Packing Patterns into $\pi\pi$ and $\pi\pi^r$ words

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Abstract

In this paper we will discuss packing permutation patterns into words of the form $\pi\pi$, i.e., a permutation followed by itself, and $\pi\pi^r$, i.e., a permutation followed by its reverse. We consider the optimal packing of patterns of lengths 3 and 4 for the former words and 3, 4, and 5 for the latter. We also discuss the methods by which these packings are obtained and characteristics of patterns which follow a strict upper bound.

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1 Introduction

The field of permutation patterns is often studied from the point of view of pattern avoidance. This was first studied by Knuth in 1968 who was motivated by determining the nature of stack-sortable permutations [5]. The specific field of pattern packing followed much later, with the first works being published by Price et. al. in 1997 [7], Burstein et al. in 2002 [3], and Albert et. al. also in 2002 [1].

Let $\mathcal{S}_n$ denote the set of all permutations of length n, and consider $\pi \in \mathcal{S}_n$ and $\rho \in \mathcal{S}_k$. We say that π contains ρ if there exist indices $1 \leqslant i_1 < i_2 < \cdots < i_{m-1} < i_m \leqslant n$ such that $\pi_{i_a} \leqslant \pi_{i_b}$ if and only if $\rho_{i_a} \leqslant \rho_{i_b}$. In this case, we say that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_n}$ is *order isomorphic* to ρ; otherwise π avoids ρ. Any word can contain multiple instances of ρ. It follows that some words have more instances ρ than others. Suppose $\hat{\pi} \in \mathcal{S}_n$, $\rho \in \mathcal{S}_k$, and $\hat{p}(\rho)$ is the number of instances of ρ in $\hat{\pi}$. If for every $\pi^* \in \mathcal{S}_n$, $\hat{p}(\rho) \geqslant p^*(\rho)$, then we say $\hat{\pi}$ is *ρ-optimal.* These ρ-optimal words are not necessarily unique per length of π.

If we wish consider the number of instances of a pattern in its optimal words as the size of the words gets large, we can also consider the packing density. The packing density of a pattern ρ, usually denoted as δ, is defined as the probability that, as $n \to \infty$, a randomly chosen combination of k letters forms an instance of ρ. As we are packing patterns into words where each letter is in the word twice, we wish to exclude the combinations that use repeated letters. So, instead of a total of $\binom{2n}{k}$ combinations, we consider the $2^k\binom{n}{k}$ combinations of k distinct letters as the sample space.

When considering the structure of words, we can compare the position of letters relative to each other. Consider two words, $\hat{\pi}, \pi^* \in \mathcal{S}_n$. If $\hat{\pi} = \hat{\pi}_1\hat{\pi}_2\cdots\hat{\pi}_n$ and $\pi^* = \hat{\pi}_n\hat{\pi}_{n-1}\cdots\hat{\pi}_1$ then we say $\hat{\pi}$ is the reverse of π^*, denoted $\hat{\pi} = (\pi^*)^r$. Similarly, if $\pi^* = (n+1-\hat{\pi}_1)(n+1-\hat{\pi}_2)\cdots(n+1-\hat{\pi}_n)$, then we say $\hat{\pi}$ is the complement of π^*, denoted $\hat{\pi} = (\pi^*)^c$. In general throughout this paper, $n = |\pi|$ and $k = |\rho|$. In order to denote a pattern ρ such that $\rho_1 < \rho_2 < \cdots < \rho_k$, we use I_k, and likewise for ρ such that $\rho_1 > \rho_2 > \cdots > \rho_k$ we use J_k. In another direction, a permutation may be viewed as a bijection on $[n]$. When we graph the points (i, π_i) in the Cartesian plane, all points lie in the square $[0, n+1] \times [0, n+1]$.

Pattern packing up to this point has been studied in words in general [3], and in layered words [6], which will be discussed later. This paper, however, will focus wholly on words that follow two structures: $\pi\pi^r$, i.e., a permutation followed by its reverse, and $\pi\pi$, i.e., a permutation followed by itself. This work is motivated by Anderson et. al. who studied pattern avoidance in words of the form $\pi\pi^r$ [2], and Cratty et. al. who studied pattern avoidance in words of the form $\pi\pi$ [4]. The patterns discussed begin from the simplest cases, moving to increasingly longer patterns, with several interesting counting results along they way. Specifically we consider the strictly increasing pattern and patterns of length 2, 3, and 4 in words of the form $\pi\pi^r$ and $\pi\pi$. We also consider several patterns of length 5 in words of the form $\pi\pi^r$.

2 Words of the form $\pi\pi^r$

In this section, we will pack several different permutation patterns into words of the form $\pi\pi^r$.

2.1 Maximal Patterns

Let $\rho \in \mathcal{S}_k$ and let $\hat{p}(\rho)$ be the number of instances of the pattern ρ in the word $\hat{\pi}\hat{\pi}^r$. We begin by determining $\max\limits_{\hat{\pi}\in S_n} \hat{p}(\rho)$.

Theorem 1. *For any word of the form $\pi\pi^r$, the ρ-optimal word for any pattern ρ will contain at most $2\binom{n}{k}$ instances of ρ, where $n = |\pi|$ and $k = |\rho|$.*

Proof. Given any permutation π with n letters, there are $\binom{n}{k}$ ways to pick k letters of π, where $k \leqslant n$. When looking at words of the form $\pi\pi^r$, an additional instance of each letter of π is appended to the word in reverse order. It follows that any word of this form has $\binom{n}{k}$ ways to choose k distinct letters and if there is an instance of ρ using these chosen letters, there will be exactly two ways to form ρ with that combination of letters. This is due to the fact that the letter closest to the end of π in ρ can be replaced with the same letter in π^r. If any other letter in ρ were to be replaced in this way, it would no longer form ρ because the letters in ρ would no longer be considered in the order in which they were picked. Therefore, given a pattern ρ of length k, and a word formed of a permutation π and its reverse, any ρ-optimal word will contain at most $2\binom{n}{k}$ instances of ρ. □

Let ω be the most instances of any pattern of length k that can fit into a word. Should the number of instances of ρ in a ρ-optimal word equal ω, we then can call ρ maximal. By Theorem 1, for words of the form $\pi\pi^r$, we know this value to be equal to $2\binom{n}{k}$.

Due to the symmetries of $\pi\pi^r$ words, we know that if ρ is maximal, then so is ρ^r, ρ^c, and $(\rho^r)^c$ as shown in Theorem 2. However, $\rho^r, \rho^c,$ and $(\rho^r)^c$ may not be distinct, as ρ^r may be equal to ρ^c.

Theorem 2. *If $\rho \in \mathcal{S}_k$ is maximal, then $\rho^r, \rho^c,$ and $(\rho^r)^c$ are also maximal.*

Proof. Assume a pattern ρ is maximal. Suppose that a word w of the form $\pi\pi^r$ is optimal for ρ, it follows that the word will also be optimal for ρ^r. In fact, every word that is optimal for ρ will also be optimal for ρ^r. Because ρ is maximal, all words that are optimal for ρ will contain $2\binom{n}{k}$ instances of ρ. Due to the symmetries of words of the form $\pi\pi^r$, it follows that the reverse of w will also contain $2\binom{n}{k}$ instances of ρ^r. Logically, $(\pi\pi^r)^c$ will also be optimal for and contain the same number of instances of ρ^c. Therefore, for every maximal pattern ρ, ρ^r, ρ^c and $(\rho^r)^c$ are also maximal. $\square$

We have defined maximalness in words of the form $\pi\pi^r$ and observed symmetries that preserve maximalness, but they fail to characterize which patterns are maximal. To this end; let ρ_i be a letter in ρ at position i. If ρ_{i-1} and ρ_{i+1} both exist, and either $\rho_{i-1} < \rho_i > \rho_{i+1}$ or $\rho_{i-1} > \rho_i < \rho_{i+1}$, we say ρ_i is an *extreme point*. For example, in Figure 1, the extreme points of each word are squares.

