 3.3. Elsewise, we know categorically that $N \times \mathbb{Z}$ is a subgroup of some 3-dimensional crystallographic group because it is simply a 2-dimensional point group coupled with an orthogonal translation. We then determine $\operatorname{Aut} N$ and group these automorphisms into conjugacy classes in accordance with Theorem 3.5. One representative from each conjugacy class is chosen; as all are equivalent, those which are computationally simplest are the best choices. The requirements of Theorem 3.6 provide our final exclusion; if they are not satisfied for a given representative automorphism, its conjugacy class cannot contribute to our listing, but if they are satisfied then its conjugacy class will be found as a subgroup of some 3-dimensional crystallographic group.

2.3. **Results.** In 3 we have obtained several exclusion theorems on the virtually cyclic subgroups of 3-dimensional crystallographic groups as obtained through semidirect products. Additionally, we have a complete listing of these semidirect products, including their supergroups and information from the International Tables where applicable.

3. Exclusion Theorems for Semidirect Products

To begin the process of exclusion we require the following lemma.

Lemma 3.1. *In a crystallographic group G, let $f = (0, B)$ be a finite-ordered element and $x = (v, I)$ be a translation. Then f commutes with x if and only if B fixes v.*

Proof. First, we assume that f commutes with x.

$$fx = xf$$
$$(0, B)(v, I) = (v, I)(0, B)$$
$$(Bv, B) = (v, B)$$
$$Bv = v$$

Hence, that f commutes with x implies that B fixes v. If we conversely assume that B fixes v, it follows readily by the same process that f commutes with x. □

With this lemma we establish a theorem which will exclude many of the large candidate finite subgroups.

Theorem 3.2. *$F \times \mathbb{Z}$ is a subgroup of some d-dimensional crystallographic group if and only if F is a subgroup of some $(d-1)$-dimensional crystallographic group.*

Proof. First, assume that F is a subgroup of some $(d-1)$-dimensional crystallographic group G. Then $F \times \mathbb{Z}$ is a subgroup of the d-dimensional crystallographic group $G \times \mathbb{Z}$.

Conversely, assume that $F \times \mathbb{Z}$ is a subgroup of some d-dimensional crystallographic group H. Since in this direct product the generator $x \in \mathbb{Z}$ commutes with all $f \in F$ and $x \in T$, T the Bieberbach lattice of H, we see that there may be at most $d-1$ independent elements $t \in T$ which do not commute with a given $f \in F$. Therefore, F must be a subgroup of some $(d-1)$-dimensional crystallographic group. □

The following well-known result will help us to exclude the semidirect products of a candidate finite group based only on its direct product.

Lemma 3.3. *If a crystallographic group G admits a subgroup $F \rtimes_\alpha \mathbb{Z}$ for some finite group F and some homomorphism $\alpha : \mathbb{Z} \to \operatorname{Aut} F$, then G also admits a subgroup $F \times \mathbb{Z}$.*

Proof. Since $\mathbb{Z}$ is cyclic, α is completely determined by the automorphism $\alpha(1) = \phi$. Since F is finite, $\operatorname{Aut} F$ is also finite, so $\phi^n = I$ for some finite n. If we then consider only those elements $y = x^{kn} \in \mathbb{Z} \forall k \in \mathbb{Z}$, we find that $fy = yf \forall f \in F$. Hence, the subgroup of $F \rtimes_\alpha \mathbb{Z}$ generated by $f \in F$ and $y = x^{kn} \in \mathbb{Z}$ is a direct product $F \times \mathbb{Z}$. □

Note that it is the contrapositive of this theorem which provides the exclusion we need.

We also note the following corollary.

