Constructing Copoint Graphs of Convex Geometries

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Abstract

We work with copoint graphs of convex geometries. Copoint graphs can be used to study the complex and fairly recent field of convex geometries. Comparing copoint graphs and their convex geometries helps identify properties. We demonstrate that multiple convex geometries have the same underlying copoint graph. All graphs on one to five vertices can be represented as possible copoint graphs of some convex geometry. Furthermore, we construct several infinite classes of copoint graphs including the complete k-partite graph, path graph, centipede graph, ladder graph, comb graph, pom-pom graph, sharkteeth graph, and broken wheel graph.

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1 Introduction

In order to best understand our research, we need to introduce some preliminary definitions. We need to understand what a convex geometry is to construct copoint graphs of convex geometries. A finite set X together with $\mathscr{L}$ that is anti-exchange is a *convex geometry*, where X is the *ground set* of the convex geometry [EJ85]. A convex geometry is denoted as $(X, \mathscr{L})$. $\mathscr{L}$, a *closure operator*, is a function that takes subsets of X to subsets of X. If $\mathscr{L}$ is *anti-exchange* and there is a closed set K with $p, q \in X - K$, then $q \in \mathscr{L}(K \cup p)$ implies that $p \notin \mathscr{L}(K \cup q)$. The paper from Edelman and Jamison [EJ85] shows many results and equivalent definitions of convex geometries. We work with the congruent definition that defines a convex geometry as a finite set, X, together with an alignment $\mathscr{L}$ that has the greedy property. Thus, this equivalent definition of convex geometries satisfies the following properties: 1. A is a subset of $\mathscr{L}(A)$, 2. if A is a subset of B then $\mathscr{L}(A)$ is a subset of $\mathscr{L}(B)$ and 3. $\mathscr{L}\big(\mathscr{L}(A)\big)$ is equal to $\mathscr{L}(A)$. A set is *closed* if $\mathscr{L}(A)$ is equal to A.

If the closure operator $\mathscr{L}$ is an *alignment*, then for all the sets of X in $\mathscr{L}$, the intersections of the subsets of X in $\mathscr{L}$ are also in $\mathscr{L}$. The closure operator, $\mathscr{L}$, has the *greedy property* if for any closed subset C, where C does not equal X, there can be a point added to C; that is, there is a closed set of the form $C \cup p$ for $p \notin C$. The greedy property works exclusively with closed sets.

A *copoint* is a maximal closed set C in $X - p$; in convex geometries, each copoint is attached to the unique point p, where $p = \alpha(C)$. In a ground set of size X, each point in the ground set will have at least one copoint attached to it. *Copoint graphs* are graphs where the vertices are labeled as an ordered pair, $\big(\alpha(A), A\big)$, where A is the copoint and $\alpha(A)$ is the point attached to the copoint. Copoint graphs were introduced by Morris [Mor06] for planar point sets and later generalized by Beagley [Bea13] to all convex geometries. A copoint graph is denoted as $\mathcal{G}(X, \mathscr{L})$. *Edges* exist in the graph if the point attached to A is in the copoint B and the point attached to B is in the copoint A, or $\alpha(A) \in B$ and $\alpha(B) \in A$. In this paper, when it is said that a copoint is adjacent or appended to another copoint, there exists an edge between the copoints in the ordered pair notation.

Hasse diagrams are a type of mathematical diagram used to represent partially ordered sets. In our research, Hasse diagrams display all the closed sets of the convex geometry ordered by inclusion. The closed sets of size $|X| - 1$ are called the *extreme points* of the convex geometry.

A *clique* is a maximally connected subgraph of size n. The largest clique of size n found in the graph is the *clique number* of the graph.

It is important to note that in this paper, we focus on the connected copoint graphs and their convex geometries. If we study disconnected copoint graphs, the corresponding Hasse diagram will be a stacking of the Hasse diagrams of each connected component. An example of the convex geometry of two connected components is in Figure 1 with the corresponding copoint graph in Figure 2.

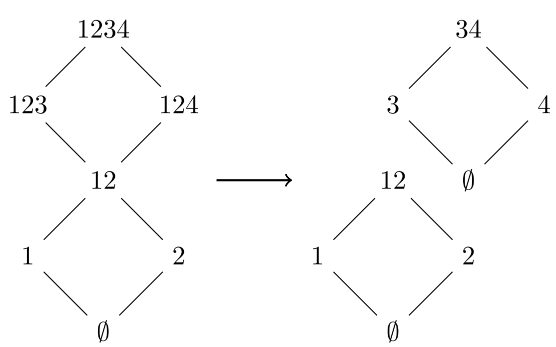


Figure 1: Hasse diagram stacking

A set of vertices that are pairwise non-adjacent in a graph is *vertex independent* or *independent*. For a convex geometry, $(X, \mathscr{L})$, a set I is *convexly independent* if for all x in I, X is not in $\mathscr{L}(I - \{x\})$. The clique number is also the size of the largest convexly independent set [Mor06, Proposition 1.2]. Throughout the

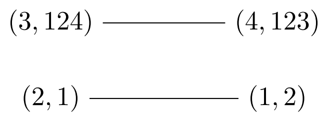
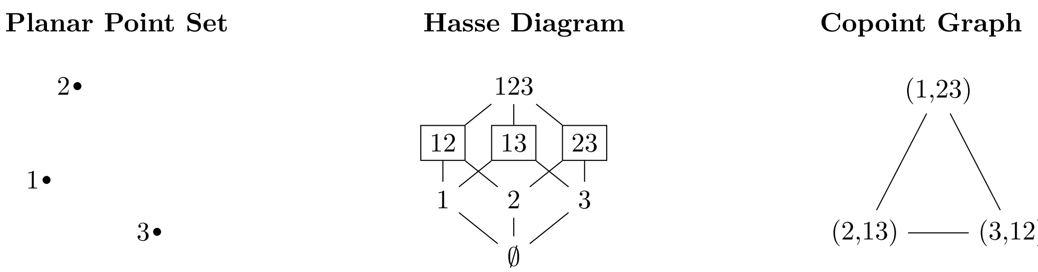


Figure 2: Labeled disconnected copoint graph of Figure 1

rest of the paper, when we reference independence, we are referencing vertex independence and will clarify otherwise.

This project is motivated by the relationship between planar point sets and their copoints. Morris [Mor06] gives an algorithm to find copoints from a planar point set. However, it is outside the scope of this paper.



In a Hasse diagram, the copoints are easily located. They are the closed sets that contain only one element above them. All of the copoints in the Hasse diagrams are circled or boxed throughout this paper. We denote the set $\{1, 2, 3, ..., n\}$ as $[n]$.

In this paper, we will demonstrate the non-uniqueness of copoint graphs. We will also show the existence of all copoints graphs from one to five vertices. In addition, we will prove the existence of several infinite families of graphs including the complete k-partite graph, path graph, centipede graph, ladder graph, comb graph, pom-pom graph, sharkteeth graph, and broken wheel graph. We will also pose questions for future research.

2 General Findings

Theorem 2.1. *If the ground set $1 \leq |X| \leq 5$, there exists a copoint graph of a convex geometry.*

This was proven by brute force: By finding at least one labeled copoint graph and proving that its corresponding Hasse diagram was of a convex geometry, we are able to show that all connected graphs are possible within this given ground set. For reference, please see Appendix A-D beginning on page 27. During this process, we found that we could find multiple convex geometries for a given graph. This led us to the following theorem.

Theorem 2.2. *The convex geometry that represents any existing copoint graph is not unique.*

See Figure D.2 and Figure D.3 on page 32 for an example of the same copoint graph with different labeling and convex geometries. Notice how the same graph on 5 vertices is represented in a ground set, X, of size 4 and 5.

3 Constructing Copoint Graphs

Theorem 3.1. *There exists a convex geometry whose copoint graph is a complete $K_{n,m}$ graph for all positive integers n and m.*

A complete bipartite graph is a graph composed of two disjoint sets where everything in one disjoint set is adjacent to everything in the other disjoint set. A $K_{n,m}$ bipartite graph has n elements in the first disjoint set and m elements in the second disjoint set. See Figure 3 for reference.

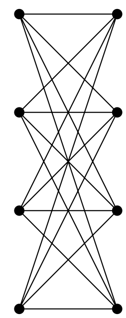


Figure 3: $K_{4,4}$ copoint graph

Proof:

We construct a convex geometry, $(X, \mathscr{L})$, such that its copoint graph, $\mathcal{G}(X, \mathscr{L})$, is a complete $K_{n,m}$ graph with a ground set $|X| = n + m$. If there is a generic closed set $C \in \mathscr{L}$, then $C = I_1 \cup I_2$.

$$\begin{aligned} &I_1 = \{1, .., k\}, && 1 \leq k \leq n \\ &I_2 = \{n+1, ..., n+\ell\}, && 1 \leq \ell \leq m \\ &\emptyset \end{aligned}$$

are all of the possible closed sets in $\mathscr{L}$.

$\mathscr{L}$ is an alignment because $C_1, C_2 \in \mathscr{L}$ where

$$\begin{aligned} C_1 &: \{1...k_1\} \cup \{n+1...n+\ell_1\} \\ C_2 &: \{1...k_2\} \cup \{n+1...n+\ell_2\} \\ C_1 \cap C_2 &: \{1, ..., \min(k_1, k_2)\} \cup \{n+1, ..., \min(\ell_1, \ell_2)\} \in \mathscr{L}. \end{aligned}$$

There is a $C \in \mathscr{L}$ where $C \neq [n+m]$ such that one of the following, or both, is in the convex geometry $(X, \mathscr{L})$:

$$\begin{aligned} C \cup \big(k+1\big) &\in \mathscr{L}, \\ C \cup \big(n+\ell+1\big) &\in \mathscr{L}. \end{aligned}$$

Thus, $\mathscr{L}$ has the greedy property.

Next, we consider the copoints of $(X, \mathscr{L})$. $A \in \mathscr{L}$ is a copoint if there is a unique p such that $(A \cup p) \in \mathscr{L}$. So, either

$$A : \{1, ..., n\} \cup \{n+1, ..., n+\ell\}$$

in which case, $\alpha(A) = n + \ell + 1$,

or

$$B : \{1, ..., k\} \cup \{n+1, ..., n+m\}$$

in which case, $\alpha(B) = k + 1$.

Any other closed set has two such p's, $n+\ell+1$ and $k+1$, and are not copoints. Therefore the above cases consider all copoints.

Now, we verify that the copoint graph, $\mathcal{G}(X,\mathscr{L})$, is a complete $K_{n,m}$ graph. The set of A's is an independent set and the set of B's is an independent set. These sets of A's and B's are convexly independent to each other and therefore adjacent in their copoint graph. A and B are two disjoint sets of n and m elements respectively, where one disjoint set is adjacent to every element in the other disjoint set.

Thus, we have constructed a convex geometry whose copoint graph is a complete $K_{n,m}$ graph.

