ON COPOINT GRAPHS

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Abstract. A convex geometry is a discrete abstraction of convexity defined by a meet-distributive lattice on a finite set. In particular, we study a graph formed from the copoints of a convex geometry. A graph that can be realized in this way from some convex geometry is called a copoint graph. We demonstrate existence and non-existence for several infinite families of graphs as copoint graphs. We show that the graph join of any non-copoint graph with an arbitrary graph is not a copoint graph. Further, we provide a construction to show that the complement of a copoint graph need not be a copoint graph. We conclude that not all trees are copoint graphs and argue that the Hasse diagram of a convex geometry has a 'rhomboidal' structure if and only if its copoint graph is a tree.

1. Introduction

Before we can outline the main goals of this paper, we give a few definitions. First, if we let X be a finite set, then $\mathscr{L} : 2^X \to 2^X$ is a **closure operator** if $\mathscr{L}$ satisfies the following properties:

(1) $\mathscr{L}(\emptyset) = \emptyset$;
(2) $\mathscr{L}(X) = X$;
(3) for all $A \subset X$, $A \subset \mathscr{L}(A)$ and $\mathscr{L}(\mathscr{L}(A)) = \mathscr{L}(A)$;
(4) for all $A, B \subset X$, $A \subset B \implies \mathscr{L}(A) \subset \mathscr{L}(B)$.

Throughout the rest of this paper, we will write $\mathscr{L}(2^X) := \mathscr{L}$ and we will also say that sets in $\mathscr{L}$ are **closed**. Note that throughout this paper we will denote the collection of closed sets of a given size $|X| - k$ as R_k.

Next, we say that a closure operator $\mathscr{L}$ is an **alignment** if $A, B \in \mathscr{L}$ implies that $A \cap B \in \mathscr{L}$. A closure operator $\mathscr{L}$ has the **greedy property** if for every set $K \in \mathscr{L} - X$, there exists $p \in X - K$ such that $K \cup p \in \mathscr{L}$. With this notion of a closure operator and these additional properties, we can define a **convex geometry** to be the pair of a finite set X along with a closure operator $\mathscr{L}$, denoted $(X, \mathscr{L})$ that both is an alignment and has the greedy property. Note that a convex geometry $(X, \mathscr{L})$ forms a poset ordered by inclusion, and thus we can use a Hasse diagram to visualize $(X, \mathscr{L})$.

A set $C \in \mathscr{L}$ that is maximal in $X - p$ for some $p \in X$ is called a **copoint attached to** p. Throughout our paper we will denote the collection of copoints in $(X, \mathscr{L})$ as $M(X, \mathscr{L})$. In [EJ85], Edelman and Jamison show that if C is a copoint attached to both p, q, then $p = q$. As a result, we are able to define the mapping $\alpha : M(X, \mathscr{L}) \to X$ by $C \mapsto \alpha(C) := p$ where C is attached to p. Another result of [EJ85] is that α is a surjective mapping. In general though, α is not injective.

We refer to copoints of size $|X| - 1$ as **extreme copoints** and the points that they are attached to are called **extreme points**.

For a given convex geometry $(X, \mathscr{L})$, we define the **copoint graph** of $(X, \mathscr{L})$ to be $\mathcal{G}(X, \mathscr{L}) := (V, E)$ where $V := M(X, \mathscr{L})$ and $E := \{\{A, B\} : A, B \in M(X, \mathscr{L}), \alpha(A) \in B, \alpha(B) \in A\}$.

The primary objective of this report is to consider a graph of interest G and determine if there exists a convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L}) = G$. Some preliminary work has been done in this area already, as the following theorem indicates.

Theorem 1.1. [Bea13] *Let C_n be the cycle graph on n vertices. For $n \geq 6$, there does not exist a convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L}) = C_n$.*

Note, C_n refers to a cycle graph only in Section 6. Elsewhere we will sometimes use the notation C_n to refer to a copoint of size $|X| - n$.

Morris first introduced the notion of a copoint graph in [Mor06]. Given $X \subset \mathbb{R}^2$ finite and in general position, defining $\mathscr{L} := \{\text{conv}(A) \cap X : A \subset X\}$ gives a convex geometry $(X, \mathscr{L})$. Morris [Mor06] showed that if the clique number of copoint graphs arising from planar point sets can be made sufficiently large, one could answer a long standing problem of Erdős and Szekeres posed in [ES35]. That is, how large should $|X|$ be for X to contain a convex n-gon?

There is a need to understand the structure of copoint graphs in relation to the structure of their convex geometries. Learning about the types of copoint graphs that are possible may lead to greater understanding about graph invariants such as clique number and chromatic number, which is the key to furthering the work of Morris.

2. Graph joins

Let $(X, \mathscr{L})$ be a convex geometry. Let $Y \in \mathscr{L}$ and consider $\mathscr{L}|_Y := \{C \cap Y : C \in \mathscr{L}\}$ and $\mathscr{L}/Y := \{C \subset X - Y : C = \mathscr{L}(D \cup Y) - Y \text{ for some } D \subset X - Y\}$. The pairs $(Y, \mathscr{L}|_Y)$ and $(X - Y, \mathscr{L}/Y)$ are convex geometries [EJ85]. The former is called the **deletion** of Y and the latter is called the **contraction** on Y. Note that $\mathscr{L}|_Y$ induces a convex geometry for any $Y \subset X$ and Y need not be closed.