Corollary 3.4. *For any semidirect product $F \rtimes_\alpha \mathbb{Z}$ a subgroup of a crystallographic group with $\mathbb{Z} \cong \langle x \rangle$, the elements $f \in F$ fix the shift vectors $x^k \in T$.*

Proof. Given $G = F \rtimes_\alpha \mathbb{Z}$ a subgroup of a crystallographic group, we consider the subgroup $H < G$ such that $H \cong F \times \mathbb{Z}$ shown to exist in the proof of Lemma 3.3. The elements $z \in \mathbb{Z}$ of H correspond to the shift vectors $x^k \in \mathbb{Z}$ of G, and becuase H is a direct product $F \times \mathbb{Z}$ all $f \in F$ commute with all $z \in H$. By Lemma 3.1, all $f \in F$ then fix all $x^k \in G$. □

We also utilize this theorem from Alperin and Bell [1]. Its proof is reprinted here for convenience, with notation modified for consistency with our conventions.

Theorem 3.5. *Let F and H be groups, let $\alpha : H \to \operatorname{Aut}(F)$ be a homomorphism, and let $\phi \in \operatorname{Aut} F$. If $\widehat{\phi}$ is the inner automorphism of $\operatorname{Aut}(F)$ induced by ϕ, then $F \rtimes_{\widehat{\phi}\circ\alpha} H \cong F \rtimes_\alpha H$.*

Proof. Define $\theta : F \rtimes_\alpha H \to F \rtimes_{\widehat{\phi}\circ\alpha} H$ by $\theta(fh) = \phi(f)h$. We have

$$\begin{aligned}
\theta(f_1h_1f_2h_2) &= \theta(f_1\alpha(h_1)(f_2)h_1h_2) \\
&= \phi(f_1)\phi(\alpha(h_1)(f_2))h_1h_2 \\
&= \phi(f_1) \cdot (\phi \circ \alpha(h_1) \circ \phi^{-1} \circ \phi)(f_2) \cdot h_1h_2 \\
&= \phi(f_1) \cdot (\widehat{\phi} \circ \alpha)(h_1)(\phi(f_2)) \cdot h_1h_2 \\
&= \phi(f_1)h_1\phi(f_2)h_2 = \theta(f_1h_1)\theta(f_2h_2),
\end{aligned}$$

which shows that θ is a homomorphism. But the homomorphism sending $fh \in F \rtimes_{\widehat{\phi}\circ\alpha} H$ to $\phi^{-1}(f)h \in F \rtimes_\alpha H$ is inverse to θ, and therefore θ is an automorphism. □

This result implies that, if we identify the conjugacy classes of automorphisms of a given group, we need to consider only one element of each class to evaluate the candidacy of all automorphisms in that class.

Now we develop a test to determine whether a given semidirect product of a group is or is not a subgroup of any d-dimensional crystallographic group. $\widehat{a}$ will denote conjugation by a.

Theorem 3.6. *Let F be a finite subgroup of some 2-dimensional crystallographic group, $\alpha : \mathbb{Z} \to \operatorname{Aut} F$ be a homomorphism, and $\alpha(1) = \phi \in \operatorname{Aut} F$. Then $F \rtimes_\alpha \mathbb{Z}$ is a subgroup of some 3-crystallographic group if and only if $\phi = \widehat{m}$ for some $m \in M$, with M also a finite subgroup of some 2-dimensional crystallographic group and $F < M$.*

Proof. In $F \rtimes_\alpha \mathbb{Z}$ with $\mathbb{Z} \cong \langle x \rangle$ and $f \in F$ we have the relation $x^{-1}fx = \phi(f)$. Considered linear-algebraically, $x = (v, A)$ and $f = (0, B)$ for some vector v and orthogonal 3×3 matrices A, B. Following the implications of this relation, we have:

$$x^{-1}nx = \phi(n)$$
$$(-A^{-1}v, A^{-1})(0, B)(v, A) = (0, \psi(B))$$
$$(-A^{-1}v + A^{-1}Bv, A^{-1}BA) = (0, \psi(B))$$

So $A^{-1}BA = \psi(B)$. Accordingly, since $\phi = \widehat{x}$ implies $\psi = \widehat{A}$, if we let $y = (0, A)$ we also have $\phi = \widehat{y}$. Note that no compound operation $z = (v', A')$ in the 3-dimensional crystallographic groups has v' not fixed by A'. Now consider the group M generated by y and all $f \in F$. M must also be a finite subgroup of some 2-dimensional crystallographic group because, for all $z = (v', A')$ in the 3-dimensional crystallographic groups, A' fixes v'.