■

3.1 Example of $K_{n,m}$

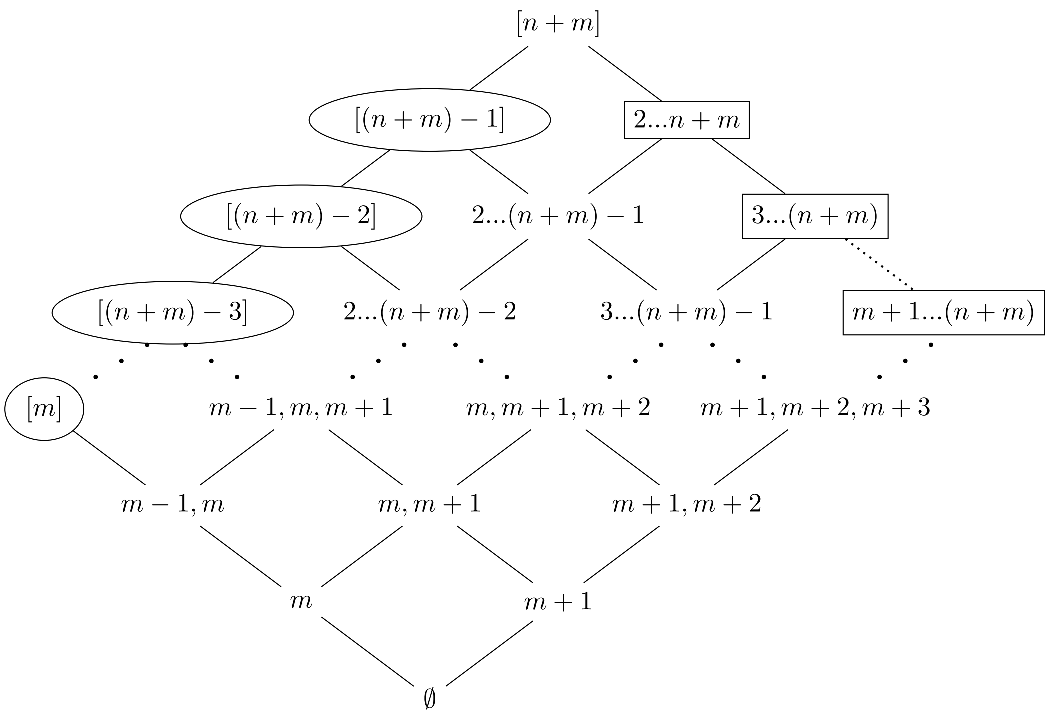


Figure 4: Hasse diagram of a convex geometry whose copoint graph is $K_{n,m}$

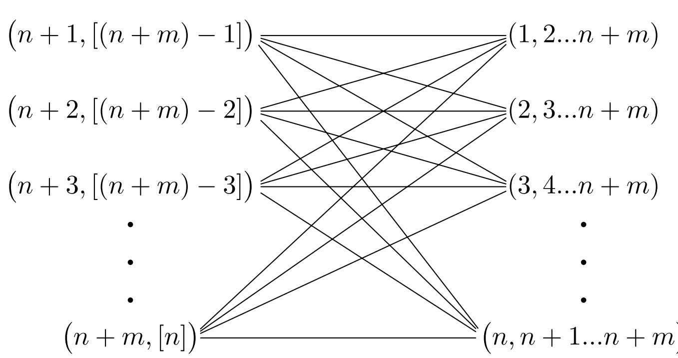


Figure 5: Labeled copoint graph of $K_{n,m}$

Corollary 3.2. *There exists a convex geometry whose copoint graph is the complete k-partite graph for any $k \geq 2$.*

A k-partite graph has k disjoint sets where each element is adjacent to all the elements of the other disjoint sets, but not adjacent to any element within its own disjoint set. This can be proven in a similar fashion as Theorem 3.1 with k arbitrary disjoint sets.

Theorem 3.3. *There exists a convex geometry whose copoint graph is a n, m pom-pom where $n \geq m$.*

A n, m pom-pom graph is a graph with a two clique base graph. Appended to one endpoint of the two-clique base graph are n leaves and appended to the other endpoint of the two-clique base graph are m leaves. See Figure 6 for reference.

Figure 6: 6,5 Pom-pom copoint graph

Proof:

We construct a convex geometry, $(X, \mathscr{L})$, such that the copoint graph, $\mathcal{G}(X, \mathscr{L})$, is a n, m pom-pom graph with a ground set $|X| = n + 2$. If there is a generic closed set $C \in \mathscr{L}$, then $C = I_1$.

$$I = \{a...b\}, \quad \begin{array}{l} 1 \leq a \leq b \leq n+2 \\ a \leq m+2 \end{array}$$

where the set $\{a...b\}$ is a consecutively ordered set.

I_1 and $\emptyset$ are all the closed sets in $\mathscr{L}$. We say $\mathscr{L}$ is an alignment because if we take the intersection of any two closed sets,

$$\{a...c_2\} \cap \{c_1...b\}$$

then $\{c_1...c_2\}$ or $\emptyset$ is $\in \mathscr{L}$.

There exists a C where $C \neq [n+2]$ such that

$$C = \{a...b\}.$$

One of the following, or both, is in the convex geometry $(X, \mathscr{L})$:

$$C \cup \{a-1\} \in \mathscr{L},$$
$$C \cup \{b+1\} \in \mathscr{L}.$$

Thus, $\mathscr{L}$ has the greedy property.

Next, we consider the copoints in $(X, \mathscr{L})$. $A \in \mathscr{L}$ is a copoint if there is a unique p such that $(A \cup p) \in \mathscr{L}$. So, either

$$A : \{1...c_1\}$$

in which case, $\alpha(A) = c_1 + 1$ where $c_1 \leq n + 1$,

or

$$B : \{c_2...m+2...n+2\}$$

in which case, $\alpha(B) = c_2 - 1$ where $c_2 \geq 2$.

Any other closed set has two such p's, $c_1 + 1$ and $c_2 - 1$, and therefore are not copoints. Therefore the above case considers all copoints.

Now, we verify that the copoint graph, $\mathcal{G}(X, \mathscr{L})$, is a n, m pom-pom graph. First, we denote the following copoints:

$$C : \{1...n+1\}$$

attached to $n + 2$.

$$D : \{2...n+2\}$$

attached to 1.

These copoints are convexly independent with each other and therefore adjacent in the copoint graph. They make up the base two-clique graph to which the leaves are appended. The independent set of A's is convexly independent with the copoint D and therefore adjacent in the copoint graph. This creates the n leaves appended to the copoint D. The independent set of B's is convexly independent with the copoint C and therefore adjacent in the copoint graph. This creates the m leaves appended to the copoint C.

Thus, we have constructed a convex geometry whose copoint graph is a n, m pom-pom.

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3.2 Example of Pom-pom

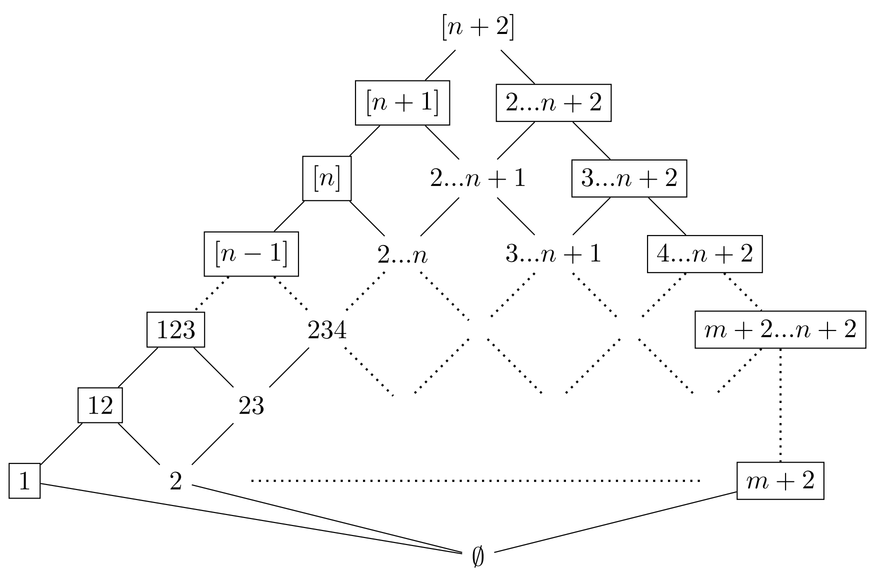

Figure 7: Hasse diagram of a convex geometry whose copoint graph is an n, m pom-pom

4 Constructing Copoint Graphs by Induction

Theorem 4.1. *There exists a convex geometry whose copoint graph is a path, P_n for all $n > 0$.*

A path graph is a graph of consecutively appended vertices where the degree of the vertex is either one or two. See Figure 8 for reference.

Figure 8: Path copoint graph on 7 vertices

Proof:

Given the closed sets

$$\begin{array}{cccc} \emptyset & \{1\} & \{12\} & \{123\} \\ & \{2\} & \{13\} & \end{array}$$

where the following are copoints

$$\begin{array}{cc} \{2\} & \{12\} \\ & \{13\}, \end{array}$$

we have constructed a convex geometry $([3], \mathscr{L})$. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and the sets have the greedy property. Note that this graph is equivalent to Figure A.1 on page 27, though the labels will be permutated. Figure A.1 verifies that this is a path on two vertices.

Based on the above case, we assume $([n], \mathscr{L})$ is a convex geometry. Consider the inductive case, $\mathscr{L}'$, where $n + 1$ is added to the ground set. The additional closed sets in $\mathscr{L}'$ are

$$[n-1] \cup \{n+1\}$$
$$[n] \cup \{n+1\}$$

and the following are copoints

$$[n]$$
$$[n-1] \cup \{n+1\}$$

where $[n-1]$ is no longer a copoint.

All of the intersections of sets in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment and all the sets have the greedy property.

Now, we verify that the copoint graph, $\mathcal{G}(X, \mathscr{L}')$ is a path. Consider a copoint A of size k that is not an endpoint. The copoint A contains all other copoints of size less than or equal to $k-2$ and is contained in all copoints of size greater than or equal to $k+2$. Thus, the copoint A is only convexly independent to two copoints, a copoint B of size $k+1$ and a copoint C of size $k-1$. Thus, A is adjacent to B and A is adjacent to C in the copoint graph. We know that B contains C and therefore they are non-adjacent in the copoint graph. We can consider any non-endpoint copoint of $\mathscr{L}'$ in a similar fashion. The copoint of size 1 will not have any $(k-1)$-sized copoint to be convexly independent to because it is not a closed set in $\mathscr{L}'$.

Now, we consider the special cases of the endpoint created by the extreme points. The extreme points are copoints of the same size and it is obvious that they are convexly independent and therefore adjacent in the copoint graph.

Thus, $([n+1], \mathscr{L}')$ is a convex geometry whose copoint graph is a path.