Let $(Z, \mathscr{L}_Z)$ and $(Y, \mathscr{L}_Y)$ be convex geometries with $Z \cap Y = \emptyset$. Define $X = Z \sqcup Y$ and $\mathscr{L} := \{\mathscr{L}_Z(C \cap Z) \sqcup \mathscr{L}_Y(C \cap Y) : C \subset X\}$. $(X, \mathscr{L})$ is called the **direct sum** of $(Z, \mathscr{L}_Z)$ and $(Y, \mathscr{L}_Y)$ and we write $(X, \mathscr{L}) = (Z, \mathscr{L}_Z) \oplus (Y, \mathscr{L}_Y)$. Beagley [Bea13] showed that $(X, \mathscr{L})$ is a convex geometry. For an m-fold direct sum of convex geometries $(X_1, \mathscr{L}), ..., (X_m, \mathscr{L}_m)$ this should be evaluated left to right as $((X_1, \mathscr{L}) \oplus (X_2, \mathscr{L}_2)) \oplus ...) \oplus (X_m, \mathscr{L}_m)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_2 \cap V_1 = \emptyset$. Set $V = V_1 \sqcup V_2$ and $E := \{\{v, w\} : \{v, w\} \in E_1 \sqcup E_2 \text{ or } v \in V_1 \implies w \in V_2 \text{ or } w \in V_1 \implies v \in V_2\}$. The **join** of G_1 and G_2, written $G_1 \vee G_2$ is the pair (V, E). Informally, we retain all edges from G_1 and G_2, and gain all pairs of vertices For an m-fold join $G_1 \vee G_2 \vee G_3 \vee ... \vee G_m$, this is evaluated left to right as $(...(G_1 \vee G_2) \vee G_3) \vee ...) \vee G_m$. The following proposition relates these two constructions.

Proposition 1 ([Bea13], Proposition 3.4)**.** *For all $1 \leq i \leq m$ let $(X_i, \mathscr{L}_i)$ be convex geometries with $X_i \cap X_j = \emptyset$ for all $i \neq j$. $\bigoplus_{i=1}^m (X_i, \mathscr{L}_i)$ is a convex geometry and*

$$\mathcal{G}(\bigoplus_{i=1}^{m}(X_i, \mathscr{L}_i)) = \bigvee_{i=1}^{m} \mathcal{G}(X_i, \mathscr{L}_i).$$

Theorem 2.1. *Let $(X_1, \mathscr{L}_1), (X_2, \mathscr{L}_2)$ be convex geometries such that $X_1 \cap X_2 = \emptyset$ and put $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2) := (X, \mathscr{L})$. If $Y \subseteq X_1$, then $(Y, \mathscr{L}_1|_Y) \oplus (X_2, \mathscr{L}_2) = (Y \sqcup X_2, \mathscr{L}|_{Y \sqcup X_2})$*

Proof. First let $(Y \sqcup X_2, \mathscr{L}') = (Y, \mathscr{L}_1|_Y) \oplus (X_2, \mathscr{L}_2)$. In order to show that $(Y \sqcup X_2, \mathscr{L}') = (Y \sqcup X_2, \mathscr{L}|_{Y \sqcup X_2})$, it is sufficient to show that for any $D \in Y \sqcup X_2$, $\mathscr{L}'(D) = \mathscr{L}|_{Y \sqcup X_2}(D)$. We see that

$$\mathscr{L}'(D) = \mathscr{L}_1|_Y(D \cap Y) \sqcup \mathscr{L}_2(D \cap X_2)$$

. Then

$$\begin{aligned}\mathscr{L}|_{Y \sqcup X_2}(D) =& \mathscr{L}(D) \cap (Y \sqcup X_2) \\ =& (\mathscr{L}_1(D \cap X_1) \sqcup \mathscr{L}_2(D \cap X_2)) \cap (Y \sqcup X_2) \\ =& ((\mathscr{L}_1(D \cap X_1)) \cap (Y \sqcup X_2)) \sqcup ((\mathscr{L}_2(D \cap X_2)) \cap (Y \sqcup X_2)) \\ & .\end{aligned}$$

Since the codomain of $\mathscr{L}_2$ is X_2 and $X_2 \subseteq Y \sqcup X_2$, we know that

$$\begin{aligned}&((\mathscr{L}_1(D \cap X_1)) \cap (Y \sqcup X_2)) \sqcup ((\mathscr{L}_2(D \cap X_2)) \cap (Y \sqcup X_2)) \\ &= ((\mathscr{L}_1(D \cap X_1)) \cap (Y \sqcup X_2)) \sqcup \mathscr{L}_2(D \cap X_2).\end{aligned}$$

Further, because $D \subseteq Y \sqcup X_2$, $X_1 \cap X_2 = \emptyset$, and $Y \subseteq X_1$, we know that $D \cap X_1 = D \cap Y$ and thus

$$\begin{aligned}
&((\mathscr{L}_1(D \cap X_1)) \cap (Y \sqcup X_2)) \sqcup \mathscr{L}_2(D \cap X_2) \\
&= ((\mathscr{L}_1(D \cap Y)) \cap (Y \sqcup X_2)) \sqcup \mathscr{L}_2(D \cap X_2) \\
&= ((\mathscr{L}_1(D \cap Y) \cap Y) \sqcup (\mathscr{L}_1(D \cap Y) \cap X_2)) \sqcup \mathscr{L}_2(D \cap X_2) \\
&= ((\mathscr{L}_1|_Y(D \cap Y)) \sqcup (\mathscr{L}_1(D \cap Y) \cap X_2)) \sqcup \mathscr{L}_2(D \cap X_2).
\end{aligned}$$

Since the codomain of $\mathscr{L}_1$ is X_1, $\mathscr{L}_1(D \cap Y) \cap X_2 = \emptyset$ and we can see

$$\begin{aligned}
&((\mathscr{L}_1|_Y(D \cap Y)) \sqcup (\mathscr{L}_1(D \cap Y) \cap X_2)) \sqcup \mathscr{L}_2(D \cap X_2) \\
&= \mathscr{L}_1|_Y(D \cap Y) \sqcup \mathscr{L}_2(D \cap X_2)
\end{aligned}$$

and thus we have shown $\mathscr{L}'(D) = \mathscr{L}|_{Y \sqcup X_2}$ proving that $(Y, \mathscr{L}_1|_Y) \oplus (X_2, \mathscr{L}_2) = (Y \sqcup X_2, \mathscr{L}|_{Y \sqcup X_2})$.