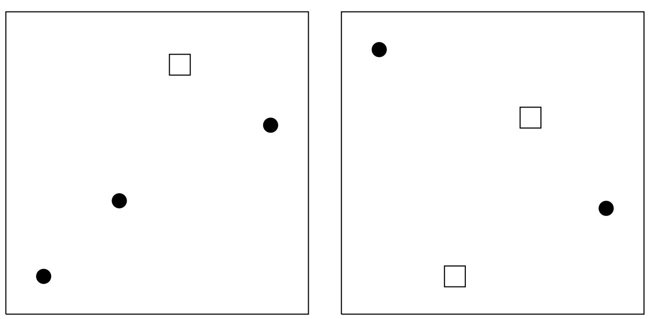

Figure 1: A graph of the word 1243 and its extreme point and 4132 and its extreme points

Theorem 3. *If a pattern $\rho \in \mathcal{S}_k$ has at most one extreme point, then any ρ-optimal word of the form $\pi\pi^r$ will have the upper bound of $2\binom{n}{k}$ instances of ρ.*

Proof. Suppose that ρ consists of a strictly increasing segment, followed by a strictly decreasing segment. Now, consider the word $w = \pi\pi^r$, where $\pi = I_n$. For any of the $\binom{n}{k}$ ways of choosing numbers i_1, i_2, …, i_k between 1 and n, there will be two ways to find ρ in w using these particular numbers. This is because, for the largest number, i_k, we can choose either of its two instances. Then, we are able to fill all remaining spots of the pattern ρ, with letters from w that occur in the correct order around i_k. Since the same logic holds for a ρ consisting of a strictly decreasing segment followed by a strictly increasing one and the word $\pi\pi^r = J_n I_n$, the ρ-optimal words for any ρ with at most one extreme point will have $2\binom{n}{k}$ instances of ρ. Thus, these ρ are maximally optimal. □

In a similar manner to Theorem 2, extreme points respect the symmetries of the square. This is discussed in Theorem 4.

Theorem 4. *A pattern ρ has the same number of extreme points as its reverse and complement.*

Proof. Let the pattern ρ have m extreme points. The reverse of ρ must have also have m extreme points. Reversal is an involution on permutations that sends peaks to peaks and valleys to valleys. Therefore it cannot have any more or any less than m extreme points. Similarly, the complement is an involution that sends peaks to valleys and valleys to peaks. Then ρ^c too must have m extreme points. □

2.2 Increasing and Decreasing Patterns

Now, we will consider patterns where $\rho = I_k$ or $\rho = J_k$. We first consider ρ where $k = 2$. Theorem 5 provides a formula to compute the number of instances of 12 in 12-optimal words.

Theorem 5. *The number of instances of the pattern $\rho = 12$ in a word of the form $\pi\pi^r$ is $n^2 - n$.*

Proof. Consider a word of the form $\pi\pi^r$, where $n = |\pi|$. There are two cases for these words. First, the first 1 can occur before some letter j where $j > 1$. Then because of the symmetry of the word, it must be that the second 1 occurs after the second j. So, the 12 is created using 1 and j exactly twice, using the first 1 and both j's. If the first j occurs before the first 1, then this letter will not be used to form a 12 pattern, rather this pattern will be formed using both 1's and the second j. So each letter greater than 1 is included in exactly two instances of the pattern $\rho = 12$ involving the letter 1. This pattern continues for all numbers up to $n-1$. So, there are exactly $2(n-1)$ instances of the pattern $\rho = 12$. Similarly, there are exactly $2(n-2)$ instances of the pattern where the smaller letter is 2. This continues up to $n-1$, where there are exactly $2(n-(n-1))$ or $2(1)$ instances of the pattern where the smaller letter is $n-1$. In total, there are $2(n-1)+2(n-2)+\cdots+2(1)$ instances of the pattern. This simplifies to

$$\begin{aligned}
&2(n-1)+2(n-2)+\cdots+2(1) \\
&= 2[(n-1)+(n-2)+\cdots+(1)] \\
&= 2\sum_{i=0}^{n-1}(n-i) \\
&= 2\left[\frac{n(n-1)}{2}\right] \\
&= n(n-1) \\
&= n^2-n.
\end{aligned}$$

So, there are n^2-n instances of the pattern $\rho = 12$ in each word of the form $\pi\pi^r$. □

Because of the structure of $\pi\pi^r$ words, similar logic can be applied to the pattern $\rho = 21$ in $\pi\pi^r$.

Theorem 6. *The number of instances of the pattern $\rho = 21$ in a word in the form $\pi\pi^r$ is n^2-n.*

Proof. From Theorem 5, it is known that the number of instances of the pattern $\rho = 12$ in any word of the form $\pi\pi^r$ is n^2-n. Since $(\pi\pi^r)^r = \pi\pi^r$ and $(12)^r = 21$, if $\pi\pi^r$ has n^2-n instances of 12, it also has n^2-n instances of 21. □

Theorem 5 provides a way to count the number of 12s in a word which is 12-optimal. Something that can be seen by looking at the proofs above and the resulting number of 12s that pack into *any* word of the form $\pi\pi^r$ is that all words of this form are optimal for both 12 and 21. This however is not always the case for patterns in $\pi\pi^r$ words. Something we do know is that the number of optimal words for any pattern is even, as discussed in Theorem 7.

Theorem 7. *Let τ denote some permutation of length n with the values x and y removed. If the $\pi\pi^r$ word $\tau xyyx\tau^r$ is ρ-optimal, then the word $\tau yxxy\tau^r$ will also be ρ-optimal.*

Proof. Consider the instances of ρ that use either x or y, as if neither is used, their relative position does not matter. If only one of x and y is used, either of the two xs or two ys could be used interchangeably. If we swap x and y, this fact holds. If both x and y are used, we can treat this as either 12 or 21 (depending on whether xy is an ascent or descent) in $\pi\pi^r$ where $n = 2$. As shown in Theorem 5, all $\pi\pi^r$ words are 12-optimal (or 21-optimal), so swapping x and y still does not change the number of instances of ρ. Since swapping x and y does not change the number of ρ in any of the cases, if $\tau xyyx\tau^r$ is ρ-optimal, so is $\tau yxxy\tau^r$. □

With all patterns of length 2 considered, we now look at longer patterns. Theorem 8 lists several properties of of I_k-optimal words.

Theorem 8. *For a word of the form $\pi\pi^r$, the following three statements are equivalent:*

(a) The word is I_k-optimal for all $k \in [3, n]$ [1].

(b) Each letter in π is either larger or smaller than all of the following letters in π.

(c) π avoids both 213 and 231.

Proof. If a word meets (b), then there is no number in the π part of the word whose value is between that of two later numbers in π. Since there is no way to form the patterns 213 and 231, π avoids these two patterns. Thus, (b) implies (c).
Similarly, the only way for a word to avoid the patterns 213 and 231 within π is for none of its numbers be between two later ones. Since this describes (b), (c) implies (b).
Suppose word w meets (b). There are $\binom{n}{k}$ ways to choose k numbers between 1 and n. Let us consider numbers $1 \leqslant p_1 < p_2 < \cdots < p_k \leqslant n$. Let us assume that p_m is the p_i closest to the end of π. For each of these $\binom{n}{k}$ ways to choose these p_i, we can form I_k like so: If $p_i < p_m$, choose its instance in π, if $p_i > p_m$,

[1] Although this is also true for $k = 2$, all $\pi\pi^r$ words are 12-optimal, so this is left out of the equivalence statement.

choose its instance from π^r, and if p_i is p_m, we can choose either instance. Thus, w has $2\binom{n}{k}$ instances of I_k. Additionally, since we have shown earlier in Theorem 1 that $2\binom{n}{k}$ is the upper bound for the number of instances of a pattern in a word of the form $\pi\pi^r$, we have reached said upper bound, so w must be I_k-optimal. This proves that (b) implies (a). Since we have shown that for a given n, there will always be some words that reach our upper bound of $2\binom{n}{k}$, it is also true that any word that falls short of this can not be I_k-optimal. Consider a word v that does not meet (b). Since (b) is equivalent to (c), we know that the first half of v contains at least one instance of 213 or 231. Consider numbers q_1, q_2, and q_3, whose instances in π form either 213 or 231. Because q_1 would always come either before both q_2 and q_3, or after both q_2 and q_3, any combination of k letters that includes q_1, q_2, and q_3 will never be able to form I_k. Thus, v falls short of $2\binom{n}{k}$, and is not I_k-optimal. Since a word not of the form described in (b) is I_k-optimal, the contrapositive is also true, so (a) implies (b). □

Unlike for $\rho = 12$, not every word of the form $\pi\pi^r$ is optimal for $123\cdots k$. Therefore it is useful to enumerate how many words of a given length are optimal for these patterns. This is done in Theorem 9.

Theorem 9. *There are 2^{n-1} I_k-optimal words of the form $\pi\pi^r$, where $|\pi| = n$.*

Proof. Looking at the form described in Theorem 8, we can see that it is either the case that $\pi_i = \max_{i\leqslant j} \pi_j$, or that $\pi_i = \min_{i\leqslant j} \pi_j$, for all i. Since there are n positions in π, we have $n-1$ such decisions, and the final number is whatever is left. Thus, there are 2^{n-1} words in the form $\pi\pi^r$ of length $|\pi| = n$ that are I_k-optimal. □

However, we do know that for any word that is I_k-optimal, the complement of that word is also optimal for I_k as explained in Theorem 10.