■

4.1 Example of Path

$$(1,2) — (2,13) — (3,12) \longrightarrow (1,2) — (2,13) — (3,124) - (4,123)$$

Figure 9: Labeled path copoint graph

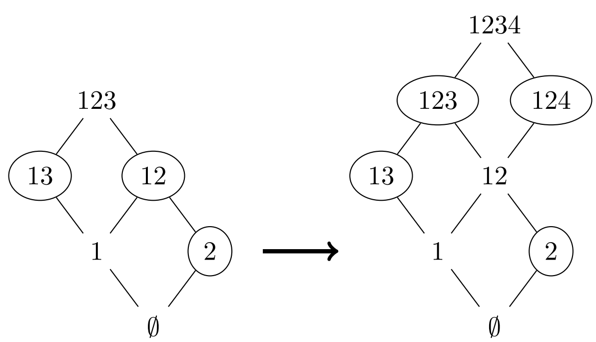

Figure 10: Hasse diagram of a convex geometry whose copoint graph is a path

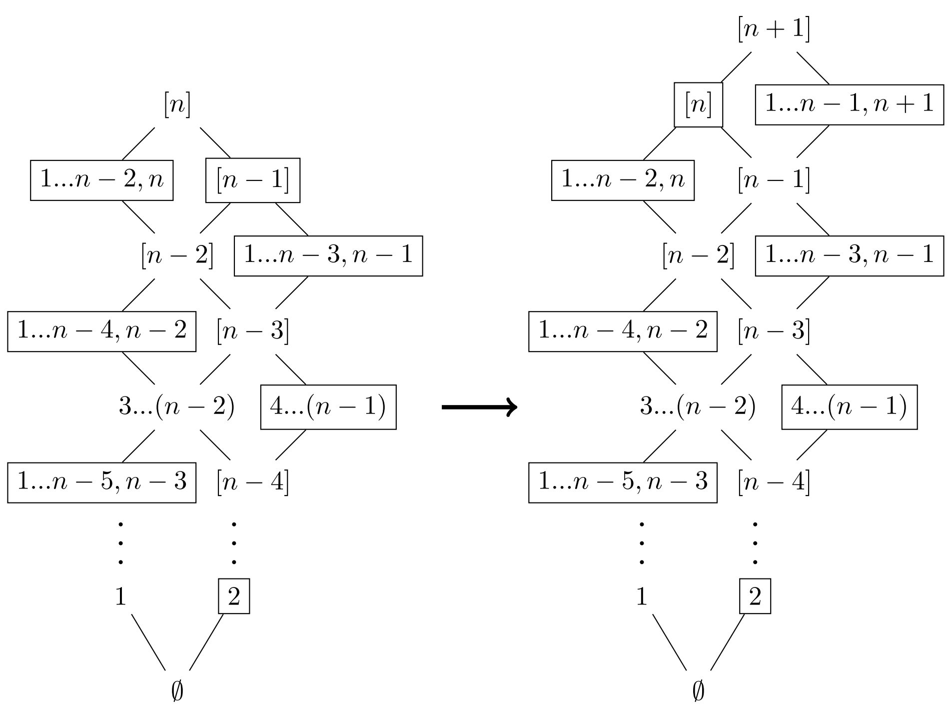

Figure 11: Hasse diagram of a convex geometry whose copoint graph is a path

Theorem 4.2. *There exists a convex geometry whose copoint graph is a broken wheel.*

A wheel graph is a cycle of vertices that is also adjacent to one vertex in the center. A broken wheel graph is a wheel graph with one edge missing between only two vertices in the cycle. See Figure 12 for reference.

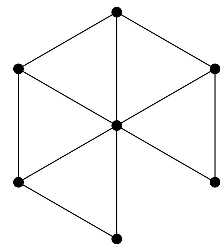

Figure 12: Broken wheel copoint graph on 7 vertices

Proof:

Given the closed sets

$\emptyset$	{1}	{12}	{123}	{1234}
	{2}	{13}	{124}	
	{3}	{23}	{234}	
		{24}		

where the following are copoints

{13}	{123}
	{124}
	{234},

we have constructed a convex geometry $([4], \mathscr{L})$. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and the sets have the greedy property. Note that this graph is equivalent to Figure C.5 on page 31, though the labels may be permutated. Figure C.5 verifies that this is a wheel on four vertices.

Based on the above case, we assume $([n], \mathscr{L})$ is a convex geometry. Consider the inductive case, $\mathscr{L}'$, where $n+1$ is added to the ground set. The additional closed sets in $\mathscr{L}'$ are

$$\{2...n-1\} \cup \{n+1\}$$
$$[n-1] \cup \{n+1\}$$
$$\{2...n\} \cup \{n+1\}$$
$$[n] \cup \{n+1\}$$

and the following are copoints

$$[n]$$
$$[n-1] \cup \{n+1\}$$
$$\{2...n\} \cup \{n+1\}$$

where $[n-1]$ and $\{2...n\}$ are no longer copoints.

All of the intersections of the sets in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment. In addition, all the sets have the greedy property.

Now, we verify that the copoint graph, $\mathcal{G}(X, \mathscr{L}')$, is a broken wheel. Consider the copoint

$$A : \{2...n+1\}$$

attached to 1.

This copoint is convexly independent to all other copoints in $\mathscr{L}'$ because it contains no other copoints in $\mathscr{L}'$. Therefore, it is adjacent to all other copoints in the copoint graph.

Now consider all other copoints. They are the same closed sets as the convex geometry of the path. Therefore, the path is a subgraph of this copoint graph. See Theorem 4.1 for proof on why the closed sets in this convex geometry are linked as a path in the copoint graph.

Thus, $([n+1], \mathscr{L}')$ is a convex geometry whose copoint graph is a broken wheel.

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4.2 Example of Broken wheel

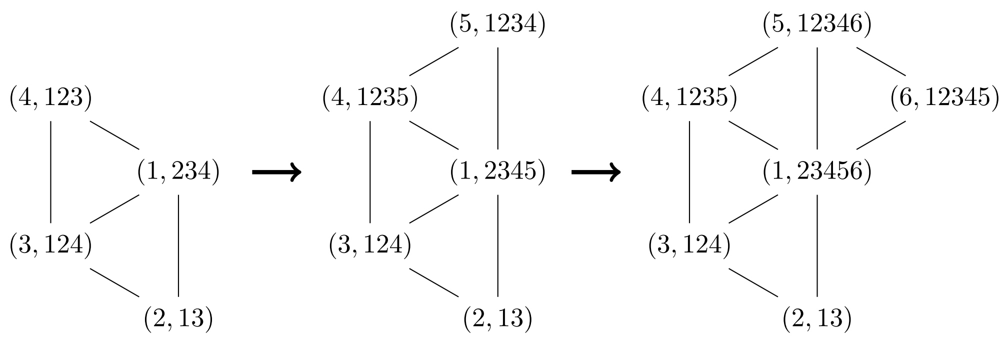

Figure 13: Labeled broken wheel copoint graph

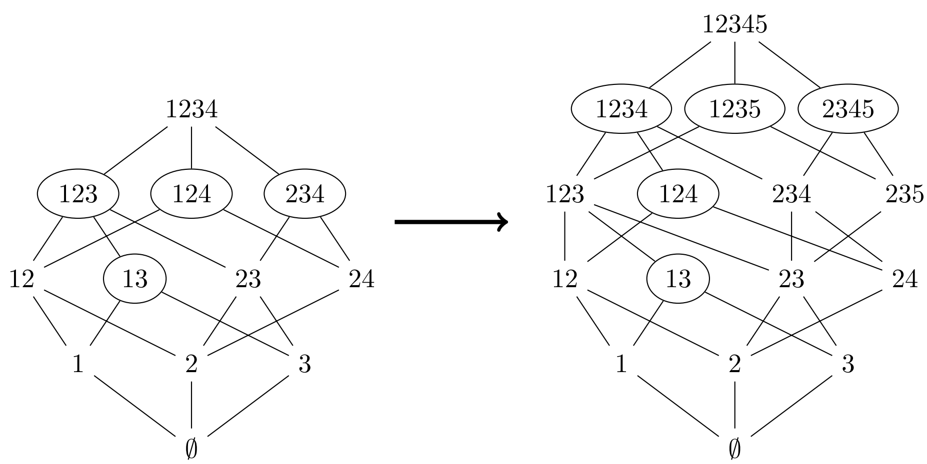

Figure 14: Hasse diagram of a convex geometry whose copoint graph is a broken wheel

Theorem 4.3. *There exists a convex geometry whose copoint graph is the sharkteeth graph.*

The sharkteeth graph is created by appending three cliques in a linear form so that the maximum degree of any vertex is four. See Figure 15 for reference.

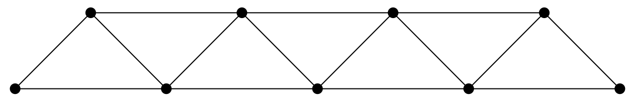

Figure 15: Sharkteeth copoint graph on 9 vertices

Proof:

Given the closed sets

$\emptyset$	$\{1\}$	$\{12\}$	$\{123\}$	$\{1234\}$	$\{12345\}$
	$\{2\}$	$\{13\}$	$\{124\}$	$\{1235\}$	
	$\{3\}$	$\{14\}$	$\{134\}$	$\{1245\}$	
		$\{23\}$	$\{125\}$		

where the following are copoints

$\{23\}$	$\{134\}$	$\{1234\}$
		$\{1235\}$
		$\{1245\}$,

we have constructed a convex geometry $([4], \mathscr{L})$. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and the sets have the greedy property. Note that this graph is equivalent to Figure D.19 on page 41, though the labels may be permutated. Figure D.19 verifies that this is a sharkteeth on five vertices.

Based on the above case, we assume $([n], \mathscr{L})$ is a convex geometry. Consider the inductive case, $\mathscr{L}'$, where $n+1$ is added to the ground set. The additional closed sets in $\mathscr{L}'$ are

$$[n-1] \cup \{n+1\}$$
$$[n-2] \cup \{n+1\}$$
$$[n-2] \cup \{n\} \cup \{n+1\}$$
$$[n] \cup \{n+1\}$$

and the following are copoints

$$[n]$$
$$[n-1] \cup \{n+1\}$$
$$[n-2] \cup \{n\} \cup \{n+1\}$$

where $[n-1]$ and $[n-2] \cup \{n\}$ are no longer copoints.

All of the intersections of the sets in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment. In addition, all the sets have the greedy property.

Now, we verify that the copoint graph, $\mathcal{G}(X, \mathscr{L}')$, is the sharkteeth graph. First we consider the three extreme points. One of the three extreme points is adjacent to only two vertices, one is adjacent to only three vertices, and the third to only four vertices.

$$A : [n]$$

attached to $n+1$ and is of degree two.

$$B : \{1, ...n-1, n+1\}$$

attached to n and is of degree three.

$$C : \{1, ..., n-2, n, n+1\}$$

attached to $n-1$ and is of degree four.

Case 1: Extreme point of degree two

It is obvious that copoint A is convexly independent to B and C. Additionally, A contains all other copoints in $\mathscr{L}'$ because all other copoints are some subset of the numbers 1 through n. Thus, A is only convexly independent to two copoints, B and C. Therefore, C is adjacent to B and C in the copoint graph and nothing else.

Case 2: Extreme point of degree three

It is obvious that copoint B is convexly independent to A and C. Because B is of size n, B is convexly independent to the copoint of size $n-1$, denoted D. Because D is contained in A, as stated in case 1, it is not contained in B. Thus, B is convexly independent to D. Therefore, B is adjacent to A, C, and D in the copoint graph and nothing else.

$$D : \{1, ..n-3, n, n+1\}$$

attached to $n-2$.

Case 3: Extreme point of degree four

It is obvious that copoint C is convexly independent to A and B. It is also clear that C is convexly independent to D because D is contained in the extreme point A (as stated in case 1). In addition, C is convexly independent to the copoint E because E is contained in the intersection of the extreme points A and B and thus cannot be contained in C. Therefore, C is adjacent to A, B, D, and E in the copoint graph and nothing else.

$$E : \{1, ...n-4, n-2, n-1\}$$

attached to $n-3$.