□

Theorem 2.2. *Let G and H be graphs. If H is not a copoint graph for any convex geometry, then $G \vee H$ is not a copoint graph for any convex geometry.*

Proof. We will assume for the sake of contradiction that there exists a convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L}) = G \vee H$. Then we let $Y = \{\alpha(C) | C \in V(G)\}$. Since there is an edge between all vertices in $V(H)$ and all vertices in $V(G)$, we can see that

$$\bigcap_{D \in V(H)} D = Y$$

and thus $Y \in \mathscr{L}$ by the alignment property. Recall that $(X - Y, \mathscr{L}/Y)$ is a convex geometry, where $\mathscr{L}/Y$ is the contraction on Y. By [Bea13, 2.6], we know that $\mathcal{G}(X - Y, \mathscr{L}/Y)$ is isomorphic to the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the copoints which contain Y. However, we know that for all $C \in V(G)$, $Y \subset \{\alpha(C)\} \not\subseteq C$ and for all $D \in V(H)$, $Y \subset D$, so $\mathcal{G}(X - Y, \mathscr{L}/Y) = H$ up to graph isomorphism. This is a contradiction because we supposed that there was no convex geometry with H as its copoint graph. Thus, we contradict the assumption that $G \vee H = \mathcal{G}(X, \mathscr{L})$. □

As a consequence of Theorem 2.2 together with Theorem 1.1 we see that $C_n \vee G$ is not a copoint graph of any convex geometry for all graphs G. For instance, the graphs in Fig. 1 and Fig. 2 are not copoint graphs of convex geometries.

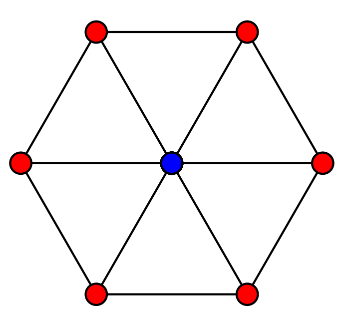


FIGURE 1. W_7:= Wheel graph on 7 vertices ($C_6 \vee K_1$)

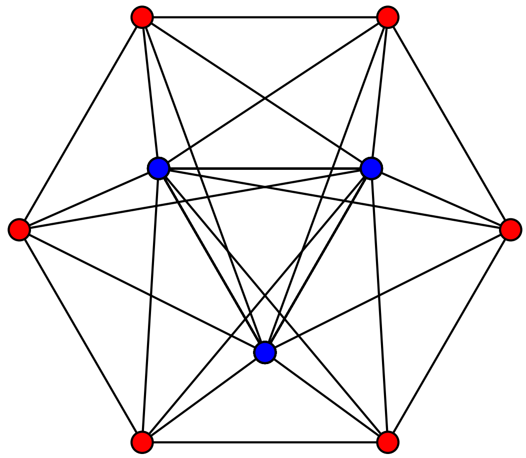


FIGURE 2. The W_6^3 graph ($K_3 \vee C_6$)

3. Kneser Graph

Definition 3.1. Kneser graphs, $K(n, m)$ are the graphs on m-subsets of an n-set, adjacent when disjoint.

Definition 3.2. Complete graphs are the graphs in which each pair of graph vertices are connected by an edge.

Proposition 2. *A convex geometry,$(X, \mathscr{L})$ whose copoint graph, $\mathcal{G}(X, \mathscr{L})$ is a complete graph can be constructed if and only if all of its copoint are extreme copoints.*

Proof. Suppose we have constructed a convex geometry $(X, \mathscr{L})$ whose copoint graph is a complete graph (K_n) on n vertices. For example see Fig. 3 that is a (K_5).

First we will assume that $|X| = k < n$. Then since we need n copoints, by the pigeon hole principle there exist at least 2 copoints, A and B, such that $\alpha(A) = \alpha(B)$. However, this implies that A is not connected to B in our copoint graph which is a contradiction. Thus $|X|= n$. By the definition of complete graph, all the copoints have degree $n-1$ in $\mathcal{G}(X, \mathscr{L})$. If all the copoints have degree $n-1$ then all the copoints attached to distinct point $p \in X$ should be contained in all the other copoints. If any copoint attached to point p is not contained in any one or more copoints than one or more edges between the copoints will be lost. Losing edges between copoints will not give rise to a copoint graph that is complete graph. Therefore, a copoint graph is a complete graph if all of it's copoints are extreme points. □

Definition 3.3. Kneser graphs of the form $K(2n, n)$ are ladder rung graphs of order $\binom{2n}{n}$.

Proposition 3. *There exists a convex geometry, $(X, \mathscr{L})$ whose copoint graph is a ladder rung graph.*

Proof. We construct the following convex geometry, $([8], \mathscr{L})$ with the following closed sets,

$\emptyset$ {1} {2} {12} {123} {123} {1234} {12345} {12346} {123456} {1234567} {1234568} {12345678}