Now we consider a case in which there is no y such that $y \in M$ for some group M with $F < M$ for which $\phi = \widehat{y}$. Since it is clear that $\phi = \widehat{x}$, $\phi \neq \widehat{y}$ implies that A does not fix v for $x = (v, A)$. But this is not the case for any 3-dimensional compound operation, so there exists no x for which this can be a valid relation. □

4. Computational Technique for Amalgamated Free Products

4.1. **Methods.** First, using the same list of finite subgroups ennumerated above, a list of all posible amalgamated free products is obtained. Using the following theorem, the list is dramatically reduced.

Theorem 4.1. *If the presentation of the amalgamated free product contains two or more elements of order two that do not commute, then the amalgamated free product is not a subgroup of any three dimensional crystallographic group.*

Proof. Note that in three dimensions there are only three possible elements of order two: inversion, 180° rotation, and reflection. It can easily be shown geometrically that all of these symmetry moves commute. Therefore, an amalgamated free product with two order two elements that do not commute cannot exist in three dimensions. □

4.2. **Results.** Examples of all cases not covered by the previous theorem are listed in Table 3.

5. Results

We have identified all the finite virtually cyclic subgroups of the 3-dimensional crystallographic groups as well as all the infinite virtually cyclic subgroups obtained through semidirect products and amalgamated free products. In accordance with 1.5,there can be no other cyclic subgroups of the 3-dimensional crystallographic groups, so this enumeration is complete.

Appendix A. Computational Tables

A.1. **Semidirect Products.** We enumerate $N \rtimes_\alpha \mathbb{Z}$ for finite subgroups N of the 3-dimensional crystallographic groups; where possible, they are identified within actual crystallographic groups.

A.1.1. *Finite groups.* All the finite subgroups of the 3-dimensional crystallographic groups are virtually cyclic. They are listed in 2.1.1.

A.1.2. *Infinite groups.* The candidate infinite virtually cyclic subgroups of the 3-dimensional crystallographic groups are presented in Table 2. When a candidate conjugacy class is viable, it is listed with its member automorphisms and an example of generators taken from the International Tables. For this table, the $\mathbb{Z}$ component of the semidirect product will always be taken to be the z-axis. Accordingly, all rotations will be measured counter-clockwise about this axis, all reflection planes will be described by lines in the xy-plane, and all compound operations will be described by "glide-reflection by ..." or "screw by ..."

Note that by Theorem 3.2 we may exclude all finite subgroups other than $\mathbb{Z}_n$ and D_n. Accordingly, the automorphisms of these groups and the semidirect products over them will not be considered.

A.2. **Amalgamated Free Products.** We enumerate $A \underset{B}{*} C$ for triples of finite subgroups A, B, C of the 3-dimensional crystallographic groups such that $[A : B] = [C : B] = 2$; where possible, they are identified within actual crystallographic groups.