Case 4: Copoints D and E

The copoint D is contained in the extreme point A and is thus convexly independent to the other extreme points, B and C. Therefore, D is adjacent to B and C in the copoint graph, but is independent to A. The copoint E is convexly independent to D because E is not contained in D. Therefore, E is adjacent to D in the copoint graph. Assume D is of size $n-1$. The copoint of size $n-3$ is contained in the intersection of the three closed sets of size $n-1$, but not D. Therefore, D is convexly independent to the copoint of size $n-3$ and adjacent in the copoint graph. The copoint D is of degree four as it is convexly independent to B, C, E and the copoint of size $n-3$ and nothing else.

The copoint E is of size $n-2$ and is also of degree four. E is convexly independent to one of the extreme points, C. In addition, it is convexly independent to the copoints of size $n-3$ and $n-4$, by similar reasoning of D. Therefore, E is adjacent to C, D, and the copoints of size $n-3$ and $n-4$ and nothing else.

Case 5: All other copoints

Consider any other copoint, K of size k. By previous argument, K will be contained in A, B or C. However, it will be convexly independent to the copoints of size $k+2$, $k+1$, $k-1$, and $k-2$. Therefore, K will be of degree four unless there does not exist a copoint of size $k-1$ or $k-2$ in $\mathscr{L}'$. There will be two such points; they will be the endpoints of the sharkteeth graph and will be of degree two and three respectively.

Thus, $([n+1], \mathscr{L}')$ is a convex geometry whose copoint graph is the sharkteeth graph.

■

4.3 Example of Sharkteeth

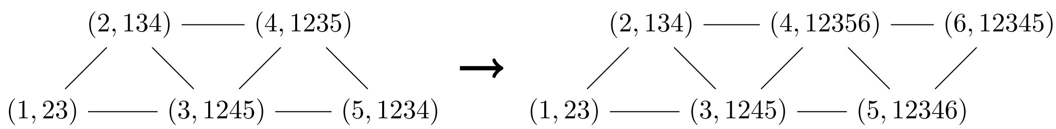

Figure 16: Labeled sharkteeth copoint graph

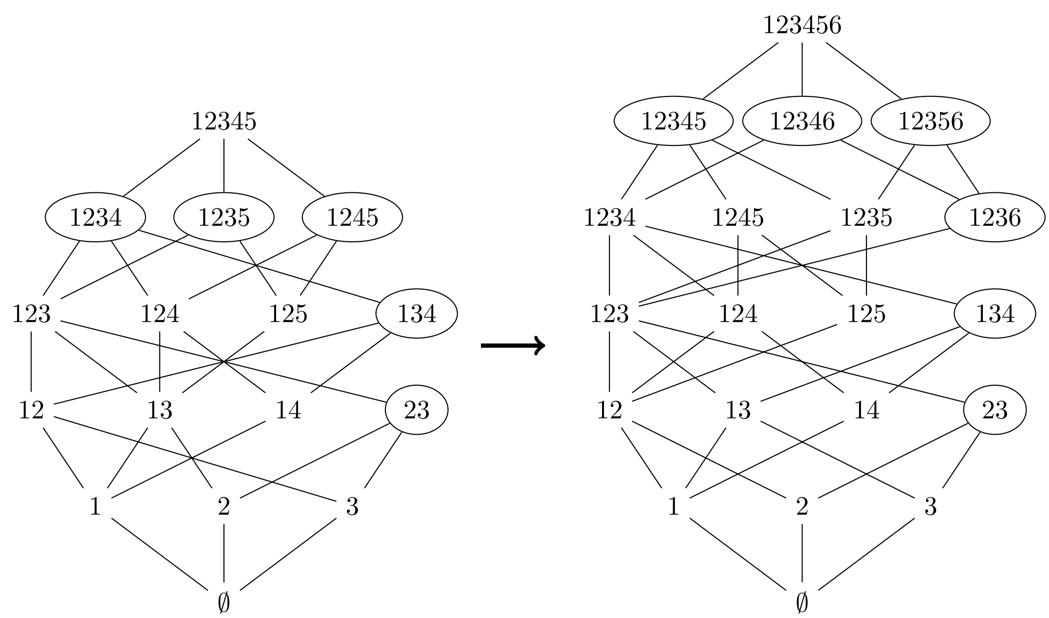

Figure 17: Hasse diagram of a convex geometry whose copoint graph is a sharkteeth graph

Theorem 4.4. *There exists a convex geometry whose copoint graph is a ladder.*

The ladder graph is a n-by-2 grid of vertices where the maximum degree of any vertex is three. See Figure 18 for reference.

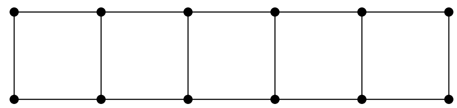

Figure 18: 6 by 2 Ladder copoint graph

Proof:

Given the closed sets

$$\begin{array}{ccccc} \emptyset & \{1\} & \{12\} & \{123\} & \{1234\} \\ & \{2\} & \{14\} & \{124\} & \\ & & \{23\} & & \end{array}$$

where the following are copoints

$$\begin{array}{cc} \{14\} & \{123\} \\ \{23\} & \{124\}, \end{array}$$

we have constructed a convex geometry $([4], \mathscr{L})$. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and the sets follow the greedy property. Note that this graph is equivalent to Figure C.2 on page 29, though the labels will be permutated. Figure C.2 verifies that this is a ladder on four vertices.

Based on the above case, we assume $([n], \mathscr{L})$ is a convex geometry. Consider the inductive case, $\mathscr{L}'$, where $n+2$ is added to the ground set. The additional closed sets in $\mathscr{L}'$ are

$$\begin{aligned} [n-2] &\cup \{n\} \cup \{n+1\} \\ [n-1] &\cup \{n+2\} \\ [n] &\cup \{n+1\} \\ [n] &\cup \{n+2\} \\ [n] &\cup \{n+1\} \cup \{n+2\} \end{aligned}$$

and the following are copoints

$$\begin{aligned} [n-2] &\cup \{n\} \cup \{n+1\} \\ [n-1] &\cup \{n+2\} \\ [n] &\cup \{n+1\} \\ [n] &\cup \{n+2\} \end{aligned}$$

where $[n-2] \cup \{n\}$ and $[n-3]$ are no longer copoints.

All of the intersections in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment and all the sets have the greedy property.

Now, we verify that the copoint graph, $\mathcal{G}(X, \mathscr{L}')$, is a ladder. Consider the copoint:

$$A : \{a_1, ..., a_2\}$$

of size k.

We claim that A is contained in or contains every copoint except for three copoints. Thus it is adjacent to these three copoints in the copoint graph.

We begin by looking at the two copoints of size $k+2$. (Note that if there are no copoints of size $k+2$, consider the copoints of size $k+1$. This is a special case where the $k+1$ sized copoints are the extreme points, and the proof is similar). We claim that A is convexly independent to only one of the copoints of sized $k+2$. We call these two copoints B and C where

$$B : \{a_1 - 1, a_1....a_2, a_2 + 1\}$$

$$C : \{a_1 + 1, ...a_2, a_2 + 1, a_2 + 2, a_2 + 3\}$$

We note that B contains A, but C does not contain A as C is does not contain a_1 in its copoint. Thus, A and C are convexly independent and therefore adjacent in the copoint graph.

Next, we consider all copoints of size k. There are two copoints of size k, A and another we call D. It is obvious that A does not contain D and that they are convexly independent. Therefore, in the copoint graph, they are adjacent to each other; D is the second copoint of three that are convexly independent to A.

Lastly, we consider the third copoint, which is one of the two copoints of size $k-2$. We call them E and F where

$$E : \{a_1 + 1, ...a_2 - 1\}$$

$$F : \{a_1 + 3...a_2 + 1\}.$$

It is clear that E is contained in A. However F includes $a_2 + 1$ and is therefore not contained in the copoint A. Thus, A is convexly independent to F and adjacent to F in the copoint graph.

It is obvious that all copoints of greater or equal to $k+4$ contain A. Similarly, all copoints less than or equal to $k+4$ are contained in A. Therefore, there are only three copoints convexly independent to A.

The endpoints of the smallest size are a special case where the $(k-2)$-sized copoints do not exist as closed sets in $\mathscr{L}'$. Note that the extreme points are convexly independent (because they are of the same size) and adjacent to each other in the copoint graph. This creates a n-by-2 grid copoint graph.

Thus, $([n+2], \mathscr{L}')$ is a convex geometry whose copoint graph is a ladder.

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4.4 Example of Ladder

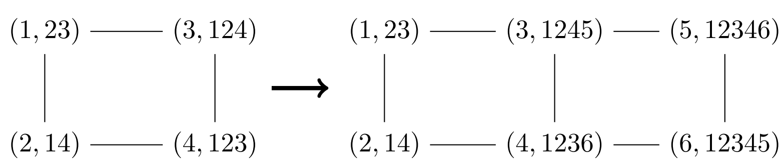

Figure 19: Labeled ladder copoint graph

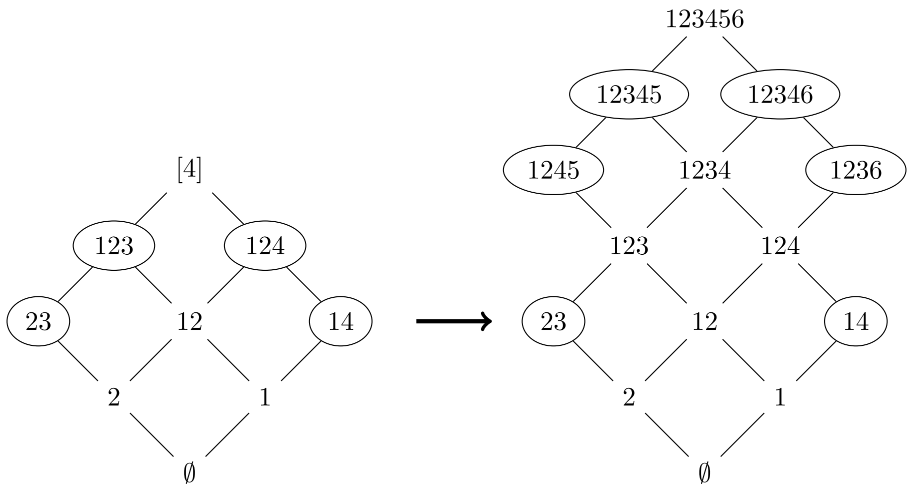

Figure 20: Hasse diagram of a convex geometry whose copoint graph is a ladder

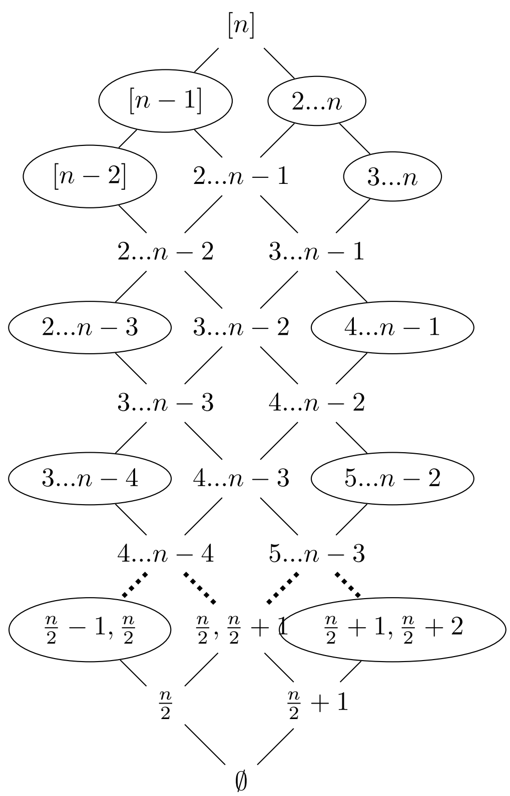

Figure 21: Hasse diagram of a convex geometry whose copoint graph is a ladder $|X| = n$

Theorem 4.5. *There exists a convex geometry whose copoint graph is a comb.*

A comb graph has a path graph as its base graph. The endpoints of this base path graph have exactly two leaves and every other vertex of the base path graph has exactly one leaf extending from it. See Figure 22 for reference.