Theorem 10. *If a word of the form $\pi\pi^r$ is I-optimal, then the word $(\pi\pi^r)^c$ is also I-optimal.*

Proof. If a word $\pi\pi^r$ is I_k-optimal, then the reverse of the word is optimal for the reverse of the pattern, so $(\pi\pi^r)^r$ is $(I_k)^r$-optimal. Since $(\pi\pi^r)^r = \pi\pi^r$ and $(I_k)^r$ is J_k, it follows that $\pi\pi^r$ is J_k-optimal. As before, if $\pi\pi^r$ is J_k-optimal, then $(\pi\pi^r)^c$ is $(J_k)^c$-optimal. Thus, $(\pi\pi^r)^c$ is I_k-optimal. □

Also following from the symmetries of the square, Theorem 11 discusses how we know that any word in $\pi\pi^r$ that is optimal for I_k is optimal for J_k.

Theorem 11. *A strictly decreasing pattern J_k is always maximal in $\pi\pi^r$, and the same words that are optimal for I_k will be optimal for J_k.*

Proof. From Theorem 3, we know that for all k there exists a word with $2\binom{n}{k}$ instances of ρ since $\rho = I_k$ has no extreme points. The reverse of every word that is optimal for I_k is optimal for the reverse of I_k, which is J_k. Since $(\pi\pi^r)^r = \pi\pi^r$, the same words that are ρ-optimal for I_k will be optimal for J_k. Therefore, J_k is maximal too. □

2.3 Non-Maximal Patterns in $\mathcal{S}_n$

In Section 2.1, we characterized patterns ρ whose optimal words contain $2\binom{n}{k}$ instances of ρ. In, this section we discuss patterns whose optimal words fail to reach this upper bound, starting with length 4 patterns and then moving to length 5 patterns.

Table 1 shows all patterns $\rho \in \mathcal{S}_n$ of length 4. For each of these patterns we give a formula enumerating a lower bound on the number of instances of the given patterns in their optimal words. Following that is a potential OEIS entry that matches the sequence.

Family of ρ	**Sequence**	**Lower Bound**	**OEIS Entry**
1243, 4312, 3421, 2134	**2, 10, 30, 70, 140**	$2\binom{n}{4}$	**A034827**
1342, 2431, 4213, 3124	**2, 10, 30, 70, 140**	$2\binom{n}{4}$	**A034827**
1234, 4321	**2, 10, 30, 70, 140**	$2\binom{n}{4}$	**A034827**
1432, 2341, 4123, 3214	**2, 10, 30, 70, 140**	$2\binom{n}{4}$	**A034827**
2143, 3412	2, 10, 28, 64, 124	(degree 4 polynomial in n)	None
1423, 3241, 4132, 2314	2, 10, 28, 60, 110	$2\binom{n-2}{3} + 2\binom{n-1}{3}$	A006331
1324, 4231	2, 10, 26, 54, 102	$2\binom{n-1}{3}$	None
2413, 3142	2, 8, 22, 48, 92	$2\binom{n-2}{3} + 2\binom{n-2}{2}$	None

Table 1: A table displaying patterns of length 4 along with a lower bound on the instances of ρ in a ρ-optimal word

The following theorems explain the logic behind the each lower bound presented in Table 1.

Theorem 12. *A lower bound on the number of instances of 2143 that can appear in 2143-optimal words of the form $\pi\pi^r$ is given by the following formula:*

$$f(n) = \begin{cases} (3n-2)\binom{\frac{1}{2}n}{3} & n \text{ is even} \\ \frac{1}{6}(3n^2-2n-9)\binom{\frac{n+1}{2}}{2} & n \text{ is odd} \end{cases}$$

Proof. Consider a $\pi\pi^r$ word where π has the following structure: If a letter's value (denoted l) is less than $n/2$, then it is immediately followed by $n+1-l$, and then by $l-1$ (unless $l=1$, in which case l is the penultimate letter), and if n is odd, $\lceil \frac{n}{2} \rceil$ is placed at the beginning (See Figure 2).

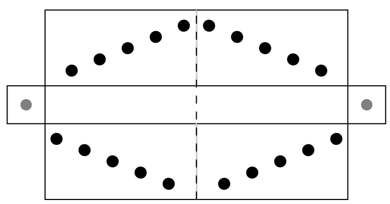

Figure 2: The word structure for the lower bound on packing the pattern 2143

Another way to see this is as two alternating sub-sequences of sizes m_1 and m_2, where $m_1 = m_2 = \frac{n-1}{2}$ when n is odd, and $m_1 + 1 = m_2 = \frac{n}{2}$ when n is even. Then the two sub-sequences can be denoted by m_1,$m_1 - 1$,...,2,1 and $n - m_2 + 1$,$n - m_2 + 2$,...$n-1$,n. Figure 3 demonstrates all the different ways of selecting a 2143 pattern from the six different usable sub-sequences along with the number of 2143s produced by each of the cases. The task of showing that these sum to the formulas given above is algebra.

□

Theorem 13. *The lower bound for the number of instances of packing the 1423 in 1423-optimal words of the form $\pi\pi^r$ is:* $2\binom{n-2}{3} + 2\binom{n-1}{3}$.

Proof. Consider the word where $\pi = 1 \oplus J_{n-2} \oplus 1$, as in Figure 4. Now, consider the letters 1 and n. At least one of them must be used in any instance of 1423. They can both be used, and either can be used without the other, but if neither is used, it is impossible to create any 1423s. First, let

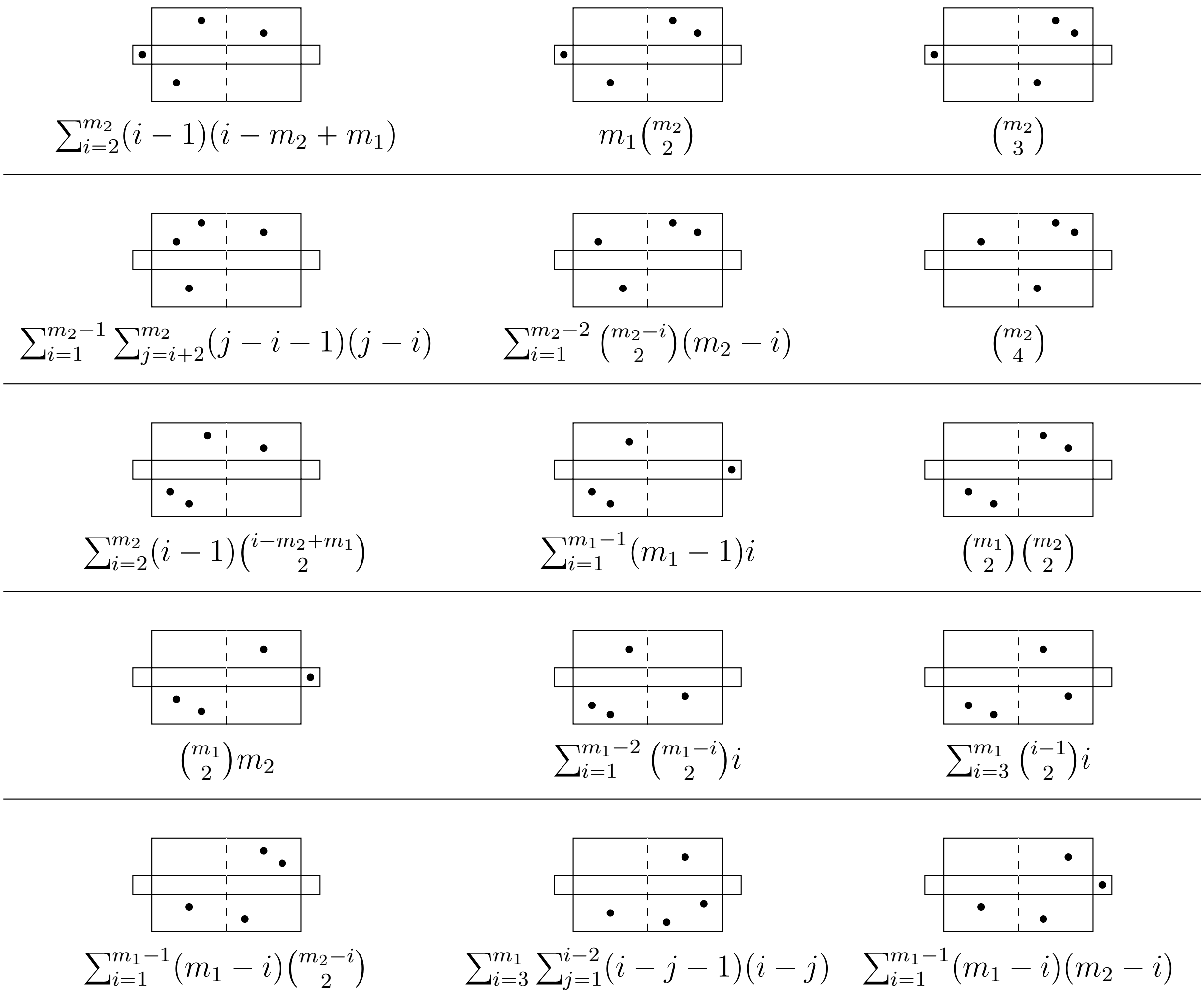

Figure 3: The 15 cases for which combinations of of sub-sequences letters can be chosen from to form 2143 patterns, along with their formulas