References

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Finite Group	Automorphism Name	Mapping	
$\mathbb{Z}_3$	ϕ_2	$x \to x^2$	
$\mathbb{Z}_4$	ϕ_2	$x \to x^3$	
$\mathbb{Z}_6$	ϕ_2	$x \to x^5$	
D_2	ϕ_2	$r \to r$	$f \to rf$
	ϕ_3	$r \to f$	$f \to r$
	ϕ_4	$r \to rf$	$f \to f$
	ϕ_5	$r \to f$	$f \to rf$
	ϕ_6	$r \to rf$	$f \to r$
D_3	ϕ_2	$r \to r$	$f \to rf$
	ϕ_3	$r \to r$	$f \to r^2f$
	ϕ_4	$r \to r^2$	$f \to f$
	ϕ_5	$r \to r^2$	$f \to rf$
	ϕ_6	$r \to r^2$	$f \to r^2f$
D_4	ϕ_2	$r \to r$	$f \to rf$
	ϕ_3	$r \to r$	$f \to r^2f$
	ϕ_4	$r \to r$	$f \to r^3f$
	ϕ_5	$r \to r^3$	$f \to f$
	ϕ_6	$r \to r^3$	$f \to rf$
	ϕ_7	$r \to r^3$	$f \to r^2f$
	ϕ_8	$r \to r^3$	$f \to r^3f$
D_6	ϕ_2	$r \to r$	$f \to rf$
	ϕ_3	$r \to r$	$f \to r^2f$
	ϕ_4	$r \to r$	$f \to r^3f$
	ϕ_5	$r \to r$	$f \to r^4f$
	ϕ_6	$r \to r$	$f \to r^5f$
	ϕ_7	$r \to r^5$	$f \to f$
	ϕ_8	$r \to r^5$	$f \to rf$
	ϕ_9	$r \to r^5$	$f \to r^2f$
	ϕ_{10}	$r \to r^5$	$f \to r^3f$
	ϕ_{11}	$r \to r^5$	$f \to r^4f$
	ϕ_{12}	$r \to r^5$	$f \to r^5f$

Table 1: Automorphisms of finite subgroups of 3DCG's

Finite Group	Conj. Class in Aut N	3DCG Supergroup	Generators
$\mathbb{Z}_2$	I	IT #3	(2)– rotation by 180° $t(0,0,1)$– translation along z-axis
$\mathbb{Z}_3$	I	IT #143	(2)– rotation by 120° $t(0,0,1)$– translation along z-axis
	ϕ_2	IT #159	(2)– rotation by 120° (6)– glide-reflection over y-axis
$\mathbb{Z}_4$	I	IT #75	(3)– rotation by 90° $t(0,0,1)$– translation along z-axis
	ϕ_2	IT #103	(3)– rotation by 90° (5)– glide-reflection over x-axis
$\mathbb{Z}_6$	I	IT #168	(6)– rotation by 60° $t(0,0,1)$– translation along z-axis
	ϕ_2	IT #184	(6)– rotation by 60° (11)– glide-reflection over x-axis
D_2	I	IT #25	(2)– rotation by 180° (3)– reflection over x-axis $t(0,0,1)$– translation along z-axis
	ϕ_2, ϕ_3, ϕ_4	IT #105	(2)– rotation by 180° (6)– reflection over y-axis (3)– screw by 90°
	ϕ_5, ϕ_6	Impossible by Theorem 3.6	
D_3	I	IT #156	(2)– rotation by 120° (5)– reflection over line $y = 2x$ $t(0,0,1)$– translation along z-axis
	ϕ_2, ϕ_3	IT #157	(2)– rotation by 120° (6)– reflection over y-axis $(2) \circ t(0,0,1)$– screw by 120°
	ϕ_4, ϕ_5, ϕ_6	IT #185	(2)– rotation by 120° (11)– reflection over x-axis (8)– glide-reflection over line $y = 2x$
D_4	I	IT #99	(3)– rotation by 90° (5)– reflection over x-axis $t(0,0,1)$– translation along z-axis
	ϕ_3	IT #99	(3)– rotation by 90° (6)– reflection over y-axis $(3) \circ t(0,0,1)$– screw by 90°
	ϕ_5, ϕ_7	IT #99	(3)– rotation by 90° (5)– reflection over x-axis $(5) \circ t(0,0,1)$– glide-reflection over x-axis
	$\phi_2, \phi_4, \phi_6, \phi_8$	Impossible by Theorem 3.6	
D_6	I	IT #183	(6)– rotation by 60° (11)– reflection over x-axis $t(0,0,1)$– translation along z-axis
	ϕ_3, ϕ_5	IT #193	(6)– rotation by 60° (12)– reflection over y-axis $(3) \circ t(0,0,1)$– screw by 60°
	ϕ_7, ϕ_8, ϕ_9	IT #183	(5)– rotation by 60° (11)– reflection over x-axis $(12) \circ t(0,0,1)$– glide-reflection over y-axis
	$\phi_{10}, \phi_{11}, \phi_{12}$	IT #183	(5)– rotation by 60° (11)– reflection over x-axis $(8) \circ t(0,0,1)$– glide-revlection over line $y = 2x$
	ϕ_2, ϕ_4, ϕ_6	Impossible by Theorem 3.6	