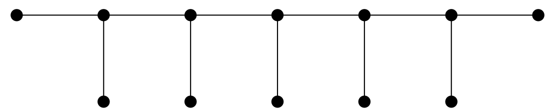

Figure 22: Comb copoint graph on 12 vertices

Proof:

Given the closed sets

$$\begin{matrix} \emptyset & \{1\} & \{12\} & \{123\} & \{1234\} \\ & \{2\} & \{23\} & \{234\} & \end{matrix}$$

where the following are copoints

$$\begin{matrix} \{1\} & \{12\} & \{123\} \\ & & \{234\}, \end{matrix}$$

we have constructed a convex geometry $([4], \mathscr{L})$. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and the sets follow the greedy property. Note that this graph is equivalent to Figure C.3 on page 30, though the labels will be permutated. Figure C.3 verifies that this is a comb on four vertices.

Based on the above case, we assume $([n], \mathscr{L})$ is a convex geometry. Consider the inductive case, $\mathscr{L}'$, where $n+2$ is added to the ground set. The additional closed sets in $\mathscr{L}'$ are

$$\begin{aligned} [n-1] &\cup \{n+1\} \\ [n] &\cup \{n+1\} \\ [n-1] &\cup \{n+1\} \cup \{n+2\} \\ [n] &\cup \{n+1\} \cup \{n+2\} \end{aligned}$$

and the following are copoints

$$\begin{aligned} &[n] \\ &[n] \cup \{n+1\} \\ [n-1] &\cup \{n+1\} \cup \{n+2\} \end{aligned}$$

where $[n-1]$ is no longer a copoint.

All of the intersections of the sets in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment and all the sets have the greedy property. We verify that this copoint graph is a centipede with the proof of $(n_1, n_2, ..., n_k)$-centipede in Theorem 5.1 on page 23.

Thus, $([n+2], \mathscr{L}')$ is a convex geometry whose copoint graph is a comb.

■

4.5 Example of Comb

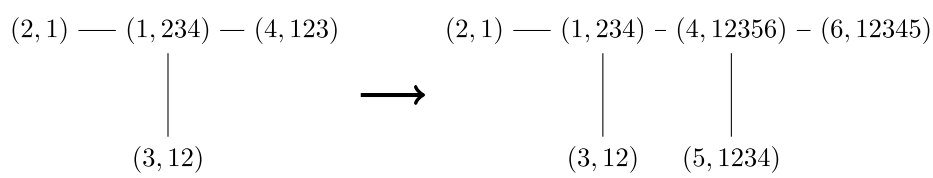

Figure 23: Comb copoint graph

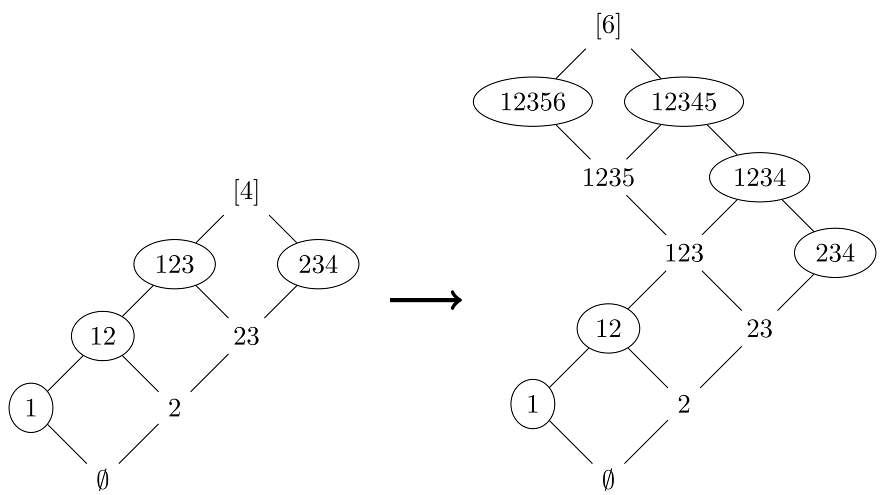

Figure 24: Hasse diagram of a convex geometry whose copoint graph is a comb

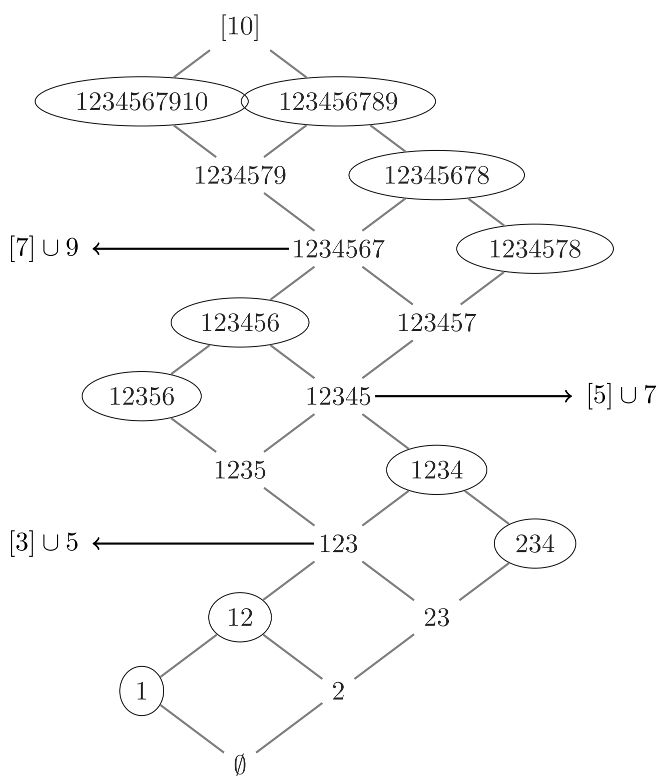

Figure 25: Hasse diagram for comb copoint graph. $X = 10$

Theorem 4.6. *There exists a convex geometry whose copoint graph is a centipede.*

A centipede graph has a path graph as its base graph. Every vertex of the base path graph has exactly one leaf extending from it.

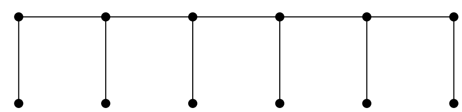

Figure 26: Centipede copoint graph on 12 vertices

Proof:

Given the closed sets

$$\begin{array}{ccccccc} \emptyset & \{1\} & \{12\} & \{123\} & \{1234\} & \{12345\} & \{123456\} \\ & \{2\} & \{23\} & \{124\} & \{1246\} & \{12346\} & \end{array}$$

where the following are copoints

$$\begin{array}{ccccc} \{1\} & \{23\} & \{123\} & \{1246\} & \{12345\} \\ & & & & \{12346\}, \end{array}$$

we have constructed a convex geometry $([6], \mathscr{L})$. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and the sets follow the greedy property. We verify that this copoint graph is a centipede with the proof of $(n_1, n_2, ..., n_k)$-centipede in Thereom 5.1 on page 23.

Based on the above case, we assume $([n], \mathscr{L})$ is a convex geometry. Consider the inductive case, $\mathscr{L}'$, where $n + 2$ is added to the ground set. The additional closed sets in $\mathscr{L}'$ are

$$\begin{aligned} [n-1] &\cup \{n+1\} \\ [n] &\cup \{n+1\} \\ [n] &\cup \{n+2\} \\ [n] &\cup \{n+1\} \cup \{n+2\} \end{aligned}$$

and the following are copoints

$$\begin{aligned} [n-1] &\cup \{n+1\} \\ [n] &\cup \{n+1\} \\ [n] &\cup \{n+2\} \end{aligned}$$

where $[n-1]$ is no longer a copoint.

All of the intersections in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment and all the sets have the greedy property. We verify that this copoint graph is a centipede with the proof of $(n_1, n_2, ..., n_k)$-centipede in Theorem 5.1 on page 23.

Thus, $([n+2], \mathscr{L}')$ is a convex geometry whose copoint graph is a centipede.

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4.6 Example of Centipede

Because we know all graphs of size 1-5 exist, we begin looking at the graph of 6 vertices.

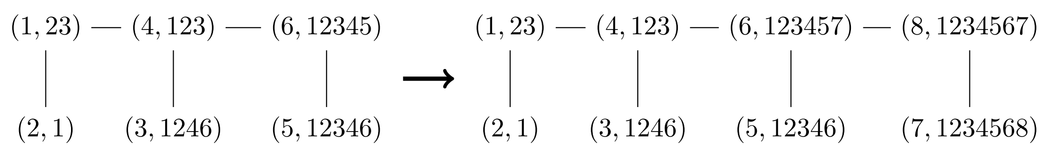

Figure 27: Labeled centipede copoint graph

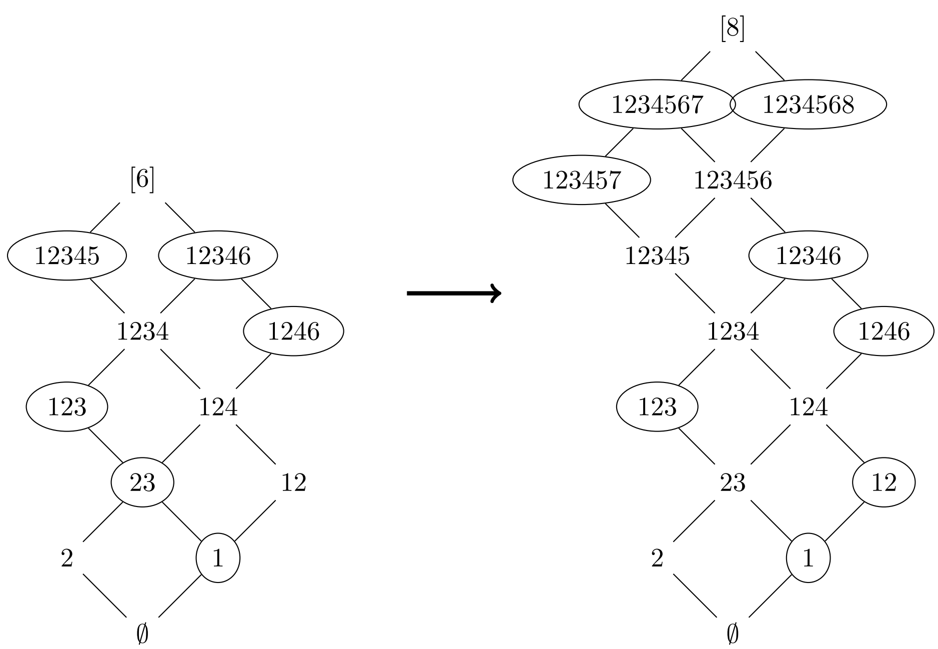

Figure 28: Hasse diagram of a convex geometry whose copoint graph is a centipede

5 Results on Trees

Trees, or acyclic graphs where there is exactly one path between two vertices, sometimes exists as copoint graphs of convex geometries.