us consider only the cases where the first 1 is used. By choosing any three of the remaining $(n-1)$ letters, we can form two instances of 1423 by using the instance of the largest chosen letter from π, and the other two from the reverse. Then, a second can be formed by using the reverse of one of the letters (n if it was chosen, otherwise the smallest of those selected). Thus, if the first 1 is used, we can form 1423 in $2\binom{n-1}{3}$ ways.
Next, if we do not use the first 1, then we must use n. From the $(n-2)$

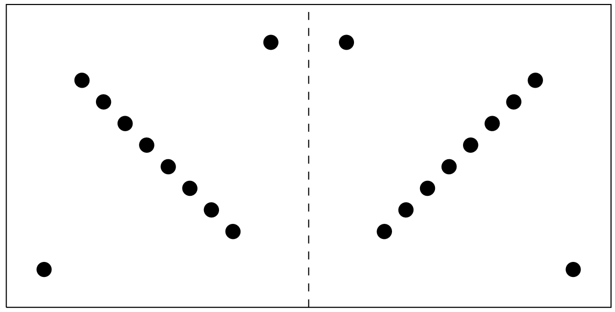

Figure 4: The word structure for the lower bound on packing the pattern 1423

letters that are neither 1 nor n, we can choose any 3 letters, and by taking the smallest from π and the remaining two from the reverse, we can form a 1423 pattern. But since either n can be used interchangeably, we really can make twice this amount. Thus, if the first 1 is not used, there are $2\binom{n-2}{3}$ ways to form 1423. $\square$

Similar to Theorem 12, we are convinced that this bound is tight. However, the following lower bound is not tight. In fact, we know that it strictly underestimates the number of instances of 1324 in a 1324-optimal word.

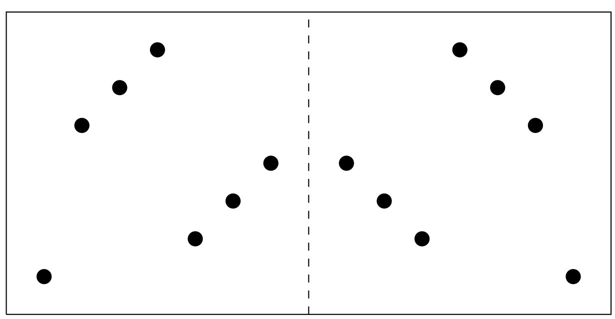

Figure 5: The word structure for the lower bound on packing the pattern 1324

Theorem 14. *A lower bound on the instances the pattern 1324 in its optimal word(s) is* $2\binom{n-1}{3}$.

Proof. Consider $\pi \in \mathcal{S}_n$ and a word $w = \pi\pi^r$, of the form $1 \oplus \left(I_{\lfloor \frac{n-1}{2} \rfloor} \ominus I_{\lfloor \frac{n-1}{-2} \rfloor} \right)$, as in Figure 5. Using only the leading 1 as the 1 in our 1324's, it follows that there $\binom{n-1}{3}$ ways of picking a 213 pattern in the remaining digits of π. Allotting that any $\pi_i \in w$ which lies closest to π_n can be swapped for its corresponding digit in the other half of the word, any pattern formed can be formed twice using the same letters, thus a lower bound for the total numbers of instances of $\rho = 1324$ that can be packed into words of this form is $2\binom{n-1}{3}$. □

The lower bound described above fails to account for any instance of 1324 where the 1 in the pattern is not the leading 1 in the word. However, there is a conjectured structure that might be the optimal structure for packing 1324 into words of the form $\pi\pi^r$. Following a similar structure to that in Figure 5, the conjectured optimal is broken up into four sub-sections. The first sub-section is 1. The other three are $I_a \ominus I_b \ominus I_c$, where as n grows, $a = \lfloor \frac{n}{3} \rfloor - 1$ and $b, c = \lfloor \frac{n}{3} \rfloor$. When n is not a multiple of three, add one to b, then after incrementing n, add 1 to c.

While a lower bound is conjectured for 2413, a proof for it does not currently exist. See Section 4 for further explanation.

Having here addressed all $\rho \in \mathcal{S}_4$, the next logical step is to begin packing ρ where $\rho \in \mathcal{S}_5$. Table 2 displays the relevant information for each pattern of $\rho \in \mathcal{S}_k$ for which we have determined a lower bound on the number of instances of ρ in ρ-optimal words.

Family of ρ	**Sequence**	**Lower Bound**	**OEIS Entry**
12345,54321	**2, 12, 42, 112, 252**	$2\binom{n}{5}$	**A277935**
15234, 51432, 43251, 23415	2, 12, 40, 100, 210	$2\binom{n-2}{3} + 4\binom{n-2}{4}$	A008911
13425, 52431, 53241, 14235	2, 10, 32, 78, 162	$2\binom{n-2}{4} + 2\binom{n-2}{3} + 2\binom{n-4}{3}$	None
15423, 32451, 51243, 34215	2, 10, 30, 70, 140	$2\binom{n-1}{4}$	A034827
15324, 42351, 51342, 24315	2, 10, 30, 70, 140	$2\binom{n-1}{4}$	A034827
21534, 43512, 45132, 23154	2, 10, 30, 70, 140	$2\binom{n-1}{4}$	A034827
25314, 41352	2, 8, 20, 44, 90	$2\binom{n-2}{3} + 4\binom{n-4}{4}$	None

Table 2: A table displaying patterns of length 5 along with a lower bound on the instances of ρ in a ρ-optimal word

As with all the patterns whose count in their optimal words is enumerated where $k = 4$, the authors believe that the lower bounds are tight to the true counting formula. Each theorem below is accompanied by figures displaying the words described in the proofs. To begin with, we look at 15234, whose lower bound is derived form packing words of the form displayed in Figure 6.

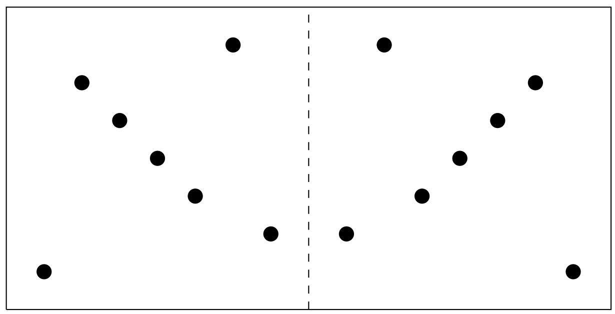

Figure 6: The word structure for the lower bound on packing the pattern 15234

Theorem 15. *There is a lower bound of $2\binom{n-2}{3} + 4\binom{n-1}{4}$ instances of the pattern* 15234 *in* 15234*-optimal words of the form $\pi\pi^r$.*

Proof. Consider words of the form $1 \oplus ((J_{n-3} \oplus n) \ominus 1)$. Then, count the copies of 15234 in these words. First consider the copies of the 15234 pattern using the 1 and the n in the π half of the word, where 1 must fill the 1 spot and n must fill the 5 spot in the pattern. Then, there are $n-2$ remaining letters in ascending order in the π^r half of the word to form the 234 portion of the pattern. If the 2 is used to form the 2 portion of the pattern, then it is possible to use the 2 from either half of the word. If not, it is possible to use n, which is the second to last letter of the π half of the word from π or π^r, so there are $2\binom{n-2}{3}$ ways to create 15234 using 1 and n. Then, consider forming the pattern using n, but specifically not using 1, where n must be the 5 in the pattern. Then, there are $n-2$ letters and 4 spots in the pattern remaining, so there are $\binom{n-2}{4}$ ways to create the 15234 pattern. Again, if the 2 is used to form the 1 in the pattern, then the 2 from either half of the word may be used. Otherwise, the n from either portion of the word may be used. So, there must be $\binom{n-2}{4}$ ways to form the pattern using the n and not the 1. Finally, consider using the 1 at the beginning of the word and not the n.