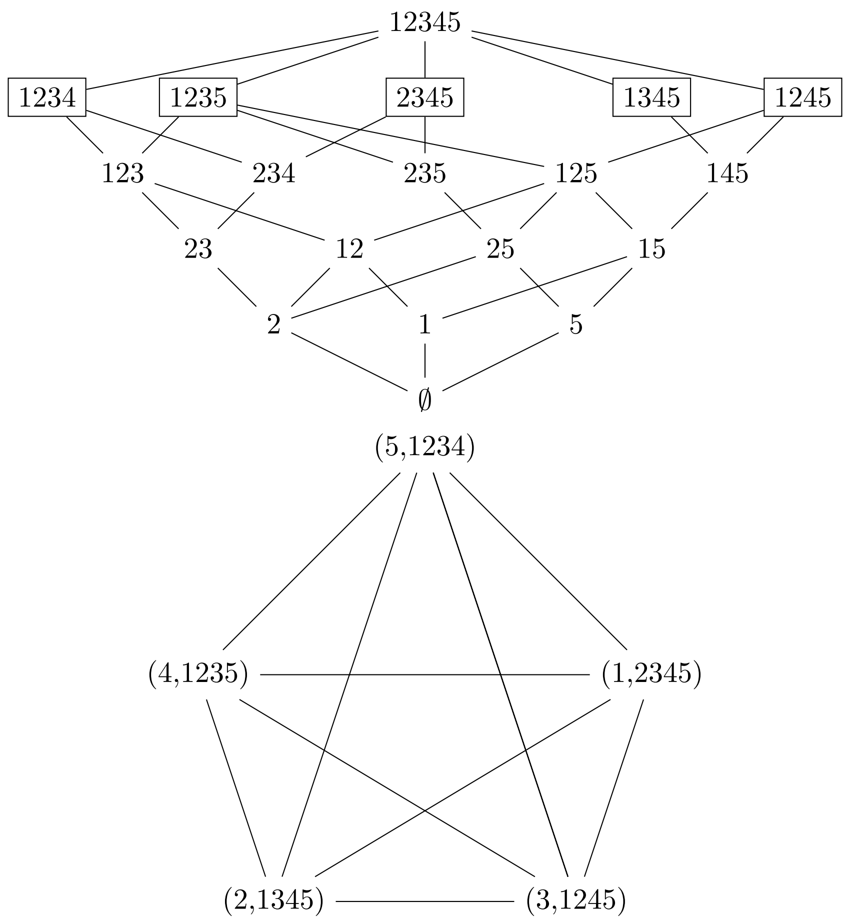


FIGURE 3. Hasse diagram of convex geometry and it's copoint graph (K_5)

where the copoints are {1} {2} {123} {124} {12345} {12346} {1234567} {1234568}. It is an alignment as the intersections of these sets are in $\mathscr{L}$ and $\mathscr{L}$ has the greedy property. The copoint graph for this convex geometry is a 4-ladder rung graph as shown below,

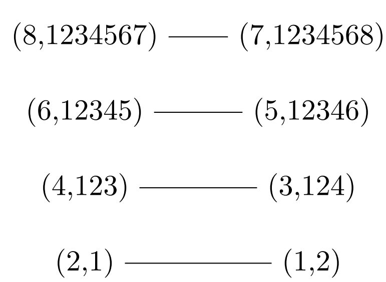


FIGURE 4. 4-ladder rung graph

Now, we assume $([n], \mathscr{L})$ is a convex geometry. The following is a convex geometry, ($[n]$, $\mathscr{L}$), that gives rise to a ladder rung graph.

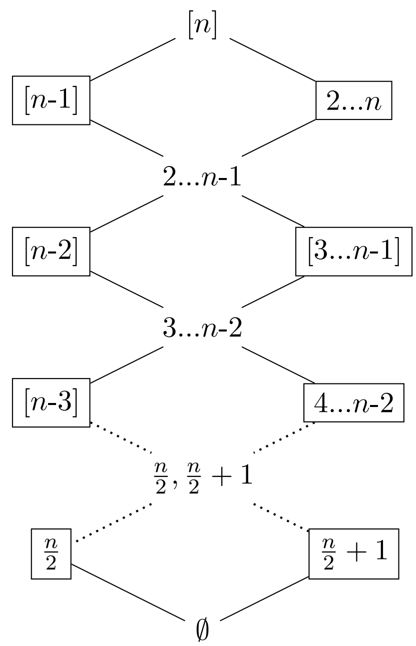


FIGURE 5. Hasse diagram of a convex geometry whose copoint graph is a ladder $|X| = n$

Note that all the intersections of sets in $\mathscr{L}$ are in $\mathscr{L}$, making $\mathscr{L}$ an alignment and all the sets have the greedy property. Notice that each copoint is contained in all other copoints except for one copoint thus all the copoints are adjacent to only one copoint. For example, copoint [n-1] is contained in all the copoints except for copoint 2...n, hence [n-1] is adjacent only to 2...n. By induction, $\mathscr{L}'$, our ground set becomes n+2. The new additional closed sets are

$[n-2] \cup\{n\} \cup \{n{+}1\}$, $[n-1] \cup[n-2], [\text{n}] \cup \{n{+}1\}$ and the new copoints are $[n-2] \cup\{n\} \cup \{n{+}1\}$ and $[n-1] \cup [n-2]$.

Note that $\mathscr{L}'$ is a convex geometry because all of the intersections of sets in $\mathscr{L}'$ are in $\mathscr{L}'$, making $\mathscr{L}'$ an alignment and all the sets have the greedy property. Now, we verify that the copoint graph, $\mathcal{G}(n+2, \mathscr{L}')$, is a ladder rung graph.

Let copoint A be $[n-2] \cup\{n\} \cup \{n{+}1\}$ and copoint B be $[n-1] \cup[n-2]$. Both A and B are on same row and have same size K. We know that A is not contained in B and convexly independent to only B. Thus, A and B are adjacent. Similarly, all the other copoints are not contained in only one copoint that is on same row and has same size. This shows that each copoint is adjacent to only one copoint. Thus, ($[n+2]$, $\mathscr{L}'$) is a convex geometry such that it's copoint graph is a ladder rung graph.