Table 2: Semidirect products in the 3DCG's

Amalgamated Free Product	3DCG Supergroup	Generators
$\mathbb{Z}_2 \underset{e}{*} \mathbb{Z}_2$	IT #25	(3)– reflection over x-axis (4)– reflection over y-axis
$\mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_4$	IT #123	(3)– rotation by 90° (11)– rotoinversion by 90°
$\mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} D_2$	IT #123	(2)– rotation by 180° (3)– rotation by 90° (13)– reflection over x-axis
$D_2 \underset{\mathbb{Z}_2}{*} D_2$	Impossible by Theorem 4.1	
$\mathbb{Z}_6 \underset{\mathbb{Z}_3}{*} \mathbb{Z}_6$	IT #191	(5)– rotation by 60° (17)– rotoinversion by 60°
$\mathbb{Z}_6 \underset{\mathbb{Z}_3}{*} D_3$	IT #191	(2)– rotation by 120° (5)– rotation by 60° (23)– reflection over x-axis
$D_3 \underset{\mathbb{Z}_3}{*} D_3$	Impossible by Theorem 4.1	
$D_4 \underset{\mathbb{Z}_4}{*} D_4$	Impossible by Theorem 4.1	
$\mathbb{Z}_4 \times \mathbb{Z}_2 \underset{\mathbb{Z}_4}{*} \mathbb{Z}_4 \times \mathbb{Z}_2$	Impossible by Theorem 4.1	
$\mathbb{Z}_4 \times \mathbb{Z}_2 \underset{\mathbb{Z}_4}{*} D_4$	Impossible by Theorem 4.1	
$D_4 \underset{D_2}{*} D_4$	IT #123	(3)– rotation by 90° (11)– rotoinversion by 90° (13)– reflection over x-axis
$D_2 \times \mathbb{Z}_2 \underset{D_2}{*} D_2 \times \mathbb{Z}_2$	Impossible by Theorem 4.1	
$D_2 \times \mathbb{Z}_2 \underset{D_2}{*} D_4$	IT #123	(2)– rotation by 180° (3)– rotation by 90° (9)– inversion through origin (13)– reflection over x-axis
$D_6 \underset{\mathbb{Z}_6}{*} D_6$	Impossible by Theorem 4.1	
$\mathbb{Z}_6 \times \mathbb{Z}_2 \underset{\mathbb{Z}_6}{*} \mathbb{Z}_6 \times \mathbb{Z}_2$	Impossible by Theorem 4.1	
$\mathbb{Z}_6 \times \mathbb{Z}_2 \underset{\mathbb{Z}_6}{*} D_6$	Impossible by Theorem 4.1	
$D_6 \underset{D_3}{*} D_6$	IT #191	(5)– rotation by 60° (17)– rotoinversion by 60° (23)– reflection over x-axis
$D_3 \times \mathbb{Z}_2 \underset{D_3}{*} D_3 \times \mathbb{Z}_2$	Impossible by Theorem 4.1	
$D_3 \times \mathbb{Z}_2 \underset{D_3}{*} D_6$	IT #191	(2)– rotation by 120° (5)– rotation by 60° (13)– inversion through origin (23)– reflection over x-axis
$D_4 \times \mathbb{Z}_2 \underset{D_4}{*} D_4 \times \mathbb{Z}_2$	Impossible by Theorem 4.1	
$D_6 \times \mathbb{Z}_2 \underset{D_6}{*} D_6 \times \mathbb{Z}_2$	Impossible by Theorem 4.1	

Table 3: Amalgamated free products in the 3DCG's

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