Again, there are $n-2$ remaining letters in the word and 4 remaining spots in the pattern, so there are $\binom{n-2}{4}$ ways to create the 15234 pattern. Then, it must be that the 152 portion of the pattern uses letters from the π half of the word and the 34 uses letters from the π^r half of the word. Then, the letter that is used as the 2 in the π half will always occur before the letters used as the 34 in the π^r half, so it is possible to use either occurrence. Therefore, there must be $2\binom{n-1}{4}$ ways to form the pattern using the 1 and not the n. In total, there are $2\binom{n-2}{3}+2\binom{n-1}{4}+2\binom{n-1}{4}$ ways to form 15234, which simplifies to $2\binom{n-2}{3}+4\binom{n-2}{4}$.

This formula holds for the complement, reverse, and reverse of the complement, 51432, 43251, and 23415 respectively. $\square$

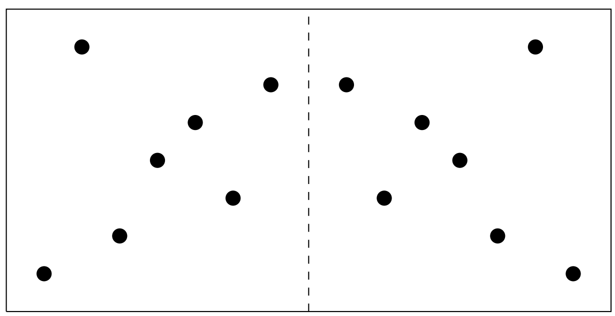

Figure 7: The word structure for the lower bound on packing the pattern 13425

Theorem 16. *There is a lower bound of* $2\binom{n-2}{4}+2\binom{n-2}{3}+2\binom{n-5}{2}$ *instances of the pattern* 13425 *in words of the form* $\pi\pi^r$.

Proof. Consider words of the form $1(n)2456\cdots3(n-1)(n-1)3\cdots6542(n)1$, as in Figure 7. Then, count the instances of 13425 in these words. First consider instances of the pattern that are formed without $n-1$. Since $n-1$ is not being used, n must be used to fill the 5 spot in every instance of the pattern 13425. Then, there are $n-2$ letters of the word and 4 spots in the pattern remaining. Since the letters in the π half of the word that would be used to fill the 3 and 4 spots of the pattern are increasing and the letter that would be used to fill the 2 spot will occur after those letters in the π^r half of the word, any combination of letters that can be picked to form the

$2, 3,$ and 4 spots of the pattern can be found exactly twice, where the letter used as the 4 can be taken from either π or π^r. So, there are $2\binom{n-2}{4}$ ways to create the 13425 pattern without using $n-1$. Then count instances of 13425 using n and the $n-1$ in π^r to fill the 5 and 4 spots of the pattern, respectively. Then, there are $n-2$ letters of the word and 3 spots in the pattern remaining. Then, because the letters that would fill the remaining 3 spots of the pattern are increasing in π and decreasing in π^r, any 3 letters that can be chosen will be found in the correct order. So, the letters that are chosen to fill the 1 and 3 spots will be found increasing in π, and the letter chosen to fill the 2 spot will be found after $n-1$ before n in π^r. In addition, since $n-1$ is the last letter in π and the first letter in π^r, either letter can be chosen. So, there are $2\binom{n-2}{3}$ ways to form the 13425 pattern using both $n-1$ and n. Finally, count instances of the pattern when $n-1$ is used and n is not. In this case, because $n-1$ must be the last letter to fill the pattern, the 3 must be used to fill the 2 spot. Then, depending on which letter is selected to fill the 1 spot, there is another letter that cannot be used. So, there are $n-4$ letters that can fill 3 spots in the pattern. Then, because $n-1$ is the last letter in π and the first letter in π^r, it can be taken from either permutation. So, there are $2\binom{n-4}{3}$ ways to form the 13425 pattern using $n-1$ and not n. In total, there are $2\binom{n-2}{4} + 2\binom{n-2}{3} + 2\binom{n-5}{2}$ ways to create the 13425 pattern in words of this form. $\square$

Theorem 17 shows the lower bound for 15423 in its optimal words. Theorems 18 and 19 count the same for 15324 and 21534. These lower bounds are all given by the same formula, explained below. We conjecture that all these lower bounds are sharp.

Theorem 17. *There is a lower bound of $2\binom{n-1}{4}$ instances of the pattern 15423 in words of the form $\pi\pi^r$.*

Proof. Consider the word $\pi = 1 \oplus (J_{n-3} \ominus 12)$, as in Figure 8. Then, count the instances of 15423 in these words. In each of the instances of the pattern, the 1 at the beginning of the word is always used to create the pattern. Then, there are $n-1$ letters left to choose from to form the remaining 5423 pattern. So, there are $\binom{n-1}{4}$ ways to create the rest of the pattern. Then consider the letters in the word, where the 15423 pattern is filled by $1dcab$ and $1 < a < b < c < d$. Consider the last letter in the pattern, b. If b can be taken from π, then it must also be able to be taken from π^r. However, if b cannot be taken from π, then it must be that a can be taken from either π or

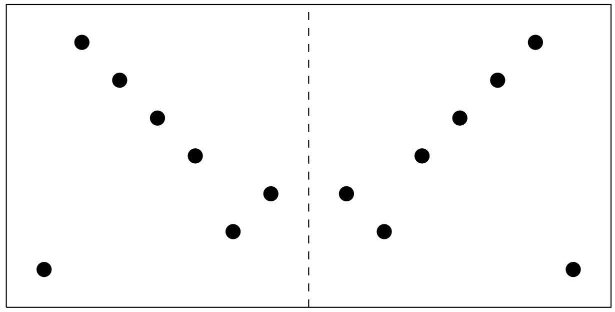

Figure 8: The word structure for the lower bound on packing the pattern 15423

π^r. This is because a is smaller than b, and the π^r half of the word is strictly increasing from 2 to n. So, if b can only be taken from π^r, then a must occur before b twice, once in π and once in π^r. So, it must be that there are always two options for either a or b used to create the 15423 pattern. Therefore, there are $2\binom{n-1}{4}$ ways to create the 15423 pattern in words of this structure.

This formula holds for the complement, reverse, and reverse of the complement, 51243, 32451, and 34215 respectively. $\square$

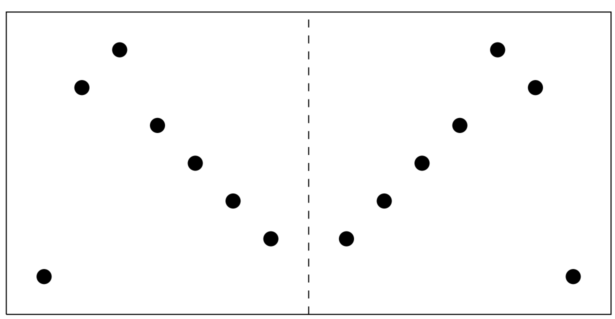

Figure 9: The word structure for the lower bound on packing the pattern 15324

Theorem 18. *There is a lower bound of* $2\binom{n-1}{4}$ *instances of the pattern* 15324 *in words of the form* $\pi\pi^r$.

Proof. Consider words of the form $1 \oplus (12 \ominus J_{n-3})$, as in Figure 9. Then, count the instances of 15324 in these words. In all of the instances of this pattern in these words, the 1 is used for the 1 in this pattern. So, there are $n-1$ letters in the word and 4 letters in the pattern remaining. It is possible to pick any 4 letters and find them in 5324 order, where the 1 position in the pattern is filled by the 1. This is due to the fact that π is decreasing from n to 2, where $n-1$ is before n, and π^r is increasing from 2 to n, where $n-1$ is after n, so any letters chosen to fill the 532 spots in the pattern will be found decreasing in the π half of the word and any letters chosen to fill the 4 spot will be found increasing in the π^r half of the word. Then, since the π^r half of the word is increasing, the letter chosen to fill the 2 spot will occur before the letter chosen to fill the 4 spot. So, there are always two different ways to create the same pattern using the two instances of the same letter used to fill the 2 spot in the pattern, thus $2\binom{n-1}{4}$.
This formula holds for the complement, reverse, and reverse of the complement, 51342, 42351, and 24315 respectively. $\square$

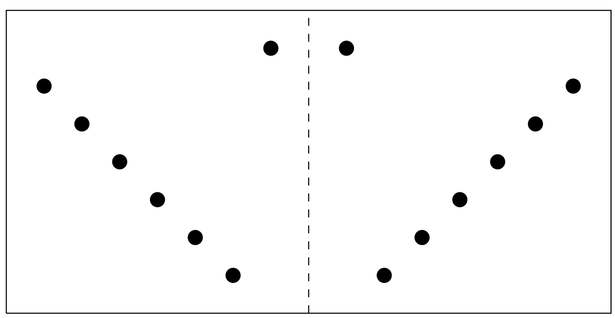

Figure 10: The word structure for the lower bound on packing the pattern 21534

Theorem 19. *There is a lower bound of* $2\binom{n-1}{4}$ *instances of the pattern* 21534 *in words of the form* $\pi\pi^r$.