□

Theorem 3.4. *There does not exist a convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L})$ is the Petersen graph (See Fig. 6).*

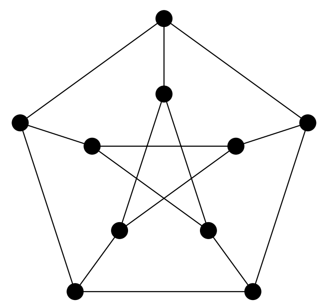


FIGURE 6. The Petersen Graph

Proof. To show that the Petersen graph cannot be a copoint graph, we will proceed by contradiction and assume there is a convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L})$ is the Petersen graph. Since the Petersen graph is connected and has no 3-cycles, we know that $(X, \mathscr{L})$ must have exactly 2 extreme copoints, A and B, that are adjacent. We also know that $A = X - \{\alpha(A)\}$ and $B = X - \{\alpha(B)\}$. Further, because any two adjacent points in a Petersen graph lie within a 5-cycle and the vertices of the Petersen graph can be arranged such that any 5-cycle is made into the exterior 5-cycle, without loss of generality we will assume that A and B lie on the exterior 5-cycle of the Petersen Graph.

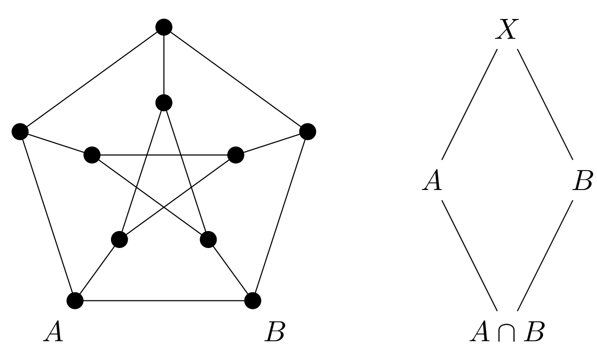


FIGURE 7. Extreme points of Petersen graph labeled with corresponding section of Hasse diagram

We need in our convex geometry a copoint adjacent to A and another copoint adjacent to B, lets call them C and D respectively. Because any 4-path in the Petersen graph lies in a 5-cycle of the Petersen graph, without loss of generality we can assume that C and D are on the exterior 5-cycle of our copoint graph. We also know that C is of the form $C = X - \{\alpha(B), \alpha(C)\}$ and D is of the form $D = X - \{\alpha(A), \alpha(D)\}$. This is because C must be adjacent to A and if C does not have B as a parent, then C will be contained in A and as a result not be adjacent to A. The same argument works for D. Further, from this we argument we know that $\alpha(C), \alpha(D) \neq \alpha(A), \alpha(B)$ and as a result, if $\alpha(C) \neq \alpha(D)$, then C and D would be adjacent in our copoint graph which is a contradiction. Thus we know that $\alpha(C) = \alpha(D)$ and we can extend our Hasse diagram as shown in Fig. 8.

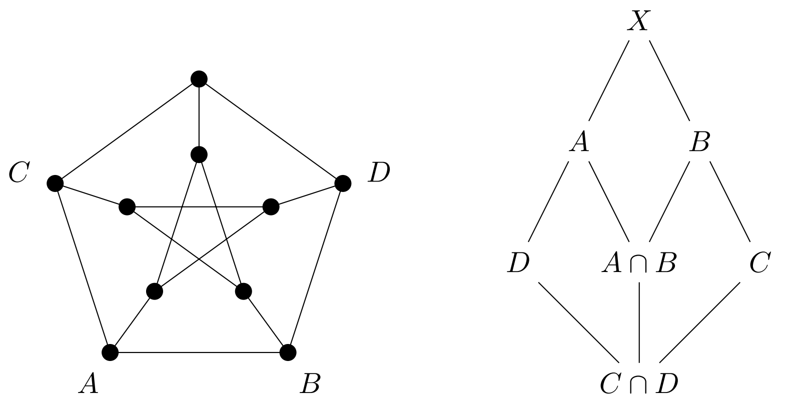


FIGURE 8. Labeled Petersen graph with corresponding section of Hasse diagram

Since we need a point adjacent to C and D that is not adjacent to A or B, we know that we need a copoint, we will call it E coming off of $A \cap B$. If the parent of E is A, then E would have to be adjacent to B and also to D. This would create a $3-$cycle in our graph which does not exist in the Petersen graph, and thus E cannot have A as a parent. The same argument also holds for B. If E has a parent that is C or D, E would not be adjacent to both C or D, and as a result we know that E must have $A \cap B$ as its parent. We then can see that E must be of the form $E = X - \{\alpha(A), \alpha(B), \alpha(E)\}$ where $\alpha(E) \neq \alpha(C)$. Thus we can continue the construction of our Hasse diagram as seen in Fig. 9.