Theorem 5.1. *There exists a convex geometry whose copoint graph is a $(n_1, n_2, ..., n_k)$-centipede graph.*

A $n_1, n_2, ..., n_k$ centipede graph is a path graph with k vertices, where n_1 leaves are adjacent to the v_1 vertex, n_2 leaves are adjacent to the v_2 vertex, and n_k leaves are adjacent to the v_k vertex. Note that $n_1, n_2, ..., n_k \geq 0$.

Proof:

We begin by considering the graph of $K_{n,1}$ as the base copoint graph. We know by Theorem 3.1 on page 5 that $K_{n,1}$ exists as a convex geometry. We append a leaf to the $K_{n,1}$ graph but place this $(n+1)^{th}$ leaf into the base path graph and call it v_2. This copoint graph becomes a $K_{n+1,1}$ graph, whose convex geometry clearly still exists. Note that n leaves and the v_2 vertex are all adjacent to v_1. They form an chain in $\mathscr{L}$ (reference Figure 29).

Now, we show that adding a leaf off of the vertex v_2 is still a convex geometry. Every time we add a leaf, we increase the ground set by 1. We add a leaf u_1 adjacent only to v_2. The leaf u_1 is independent because it contains all other copoints except for v_2. Therefore, u_1 is only adjacent to v_2 in the copoint graph.

We can also add u_2 to the set of leaves U adjacent to v_2. Similarly, u_2 is independent because it contains all other copoints; u_2 is only adjacent to v_2. $|U|$ is arbitrary. We note that $|U|$ may equal zero. To this, since we have shown that we can add another leaf. We call this v_k. To that we can add n_k leaves as before. Given v_k vertices with $n_1, n_2, ...n_k$ leaves, we can construct a $(n_1, n_2, ..., n_k)$-centipede graph.

■

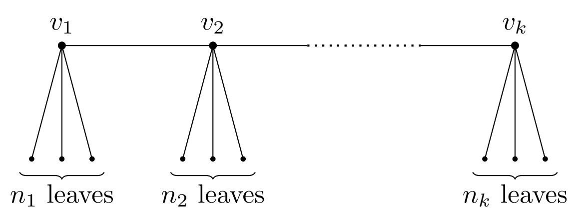

Figure 29: $(n_1, n_2, ..., n_k)$-centipede copoint graph

Note that the (n_1, n_2)-centipede graph is also a n_1, n_2 pom-pom graph like in Theorem 3.3 but with a different sized ground set. This presents an infinite number of graphs that satisfy Theorem 3.3.

6 Future Research

6.1 A Pressing Question in the Field

We ask the question, what is the maximum number of copoints in a convex geometry given a ground set of size n?

We conjecture that with a ground set of size n, the convex geometry represented by $\mathscr{L}_{n,n/2}$ gives the maximum number of copoints, where $\frac{n}{2}$ is number of extreme points in $\mathscr{L}$.

Beagley and Morris [BM14] shows that the lower bound for the maximum number of copoints in a convex geometry given a ground set of size n is

$$\binom{n}{\lfloor n/2 \rfloor} + \left\lfloor \frac{n-1}{2} \right\rfloor .$$

We pose the question, if given a convex geometry with a ground set of size n, there is only one convex geometry that gives

$$\binom{n}{\lfloor n/2 \rfloor} + \left\lfloor \frac{n-1}{2} \right\rfloor$$

number of copoints.

This led us to consider its copoint graph, $\mathcal{G}(X, \mathscr{L}_{n,n/2})$, and the following conjecture:

If $k < n$ then, $G([k], \mathscr{L}_{k,k/2}) \subseteq G([n], \mathscr{L}_{n,n/2})$.

6.2 Graphs that do not exist

Throughout our research, our team has found plenty of infinite classes of copoint graphs of convex geometries that exist and and pose the question about copoints graph of convex geometries that do not exist beyond a certain sized ground set. We challenge the reader to prove the following conjectures:

Conjecture 6.1. *A wheel on eight or more vertices do not exist as a copoint graph of a convex geometry.*

A wheel graph is a cycle of vertices that is also adjacent to one vertex in the center.

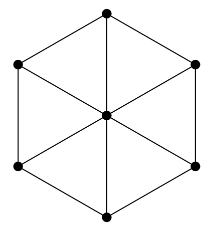

Figure 30: Wheel copoint graph on 7 vertices

Conjecture 6.2. *An exploding n-star beyond two completely-filled levels does not exist as a copoint graph of a convex geometry.*

Exploding n-star graphs are wheel graphs where the outside vertices are not connected and n is the degree of the middle vertex. The levels refers to extending the spokes of the wheel being extended to another vertex. See Figure 31 for reference.

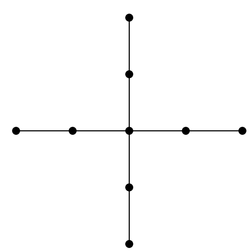

Figure 31: Exploding 4-star copoint graph

Conjecture 6.3. *A complete binary tree on ≥ 12 vertices does not exist as a copoint graph of a convex geometry.*

Complete binary trees are acyclic graphs where the vertices on every level have two leaves except possibly the last level, where all of the leaves are filled leftmost. See Figure 32 for reference.

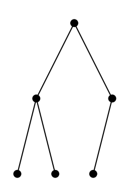

Figure 32: Complete binary tree copoint graph

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[Bea13] Jonathan E. Beagley. On the Order Dimension of Convex Geometries. *Order*, 30:837–884, 2013.

[BM14] Jonathan E. Beagley and Walter Morris. Chromatic Numbers of Copoint Graphs of Convex Geometries. *Discrete Math.*, 331:151–157, 2014.

[EJ85] Paul H. Edelman and Robert E. Jamison. The theory of convex geometries. *Geometrie Dedicata*, 19:247–269, 1985.

[Mor06] Walter Morris. Coloring copoints of a planar point set. *Discrete Applied Math.*, 154:1742–1752, 2006.

A Appendix

All copoint graphs on 2 vertices.

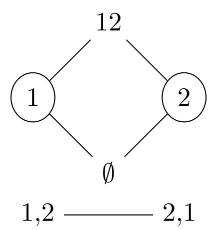

Figure A.1

B Appendix

All copoints on 3 vertices.