Proof. Consider words of the form $\pi = J_{n-1} \oplus n$, as in Figure 10. Then, count the instances of 21534 in these words. In these words, n is always used to fill the 5 spot of the pattern. This means there are $n-1$ letters to fill the remaining 4 spots of the pattern. In the π half of the word, until the n, the letters are strictly decreasing, and after n, in π^r, the letters are strictly increasing. So, there will always be a way to organize the the $n-1$ letters into the 21 before the n and the 34 after the the n. So, there are $\binom{n-1}{4}$ ways to create the 21534 pattern. Then, it is possible to use the n that is a part of either half of the word, so there are always $2\binom{n-1}{4}$ ways to create the 21534 pattern in these words. $\square$

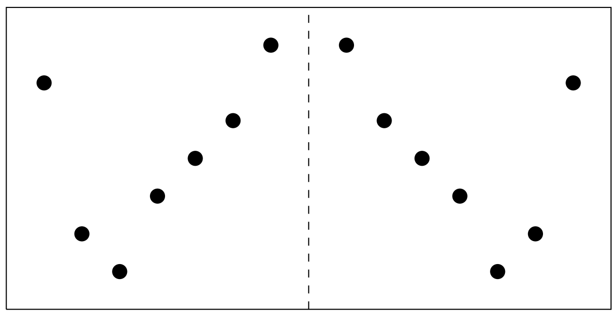

Figure 11: The word structure for the lower bound on packing the pattern 15423

The last pattern that has thus far been packed into words of the form $\pi\pi^r$ is 25314, which is explained in Theorem 20.

Theorem 20. *There is a lower bound of* $2\binom{n-2}{3} + 4\binom{n-4}{4}$ *instances of the pattern* 25314 *in words of the form* $\pi\pi^r$*.*

Proof. Consider words of the form

$$(n-1)2(n-3)(n-4)\cdots 134(n-2)(n)(n)(n-2)431\cdots(n-4)(n-3)2(n-1)$$

as in Figure 11. Then, count the instances of 25314 in these words. First, in all words where $|\pi| \leqslant 7$, all instances of the pattern use n to fill the 5 spot in the pattern and $n-1$ in the second half of the word to fill the 4 spot in the pattern. Then, that leaves $n-2$ letters in the word and 3 letters in the pattern remaining. Then, because of the structure of the word, it possible to pick any 3 remaining letters in the word and find them organized such that the 2 is before the n and the 31 are after the n, but before the $n-1$. Then, it is possible to use the n in either half of the word since they are adjacent letters, so there are $2\binom{n-2}{3}$ ways to create the 25314 pattern. However, when $|\pi| > 7$, there are some instances of the pattern where $n-2$ is used to fill the 5 spot in the pattern. In that case, because of the structure of the word, it is no longer possible to use n, $n-1$, and $n-4$ in the pattern. However, it is possible to use the $n-2$ in either half of the word since n is not able to be used. Then we can pick any 4 of the remaining $n-4$ letters to complete a 25314 pattern. So, there are $2\binom{n-4}{4}$ ways to form this pattern. In addition when $|\pi| > 7$, there are cases where $n-1$ is no longer used to fill the 4 letter in the pattern and one of the letters that previously could not be used is used. Then, because of the structure of the word and the specific letters that are used, there are 4 letters that no longer can be a part of the word, depending on which letter is picked to fill the 2 spot in the pattern. So, that leaves $n-4$ letters and 4 spots in the pattern. Then, again because of the structure of these $\pi\pi^r$ words, the letter that is used to fill the 5 spot in the pattern can be taken from either half of the word. So, there are $2\binom{n-4}{4}$ ways to form the 25314 pattern in this case. In total there are $2\binom{n-2}{3} + 2\binom{n-4}{4} + 2\binom{n-4}{4}$ ways to form the pattern 25314 in these words, which simplifies to $2\binom{n-2}{3} + 4\binom{n-4}{4}$. This formula holds for the reverse which is equal to the complement, 41352. □

3 Words of the form $\pi\pi$

In this section, we will consider several patterns in $\mathcal{S}_k$ and their packing behavior into words of the form $\pi\pi$, where $\pi \in \mathcal{S}_n$

3.1 The strictly increasing permutation I_k

As with words discussed in Section 2, the first patterns considered here are $\rho = I_k$. We also calculate the packing densities of these patterns.

Theorem 21. *There are $\binom{n}{k}(k+1)$ instances of the pattern I_k in I_k-optimal words of the form $\pi\pi$. Additionally, $\pi\pi = I_nI_n$ is the only such I_k-optimal word.*

Proof. Consider a particular combination of k letters. If they are arranged in ascending order within π, then there are $(k+1)$ possible ways for these letters to form I_k in the entire word, since there are $(k+1)$ ways to choose how many letters come from the first π. If, however, these k letters are not in ascending relative order in π, then they either form 1 or 0 instances of I_k, since, if any were possible at all, there would only be one option for where to jump from the first π to the second. Since the word where $\pi = I_n$ is the only word where all combinations of k letters are in ascending relative order in π, it must be the only I_k-optimal word, and must contain $\binom{n}{k}(k+1)$ instances of I_k. $\square$

Theorem 22. *The packing density for I_k is $\frac{k+1}{2^k}$.*

Proof. By our definition of packing density in $\pi\pi$ words and the formula given by Theorem 21, the packing density is:

$$\delta = \frac{\binom{n}{k}(k+1)}{2^k\binom{n}{k}} = \frac{k+1}{2^k}.$$

$\square$

Table 3 gives the packing densities of I_k for when $k \leqslant 10$.

k	δ
2	0.75
3	0.5
4	0.3125
5	0.1875
6	0.109375
7	0.0625
8	0.035156
9	0.019531
10	0.01074

Table 3: Packing densities of I_k for small values of k

3.2 Layered Patterns

Next we discuss packing layered patterns. Albert et al.[1] looked at the packing of layered patterns into permutations. A permutation is layered if it can be expressed as the direct sum of strictly decreasing segments, where the segments themselves can be of any length. That is, $J_a \oplus J_b \oplus J_c \cdots \oplus J_\ell$, is a layered permutation.

With this definition in mind, the first pattern we consider is 132. In particular, we find that a word that is a layered permutation followed by itself is optimal for this pattern.

Theorem 23. *There exists a word $\pi\pi$ for any length $n > 2$, with π layered, that is optimal for the pattern* 132.

Proof. Consider any three letters in π. They can have as their relative order any of the 6 length-three permutations. Table 4 below shows how many total instances of 132 a given three letters will make in $\pi\pi$ based on their order in π.

Arrangement	Instances of 132
123 123	1
132 132	4
213 213	1
231 231	0
312 312	1
321 321	1

Table 4: Number of instances of 132 a given three letters make in $\pi\pi$ based on their relative order in π

As can be seen in the table above, if we want to pack 132 into the entire $\pi\pi$ word, it suffices to pack 132 into π alone if π then also avoids 231. As it so happens, Stromquist[6] proved that there exists a layered permutation that optimally packs 132. Since layered permutations necessarily avoid 231, this fulfills our requirements. □

Now that we know that a structure for 132-optimal words, we can enumerate the number of instances of 132 in these words.