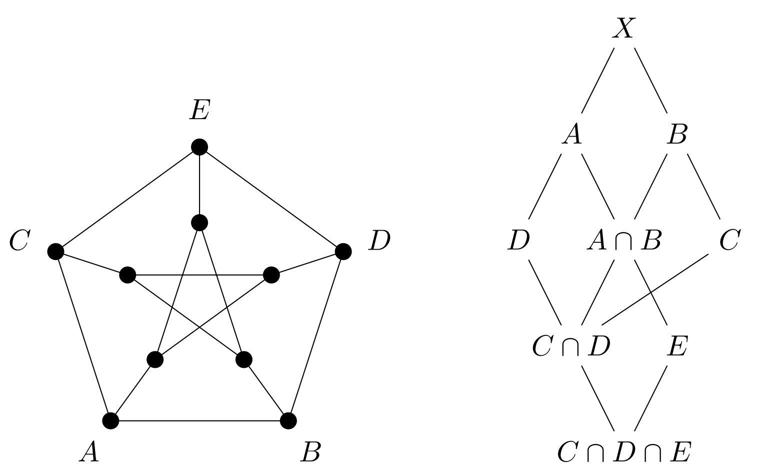


FIGURE 9. Labeled Petersen graph with corresponding section of Hasse diagram

We then need another copoint adjacent to A and another copoint adjacent to B, we will call these F and G respectively. Since they must be adjacent to A and B without being adjacent to C and D, we know that F must have C as its parent in the Hasse diagram and G must have D as its parent in the Hasse diagram. If F was not contained in C, F would have $A \cap B$, B, or E as a parent. If $A \cap B$ is the parent of F, then F would be adjacent to E which is a contradiction. Thus F must come from E or B. If B is the parent of F, then F would be adjacent to C which is a contradiction. This means that F must come from E. However, we know that $\alpha(A) \notin E$ so then F would also not contain $\alpha(A)$ and not be adjacent to A. As a result we know that F cannot come from $A \cap B$ or E so F must have C as its parent. A similar argument tells us that G has D as its parent. This then tells us that F is of the form $F = X - \{\alpha(B), \alpha(C), \alpha(F)\}$ and G is of the form $G = X - \{\alpha(A), \alpha(D), \alpha(G)\}$. If $\alpha(F) \neq \alpha(G)$, F and G are connected which is a contradiction. Thus $\alpha(F) = \alpha(G)$. From here we must work in cases, letting either $\alpha(F) = \alpha(E)$ or having $\alpha(F) \neq \alpha(E)$. In the case where $\alpha(F) = \alpha(E)$, we have a Hasse diagram of the form shown in Fig. 10.

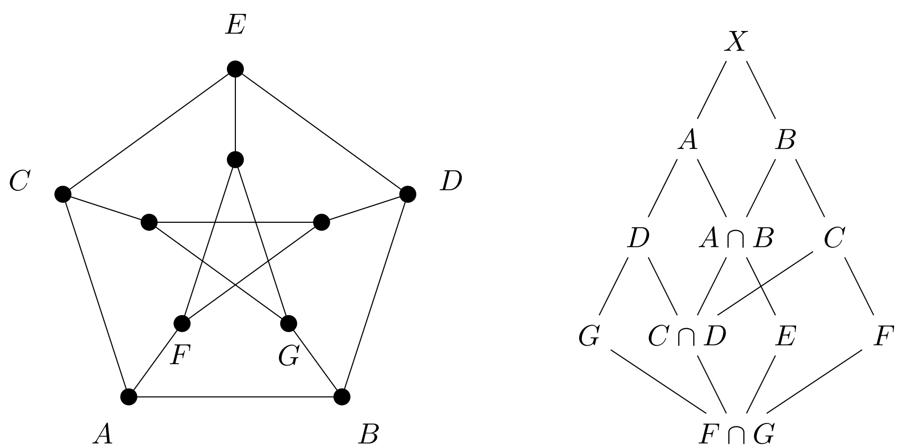
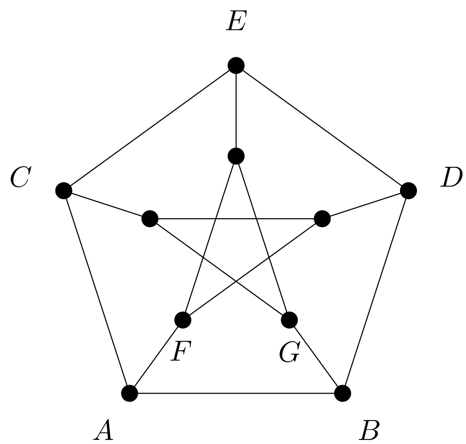


FIGURE 10. Labeled Petersen graph and corresponding Hasse diagram section when $\alpha(F) = \alpha(E)$

We will now look at adding the unlabeled point adjacent to C into our Hasse diagram. We will call this point H. Since H is adjacent to C and not adjacent to A and B, we know that $\alpha(C) \in H$ and $\alpha(A), \alpha(B) \notin H$. Since H must contain $\alpha(C)$, the parent of H must either be $A \cap B$ or E because E and $A \cap B$ are the only sets that do not contain $\alpha(A)$ and $\alpha(B)$. If the parent of H is $A \cap B$, H would be adjacent to E which is a contradiction. Thus the parent of H must be E and as a result $H \subseteq E$, and H must be of the form $H = X - \{\alpha(A), \alpha(B), \alpha(E), \alpha(H)\}$. Since $\alpha(G) = \alpha(E) \notin H$, this means H is not adjacent to G which is a contradiction and we have shown $\alpha(G) \neq \alpha(E)$. In the case where $\alpha(F) \neq \alpha(E)$, we have a Hasse diagram of the form shown in Fig. 11.



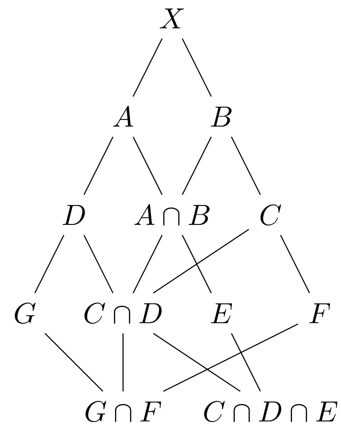


FIGURE 11. Labeled Petersen graph and corresponding Hasse diagram section when $\alpha(F) \neq \alpha(E)$

As before, we will be looking at the point H that is adjacent to C and where it needs to go in our Hasse diagram. By a similar argument in the last case, we know that the parent of H must either be $A \cap B$ or E. If the parent of H is $A \cap B$, then H is adjacent to E which is a contradiction. Thus the parent of H must be E and we get that H is of the form $H = X - \{\alpha(A), \alpha(B), \alpha(E), \alpha(H)\}$. In addition to the fact that $\alpha(H) \neq \alpha(A), \alpha(B)$, we know that because H is adjacent to C, $\alpha(H) \neq \alpha(C)$. Since $\alpha(C) \in H$, we know that $\alpha(D) \in H$. Also, because $\alpha(H) \neq \alpha(A), \alpha(D)$, we know that $\alpha(H) \in D$ and thus D is connected to H. However, this is a contradiction because H is not adjacent to D. Thus we have shown that $\alpha(E) \neq \alpha(G)$ is impossible, that $\alpha(E) = \alpha(G)$ is impossible, and, as a result, that our initial assumption is false and there exists no convex geometry with a copoint graph that is the Petersen graph.