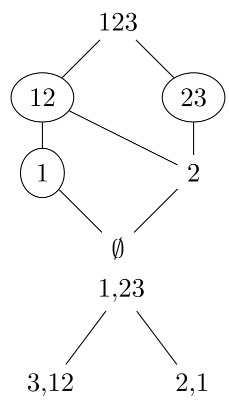

Figure B.1

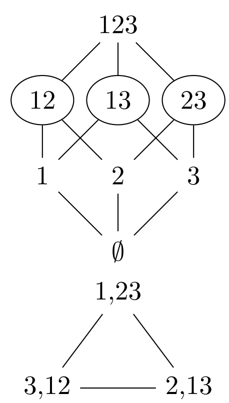

Figure B.2

C Appendix

All copoint graphs on 4-vertices

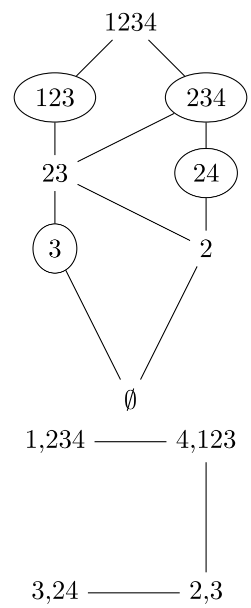

Figure C.1

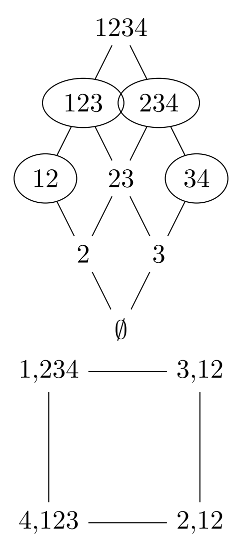

Figure C.2

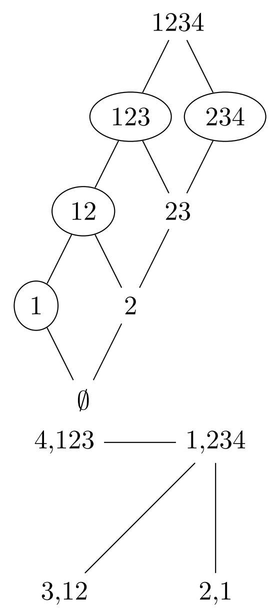

Figure C.3

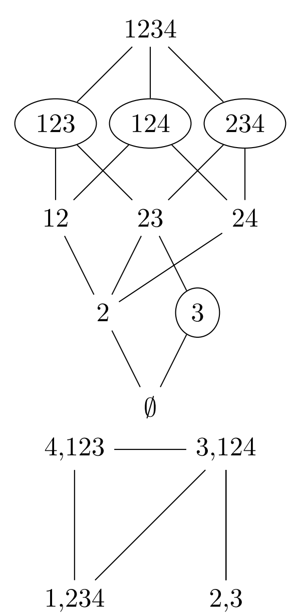

Figure C.4

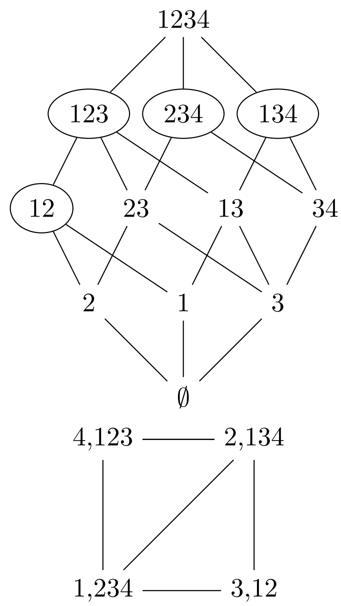

Figure C.5

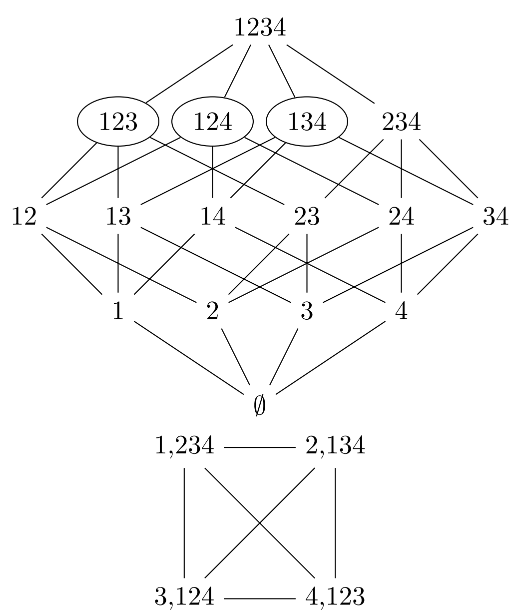

Figure C.6

D Appendix

All copoint graphs on 5-vertices

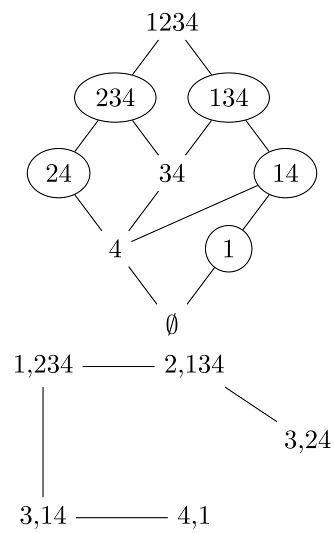

Figure D.1

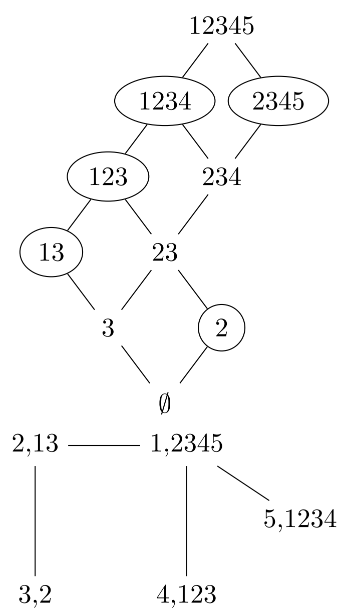

Figure D.2

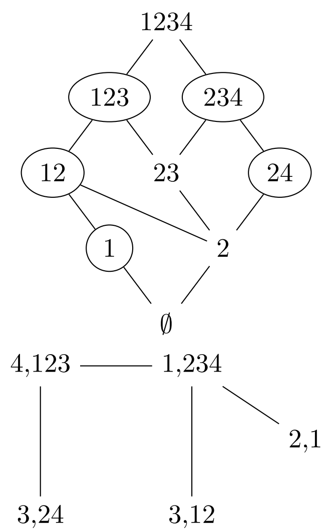

Figure D.3

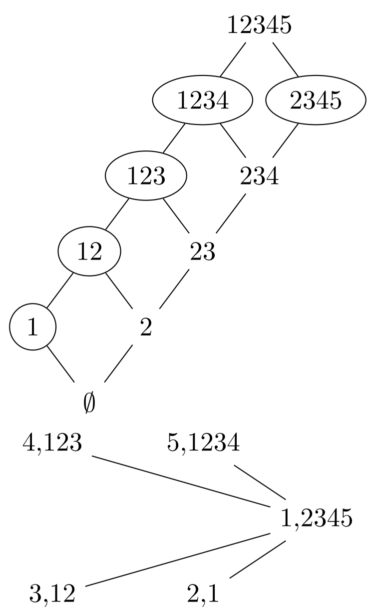

Figure D.4

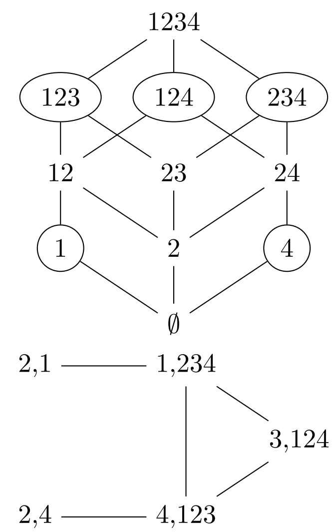

Figure D.5

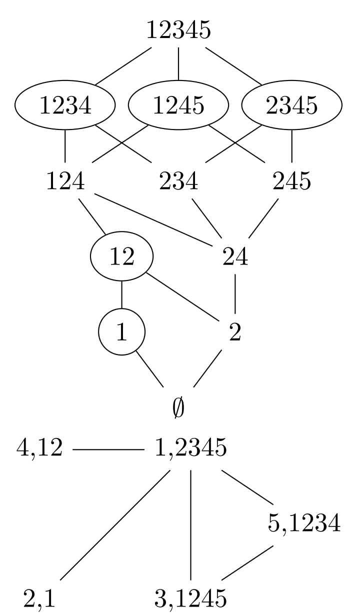

Figure D.6

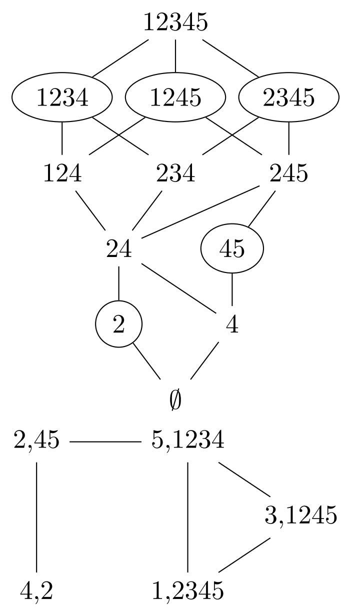

Figure D.7