Theorem 24. *The number of instances of 132 in a word of the form $\pi\pi$ is given by the following recursive function f, where $r = \frac{1}{1+\sqrt{3}} \approx 0.366$, and $q = 1 - r$:*

$$f(n) = \begin{cases} 4\left([rn]\binom{[qn]}{2}\right) + \binom{[qn]}{3} + [qn]\binom{[rn]}{2} + f([rn]) & n \geqslant 3 \\ 0 & n < 3. \end{cases}$$

Proof. If π has only two layers, which we denote as α and β (that is, $\pi = \alpha \oplus \beta$), then we can divide the instances of 132 into different cases. Obviously, there is some ratio between the size of α to the size of π that optimizes the

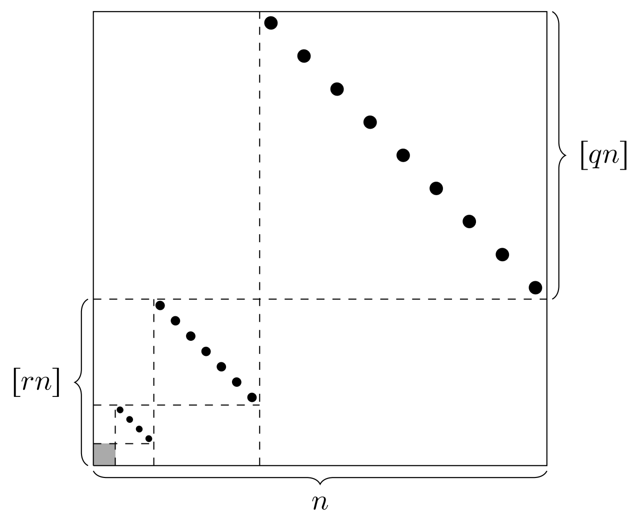

Figure 12: Ideal structure of π for 132 packing

number of 132s, but without knowing what this ratio is, we shall denote it as r. With only two layers in π, that gives us 4 sections overall $((\alpha \oplus \beta)(\alpha \oplus \beta))$.

First, there are four ways (as shown in Figure 13) to split a 132 between the four sections such that the '1' comes from an α, and the '2' and '3' come from one or both of the βs. For each of these cases, we can select '1' from any of the $[rn]$ numbers in the chosen α, and choose any two of the $[qn]$ numbers from β, and assign the larger to be '3' and the smaller to be '2'. Thus, this gives us four different ways to find $[rn]\binom{[qn]}{2}$ ways to embed 132 in $\pi\pi$ where only the '1' comes from α.

If we were to instead choose all three numbers from β, since they both are strictly decreasing, the '1' would have to be from the first β, while the '2' and '3' would have to come from the second β. Since any 3 of the $[qn]$ numbers can be arranged in this way, we have $\binom{[qn]}{3}$.

Now, if we instead choose to take only the '3' from a β, it would have to be in the first one, but could be any of the $[qn]$ numbers there. Meanwhile, any 2 of the $[rn]$ could form such a 132, but the lower number would have to come from the first α, while the higher from the second. Thus, there are $[qn]\binom{[rn]}{2}$ such 132s.

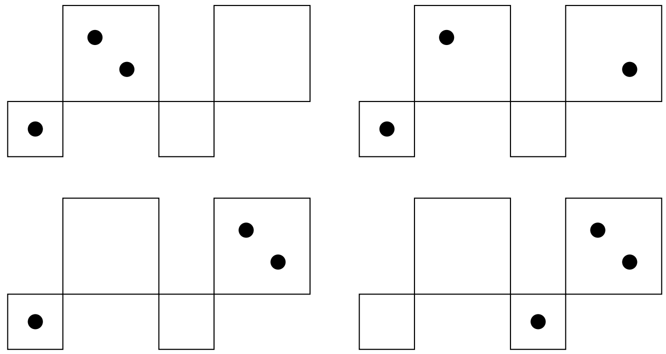

Figure 13: Cases where only the smallest letter of a 132 pattern comes from α

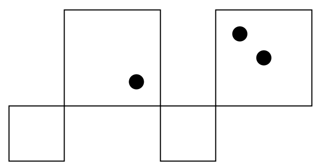

Figure 14: Packing all three letters of a 132 pattern into βs

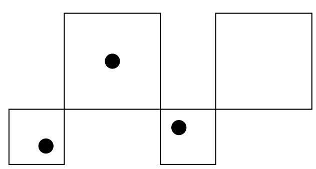

Figure 15: Packing two letters of a 132 pattern into αs and the third into a β

Finally, we can also form 132 patterns by using only the αs. However, since this is the first time we have cared about the order of the letters in α, we might as well think of it as an entire word that we wish to pack 132 into, with length $[rn]$. In this way, we instead call the entire function again with this length as the parameter. As such, we have $f([rn])$

If we do this process repeatedly, until the size of the lowest layer is less than

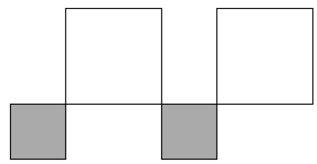

Figure 16: Case where all three letters come from either α

3, then we end up with a recursive structure, like the one shown in Figure 12. We can then find the ideal ratio r by taking the limit of f over our packing density function, as defined earlier, as n goes to infinity, and finding the r value that maximizes the result. $\square$

After taking another limit, it is simple to find that the packing density of 132 words of the form $\pi\pi$ is approximately 0.299:

$$\lim_{n\to\infty} \frac{4\left([rn]\binom{[qn]}{2}\right)+\binom{[qn]}{3}+[qn]\binom{[rn]}{2}}{\left(\binom{n}{3}-\binom{rn}{3}\right)2^3} = \frac{2\sqrt{3}+1}{4\sqrt{3}+8} \approx 0.299.$$

The proof that there is a layered 132-optimal word is fairly straightforward. We believe there is a similar argument that for any layered pattern ρ, there is a layered permutation π such that $\pi\pi$ is ρ-optimal. While we do not yet have a general proof for packing any layered pattern, we will prove that such a layered π exists for some specific length 4 layered permutations in Theorems 25 and 26, but will conjecture that such a π exists for the other patterns considered.

Conjecture 1. *If a pattern ρ is layered, then there exists a layered permutation π such that the word $\pi\pi$ is ρ-optimal.*

Recall the table we used to break down the possible arrangements of three letters to show that 132 had a layered optimizer. We now present a similar table, with all the length 4 pattern classes. Note that since we argue that the optimal words are layered, we can safely remove any of the arrangements that contain either 312 or 231 in their first half.

The first layered pattern of length four that we consider is the pattern 2143.

Arrangement	1432	2143	1243	1324
1234 1234	0	0	1	1
1243 1243	1	1	5	1
1324 1324	1	0	1	5
1432 1432	5	1	1	1
2134 2134	0	1	0	1
2143 2143	1	5	1	1
3214 3214	1	1	0	1
4321 4321	1	1	0	0

Table 5: Number of various length 4 patterns a given four letters can make in $\pi\pi$ based on their relative order in π

Theorem 25. *The number of 2143s in a 2143-optimal word of the form $\pi\pi$ is given by*

$$f(n) = 5\binom{\lfloor\frac{n}{2}\rfloor}{2}\binom{\lceil\frac{n}{2}\rceil}{2} + \binom{\lfloor\frac{n}{2}\rfloor}{4} + \binom{\lceil\frac{n}{2}\rceil}{4} + \left\lceil\frac{n}{2}\right\rceil\binom{\lfloor\frac{n}{2}\rfloor}{3} + \left\lfloor\frac{n}{2}\right\rfloor\binom{\lceil\frac{n}{2}\rceil}{3}.$$

Proof. In the same manner as with 132, it suffices to find a 2143-optimal permutation that also avoids the patterns 1234 and 1324. As demonstrated in Albert et. al. [1], since 2143 has two layers, neither of which have one element, is *only* optimized by a layered permutation, which it is fairly easy to see is a decreasing segment directly summed with an equally sized decreasing segment (which we shall again denote as $\alpha \oplus \beta$). Since this exact structure also happens to avoid 1234 and 1324, it is this same structure that is 2143-optimal in $\pi\pi$ (see Figure 17). Of note, when n is an even number, this formula simplifies down to

$$f(n) = 5\binom{\frac{n}{2}}{2}^2 + 2\binom{\frac{n}{2}}{4} + 2\left(\frac{n}{2}\right)\binom{\frac{n}{2}}{3}.$$

Since when n is odd, the extra letter can appear in either decreasing sequence, we arbitrarily decide to define $|\beta| = |\alpha| + 1$ in such a case.

First, there are 5 (or $(k+1)$) ways to divide up a 2143 pattern that appears in π between the two halves of the word. Since both α and β contain two letters, this is simply $5\binom{\lfloor\frac{n}{2}\rfloor}{2}\binom{\lceil\frac{n}{2}\rceil}{2}$.