□

4. Relation of copoint graphs to their complements

We show that even if G is a copoint graph of some convex geometry, it may not be true that $\overline{G}$ is the copoint graph of a convex geometry. To justify this, consider the following theorem in conjunction with Theorem 1.1.

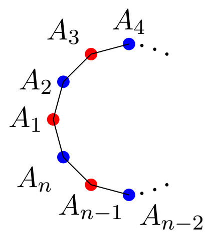


Figure 12. Labeled C_n for n even

Theorem 4.1. *There exists a convex geometry* $(X, \mathscr{L})$ *with* $\mathcal{G}(X, \mathscr{L}) = \overline{C_n}$

Proof. Ciruli et. al [CEK] demonstrated that all graphs on 5 or fewer vertices are realizable as copoint graphs of some convex geometry. Thus, we omit discussion of $\overline{C_n}$ with $n \leq 5$. Suppose first that $n \geq 6$ is even. Let $X = [n]$ and define the sets $B_{2i-1} := X - \{2i - 1\}$ for $1 \leq i \leq \frac{n}{2}$. Here we index modulo n. Note that there are $\binom{n/2}{2} \geq n/2$ sets of the form $B_{2k-1} \cap B_{2j-1}$ for $k \neq j$. Now define the sets $B_{2i} := B_{2i-1} \cap B_{2i+1} - \{2i\}$ for $1 \leq i \leq \frac{n}{2}$. Consider the collection $\mathscr{L} := \{X, \bigcap_{k \in Y} B_k : Y \subset [n]\}$.

First we have to demonstrate that $(X, \mathscr{L})$ is indeed a convex geometry. Note that $\mathscr{L}$ has the alignment property by definition. Furthermore, $\emptyset = \bigcap_{i=1}^{n} B_i \in \mathscr{L}$. We just need to see that $\mathscr{L}$ has the greedy property. Suppose that $C \in \mathscr{L} - X$. We must find a set $K = C \cup p \in \mathscr{L}$ for some $p \in X$. Suppose first that $C = \emptyset$. We can write

$$C \cup \{2k\} = \{2k\} = (\bigcap_{j=1}^{n/2} B_{2j-1}) \cap (\bigcap_{j=1}^{n/2-1} B_{2j}) = \bigcap_{j \in [X]-2k} B_j \in \mathscr{L}.$$

Now suppose that

$$C = \bigcap_{i \in Y \subset [n]} B_i$$

for some $Y \neq \emptyset$. Thus

$$C = X - \left(\bigcup_{\substack{\text{i odd} \\ i \in Y}} \{i\} \;\cup\; \bigcup_{\substack{\text{j even} \\ j \in Y}} \{j, j-1, j+1\} \right).$$

Note that we can not have the set of odds in Y empty since every set except X in $\mathscr{L}$ is contained in some B_{2i-1}. Suppose that the set of evens in Y is empty. If Y is a singleton, say $Y = \{i\}$ for an odd $i \in [n]$, then $C = B_i$ and clearly $C \cup i = B_i \cup i = X \in \mathscr{L}$. Now suppose that Y is not a singleton and pick $p \in Y$ arbitrarily. Then

$$C \cup p = (X - \bigcup_{i \in Y} \{i\}) \cup p = X - \bigcup_{i \in Y - p} \{i\} = \bigcap_{i \in Y - p} B_i \in \mathscr{L}.$$

Now we consider the case when the set of evens in Y is non-empty and $p \in Y$ is some even number. Now consider $C \cup p$. We have

$$C \cup p = X - \left(\bigcup_{\substack{\text{i odd} \\ i \in Y \cup \{p-1, p+1\}}} \{i\} \ \cup \bigcup_{\substack{\text{j even} \\ j \in Y - p}} \{j, j-1, j+1\} \right) = \bigcap_{j \in (Y-p) \cup \{p+1, p-1\}} B_j \in \mathscr{L}.$$

We have shown that $\mathscr{L}$ has the greedy property. Thus since $\mathscr{L}$ has both the alignment and greedy properties, $(X, \mathscr{L})$ is a convex geometry.

Consider the poset $(\mathscr{L}, \subset)$ and let $1 \leq k \leq n$ be given. Note that if k is odd, then $B_k = X - k$ is maximal in $X - k$, so B_k is a copoint of $(X, \mathscr{L})$. Now suppose that k is even. Suppose for the sake of contradiction that there were some closed set $C \subset X - k$ such that $B_k \subset C$. Note that $B_k \in R_3$. Then either $C \in R_1$ or $C \in R_2$. Each set in R_1 is $[n] - j$, for some odd j, while each set in R_2 is $[n] - \{j, \ell\}$ for odd j, ℓ. Both $[n] - j, [n] - \{j, \ell\} \not\subset X - k$ since $k \in [n]$ is even. This is a contradiction to the hypothesis that $B \subset X - k$. Thus B_k is maximal in $X - k$. Note that any set in R_j with $j \geq 4$ is the intersection of some B_k's. Thus no set in $\mathscr{L} - R_1 \cup R_3$ can be a copoint of $(X, \mathscr{L})$. It follows that $M(X, \mathscr{L}) = \{B_k : 1 \leq k \leq n\}$.

We claim that $\mathcal{G}(X, \mathscr{L}) = \overline{C_n}$ up to a graph isomorphism.

Consider the map $\phi : V(\mathcal{G}(X, \mathscr{L})) \to V(\overline{C_n})$ defined by $\phi(B_a) = A_a$ for each $1 \leq a \leq n$ where the A_a are as in Fig. 12. We claim that ϕ is a graph isomorphism. ϕ is clearly a bijection since each $a \in X$ corresponds to exactly one copoint B_a and each $a \in X$ also corresponds to exactly one $A_a \in V(\overline{C_n})$.