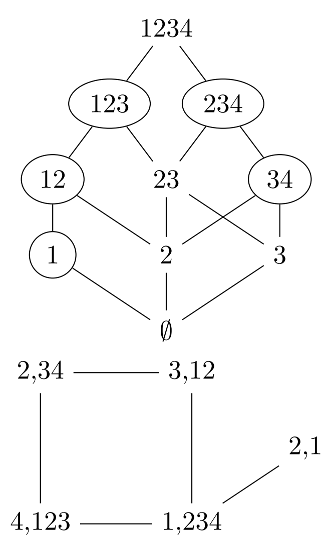

Figure D.8

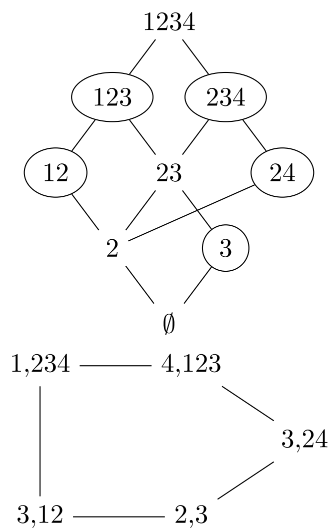

Figure D.9

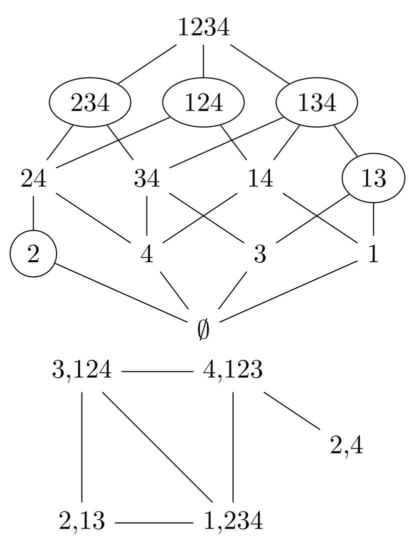

Figure D.10

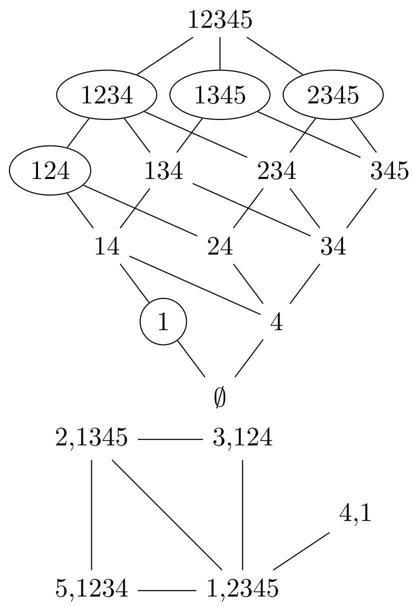

Figure D.11

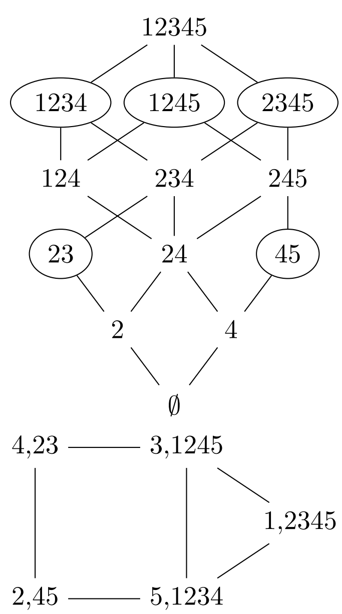

Figure D.12

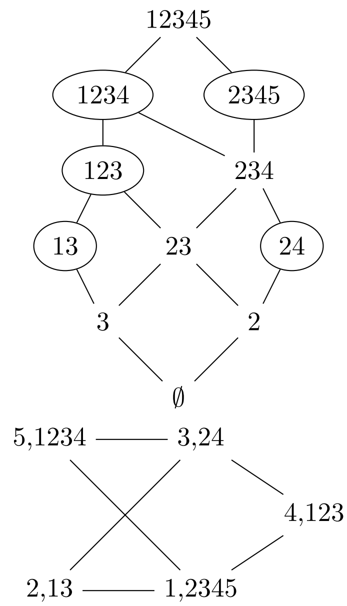

Figure D.13

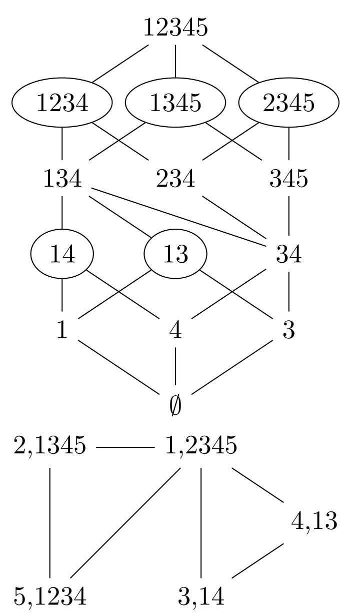

Figure D.14

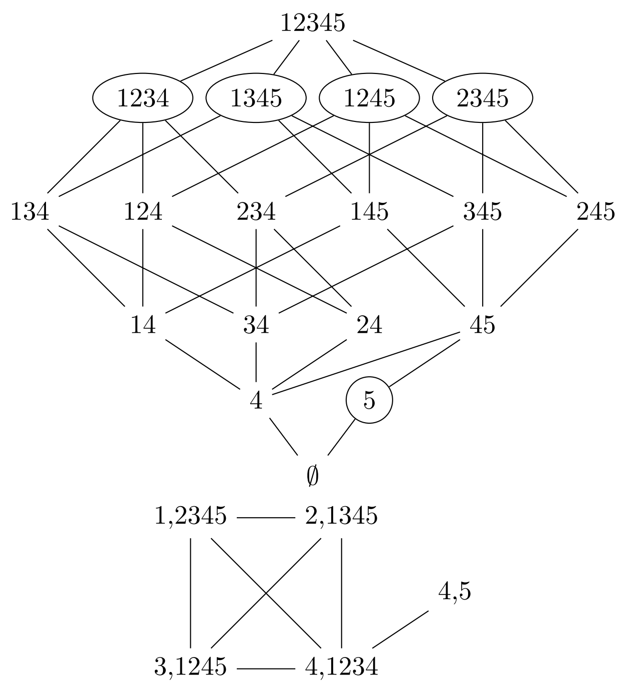

Figure D.15

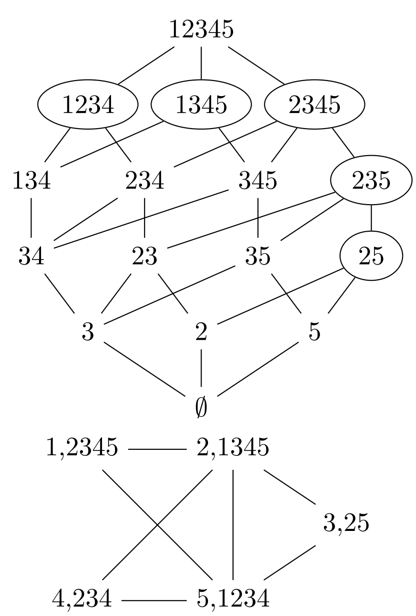

Figure D.16

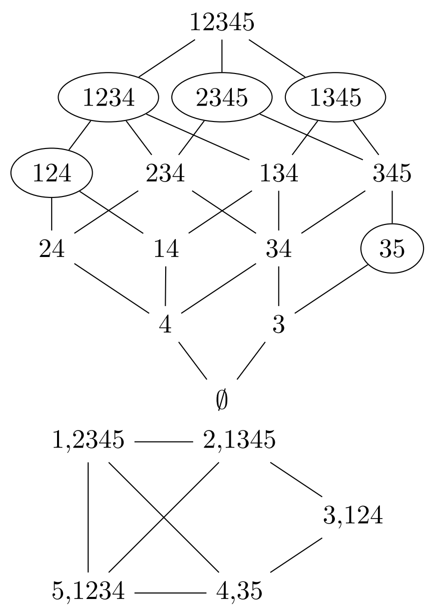

Figure D.17

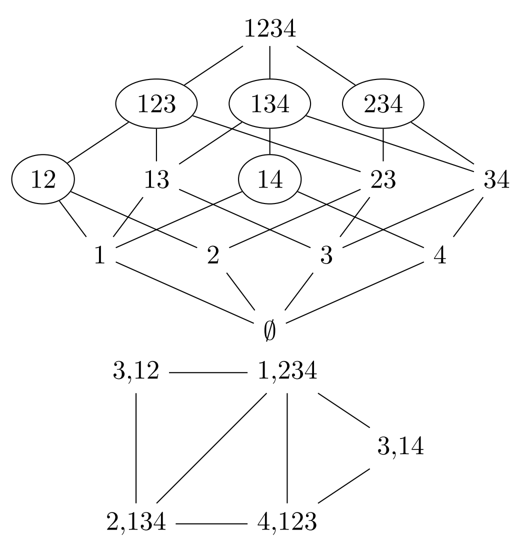

Figure D.18

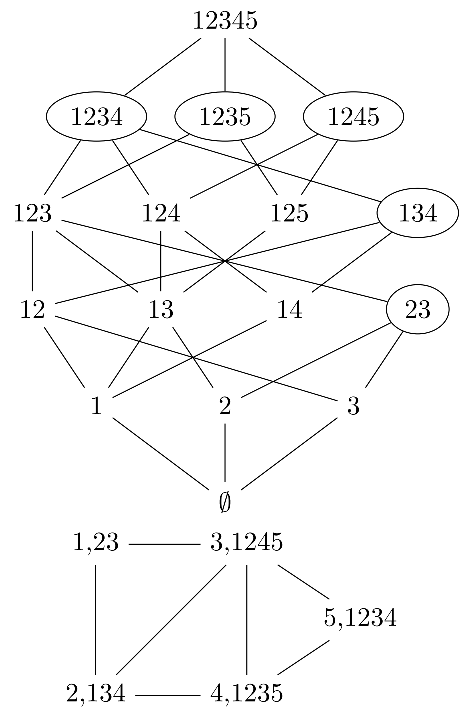

Figure D.19

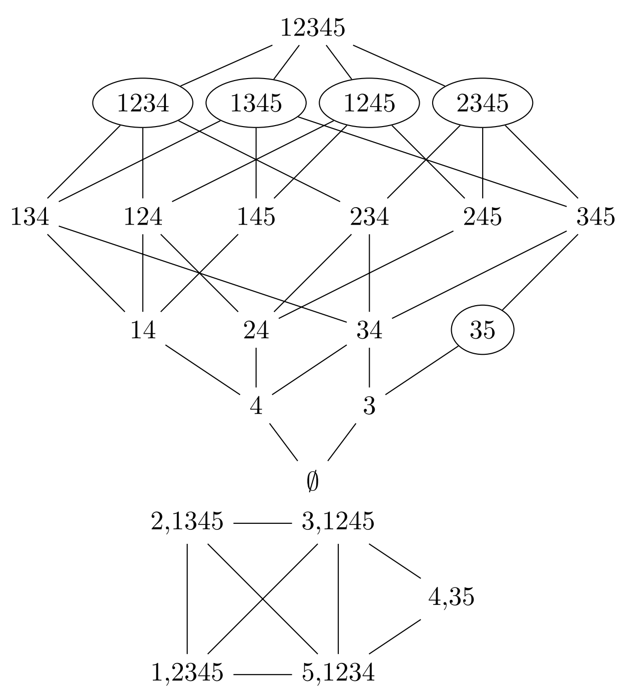

Figure D.20

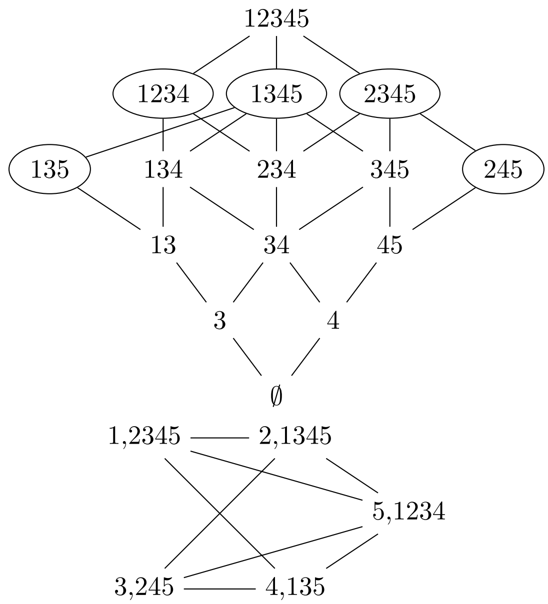

Figure D.21

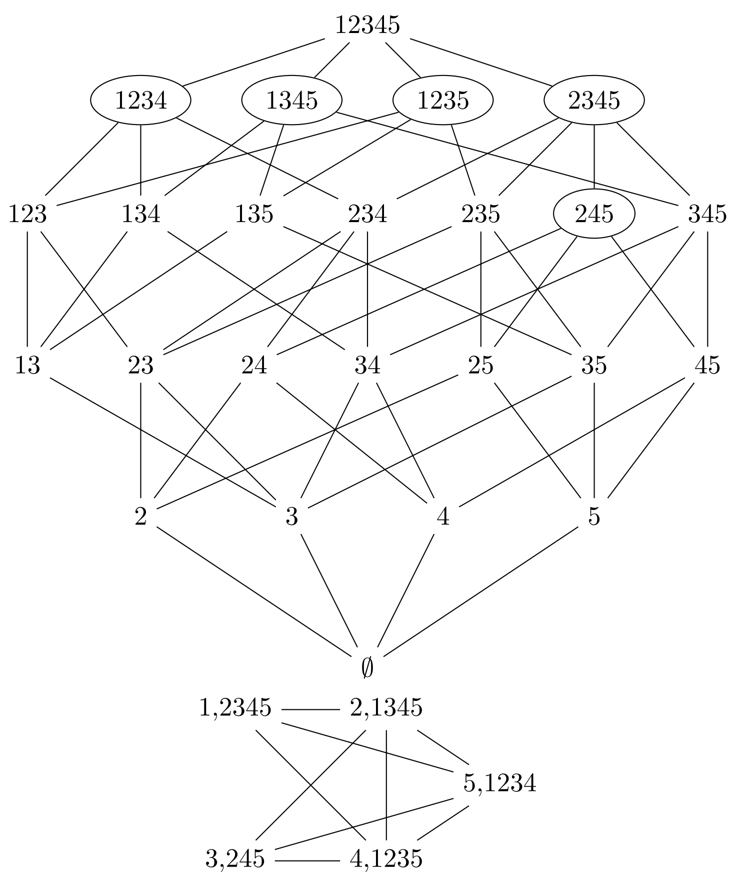

Figure D.22

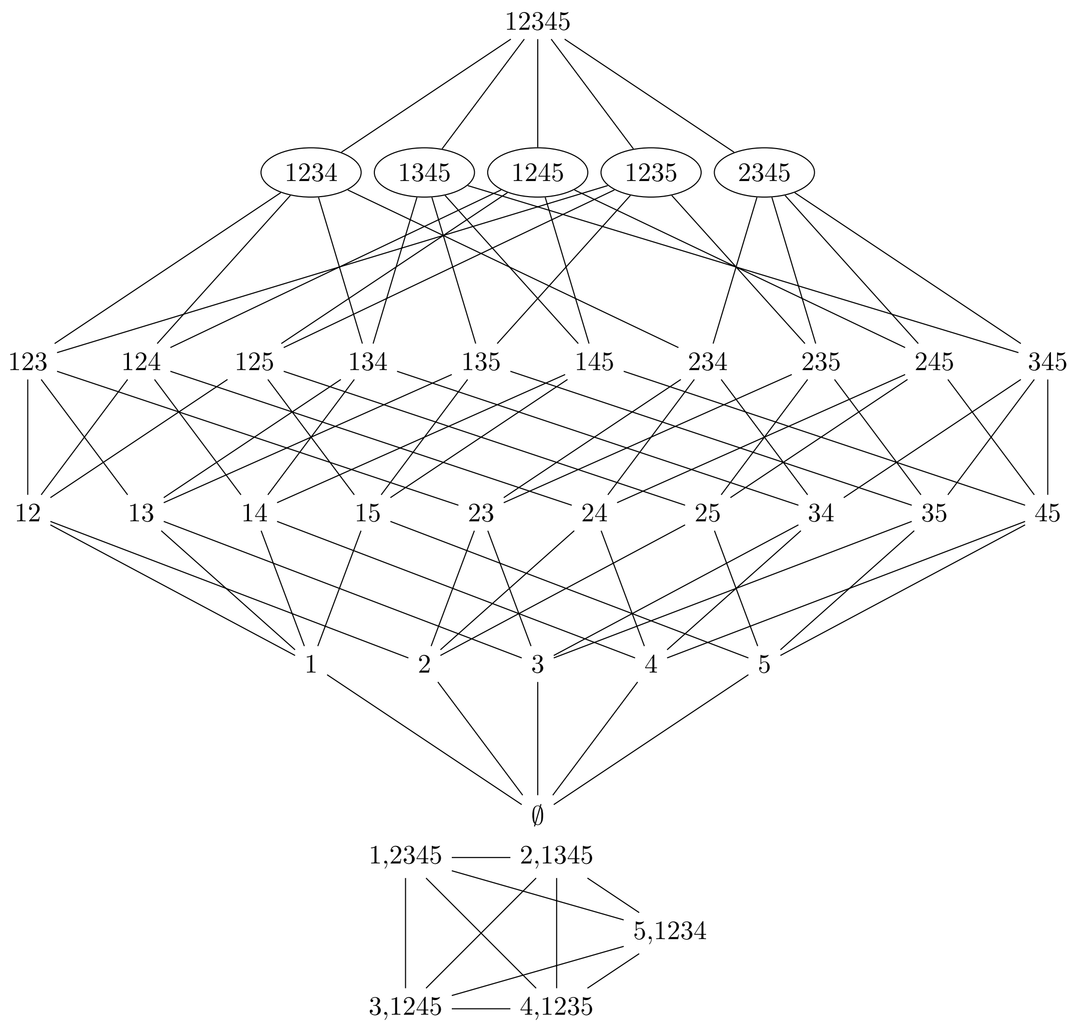
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Figure D.23