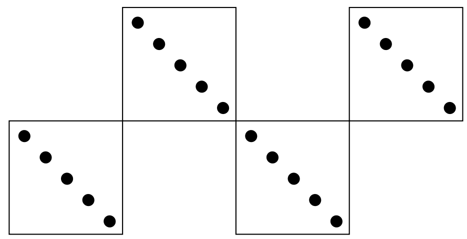

Figure 17: Structure of 2143-optimal words of the form $\pi\pi$

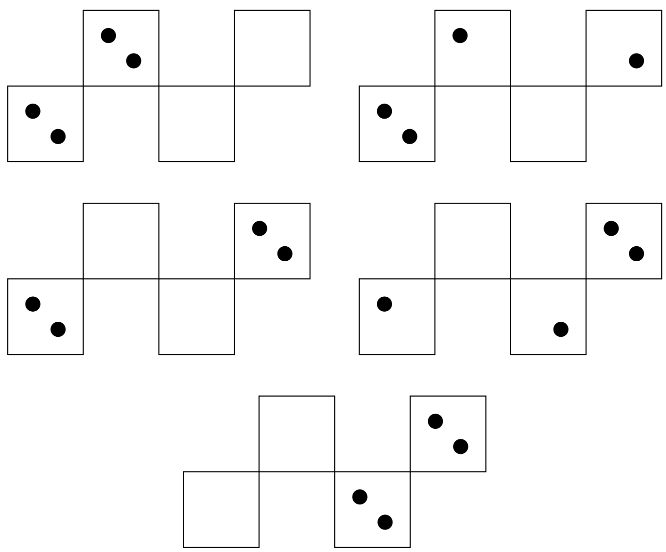

Figure 18: Cases where two letters of a 2143 come from both α and β

Next, either decreasing segment can contain all four letters that compose a 2143, but the only way to do this is for the lower two letters to be in the first instance of π, and the higher two to be in the second instance of π. Since this is choosing four letters from either α or β, there are $\binom{\lfloor \frac{n}{2} \rfloor}{4} + \binom{\lceil \frac{n}{2} \rceil}{4}$ instances in this case.

Finally, if we wish for α (or β) to contain only one letter of a 2143 pattern, then β (or α) in the same π must hold two letters, and the β (or α) in the other π must hold one, as shown in Figure 20. Since we are selecting three letters from either α or β and only one from the other, this is $\lceil \frac{n}{2} \rceil \binom{\lfloor \frac{n}{2} \rfloor}{3} + \lfloor \frac{n}{2} \rfloor \binom{\lceil \frac{n}{2} \rceil}{3}$.

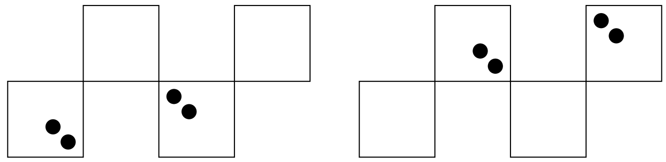

Figure 19: Cases where all four letters of a 2143 pattern are either all from α or all from β in a $(\alpha \oplus \beta)(\alpha \oplus \beta)$ structure

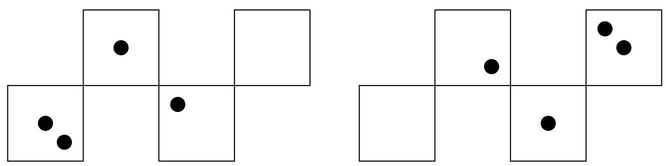

Figure 20: Cases where three letters of a 2143 pattern come from either α or β and a single letter comes from the other in a $(\alpha \oplus \beta)(\alpha \oplus \beta)$ structure

Adding all of these terms together, we end up with the formula which we wished to prove. $\square$

Of note, if we look back to Table 5 and see that every combination of four letters is either going to provide us with 5 total 2143s if it is itself one, or a single instance if it is not, we can also write this formula as $f(n) = \binom{n}{4} + 4\binom{\lfloor \frac{n}{2} \rfloor}{2}\binom{\lceil \frac{n}{2} \rceil}{2}$.

Taking the limit, we can then find that the packing density of 2143 is simply $\delta = \frac{5}{32}$.

While we have been able to prove the validity of Conjecture 1 for the patterns we have looked at so far, we now get to patterns where this remains a conjecture. Thus, the packing enumerations we describe are lower bounds. Should Conjecture 1 hold for these patterns, the lower bounds will in fact be tight.

Theorem 26. *A lower bound on the number of 1243s in a 1243-optimal $\pi\pi$ word is given by the following formula, where*

$r = \frac{5+4\sqrt{3}}{23} \approx 0.5186$:

$$f(n) = 5\binom{[rn]}{2}\binom{[(1-r)n]}{2} + \binom{[rn]}{4} + [(1-r)n]\binom{[rn]}{3} + [rn]\binom{[(1-r)n]}{3}.$$

Proof. If we again consider Table 5, we can see that the biggest factor is the number of instances of 1243 in π. If we first think only of maximizing the number of instances of 1243 in π, Albert et. al [1] show that this is achieved by the structure $\alpha \oplus \beta$, where α is strictly increasing, and β is strictly decreasing (See Figure 21).

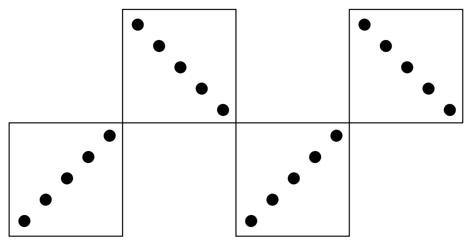

Figure 21: Structure of 1243-optimal words in $\pi\pi$

For a single permutation, the structure is optimal when both α and β are the same size. However, with the $\pi\pi$ structure, if we again look to Table 5, one can see that we are being limited by the instances of 4321 in π (we can ignore 2134 and 3214, as they never appear in this structure). If we maximize the ratio r in the formula above as n gets large, we find that r has the value $\frac{5+4\sqrt{3}}{23} \approx 0.5186$ mentioned above. Since this is very close to one half, these instances of 4321 make only a small change, as expected. Concerning which combinations of letters from α and β we can create 1243s from, these are exactly the same as 2143, with the exception that we can not create a 1243 using only letters from the two βs. $\square$

Like before, we can express this formula for the number of 1243 patterns in 1243-optimal words of the form $\pi\pi$ as:

$$\binom{n}{4} + 4\binom{[rn]}{2}\binom{[(1-r)n]}{2} - \binom{[(1-r)n]}{4}.$$

The packing density for 1243 in $\pi\pi$ words is

$$\delta = \frac{1}{194672}(5+4\sqrt{3})(388\sqrt{3}+1819) \approx 0.1526.$$

This is slightly higher than the packing density for 2143.

4 Further Research

Regarding 2413 in words of the form $\pi\pi^r$, the binomial in Table 1 is a conjectured lower bound with no proof as of yet. We also have not conjectured an optimal structure of the 2413-optimal words of this form.

Some additional avenues for future research include:

- Tighten the lower bounds and/or prove the strength of the lower bounds for packing patterns in their optimal words of the form $\pi\pi^r$.
- Continue to look at non-maximal patterns of length $k \geqslant 5$ in $\pi\pi^r$ to find lower bounds for the number of times a pattern ρ's optimal words contain ρ.
- Explore the relationship of maximal patterns to non-maximal patterns in $\pi\pi^r$.
- Look at packing non-layered patterns of length 4 in words of the form $\pi\pi$.
- Pack non-monotone patterns of length $k \geqslant 5$ in words of the form $\pi\pi$.
- Prove Conjecture 1, that a layered permutation can be optimized by a layered permutation followed by itself, either for other specific patterns, or generally.
- Find the packing densities of patterns in words of the form $\pi\pi^r$.

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References

[1] M. Albert, M. Atkinson, C. Handley, D. Holton, W. Stromquist, On Packing Densities of Permutations, *Electron. J. Combin.* **9** (2002), #R5.

[2] M. Anderson, M. Diepenbroek, L. Pudwell, and A. Stoll. *Pattern Avoidance in Reverse Double Lists*, arXiv:1704.08638.

[3] A. Burstein, P. Hästö, and T. Mansour, Packing patterns into words, *Electron. J. Combin.* **9.2** (2002-3), #R20.

[4] C. Cratty, S. Erickson, F. Negassi, and L. Pudwell. *Pattern avoidance in double lists*, Involve **10.3**(2017), 379–398.

[5] D. Knuth, *The Art of Computer Programming: Volume 1*, Addison Wesley, 1968.

[6] W. Stromquist, Packing layered posets into posets, Unpublished, 1993.

[7] A. Price, Packing densities of layered patterns, Ph. D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.