Now suppose that $\{B_k, B_\ell\} \notin E(\mathcal{G}(X, \mathscr{L}))$ for some copoints B_k, B_ℓ, $1 \leq k, \ell \leq n$. Now we can either have $B_k \in R_1$ or $B_k \in R_3$. Suppose $B_k \in R_1$. Then since $\alpha(B_k) = k$, we must have that $k = 2i-1$ for some $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$. Note that by hypothesis $B_\ell \notin R_1$, else $\{B_k, B_\ell\} \in E(\mathcal{G}(X, \mathscr{L}))$. Thus $B_\ell \in R_3$. Note that $\alpha(B_\ell) \in B$ since $B_k = X - \{2i-1\}$ and $\alpha(B_\ell)$ is even. Thus, we must have that $2i - 1 = \alpha(B_k) \notin B_\ell$. We see then that either $\ell = 2i$ or $2(i-1)$. Thus $\{\phi(B_k), \phi(B_\ell)\} = \{A_k, A_\ell\} = \{A_{2i-1}, A_{2i}$ or $\{A_{2i-1}, A_{2(i-1)}\}$. Note that in either case we have $\{A_{2i-1}, A_{2i}\} \in E(C_n)$ and $\{A_{2i-1}, A_{2(i-1)}\} \in E(C_n)$, so both are not edges in $\overline{C_n}$. The argument is symmetric if $B_k \in R_3$ since then $B_\ell \in R_1$. We have demonstrated that for any $1 \leq k, \ell \leq n$

$$\{\phi(B_k), \phi(B_\ell)\} \in E(\overline{C_n}) \implies \{B_k, B_\ell\} \in E(\mathcal{G}(X, \mathscr{L}))$$

Now suppose that for some copoints B_k, B_ℓ it was true that $\{\phi(B_k), \phi(B_\ell)\} \notin E(\overline{C_n})$. We wish to show in all cases that $\{B_k, B_\ell\} \notin \mathcal{G}(X, \mathscr{L})$. By hypothesis, we must have either that $\phi(B_k) = \phi(B_\ell)$ in which case $B_k = B_\ell$, and $\{B_k, B_\ell\} \notin E(\mathcal{G}(X, \mathscr{L}))$; Or, alternatively $\{\phi(B_k), \phi(B_\ell)\} \in E(C_n)$. In the latter case, $\ell = k \pm 1$. Suppose that k is odd. Then $k \pm 1$ are even, so $\alpha(B_\ell) = k \pm 1$ is even and by definition $B_\ell \subset B_k$. Thus,

$\{B_k, B_\ell\} \notin E(\mathcal{G}(X, \mathscr{L}))$. If k is even, then $B_k \subset B_\ell$ so $\{B_k, B_\ell\} \notin E(\mathcal{G}(X, \mathscr{L}))$.

Thus we have that for all $B_k, B_\ell \in V(\mathcal{G}(X, \mathscr{L}))$

$$\{B_k, B_\ell\} \in E(\mathcal{G}(X, \mathscr{L})) \iff \{\phi(B_k), \phi(B_\ell)\} \in E(\overline{C_n}).$$

Now we suppose that $n \geq 7$ is odd. Thus we can write $n = 2k+1$ for some $k \geq 3$. We can use the following colouring scheme for C_n.

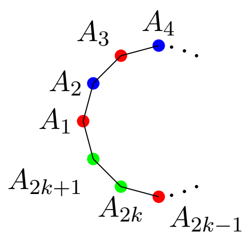


FIGURE 13. Labeled C_n for $n = 2k+1$ odd

Now set $X = [n-1] = [2k]$ and let $B_{2i-1} = X - \{2i-1\}$ for $1 \leq i \leq k$. There are $\binom{k}{2}$ sets of the form $B_{2j-1} \cap B_{2\ell-1}$ for $j \neq \ell$. Let $B_{2i} = B_{2i-1} \cap B_{2i+1} - \{2i\}$ for $1 \leq i \leq k-1$. Now let $B_{2k+1} = B_1 - \{2k\}$ and $B_{2k} = B_{2k-1} - \{2k\}$. Consider the set $\mathscr{L} := \{X, \bigcap_{j \in Y} B_j : Y \subset X \cup \{2k+1\}\}$. Note that for each $j \in [2k]$, $j \notin B_j$. Thus

$$\emptyset = \bigcap_{j \in [2k]} B_j \in \mathscr{L}.$$

$\mathscr{L}$ has the alignment property by construction. We would like to see that $\mathscr{L}$ has the greedy property. Let $C \in \mathscr{L} - X$ be given. We must show $C \cup \{p\} \in \mathscr{L}$ for some $p \in X$. If $C = \emptyset$, then

$$C \cup \{2k\} = \{2k\} = (\bigcap_{j=1}^{k} B_{2j-1}) \cap (\bigcap_{j=1}^{k-1} B_{2j}) = \bigcap_{j \in [2k]-\{2k\}} B_j \in \mathscr{L}.$$

Now suppose that

$$C = \bigcap_{j \in Y} B_j,$$

for some nonempty $Y \subset X \cup \{2k+1\}$. If $2k+1, 2k \notin Y$, then the argument to see that $C \cup p \in \mathscr{L}$ for some p, is the same as we presented for the cycles of even length. Now suppose without loss of generality that $2k \in Y$ and $2k+1 \notin Y$. Then,

$$C = (\bigcap_{j \in Y - \{2k\}} B_j) \cap B_{2k},$$

while picking $p = 2k$ gives

$$C \cup \{2k\} = (\bigcap_{j \in Y-\{2k\}} B_j) \cap (B_{2k} \cup \{2k\}) = \bigcap_{j \in (Y-\{2k\}) \cup \{2k-1\}} B_j \in \mathscr{L}.$$

If we had both $2k, 2k+1 \in Y$ we can again pick $p = 2k$, and

$$C \cup \{2k\} = \bigcap_{j \in (Y - \{2k, 2k+1\}) \cup \{1, 2k-1\}} B_j \in \mathscr{L}.$$

We see that $\mathscr{L}$ has the greedy property. Thus $(X, \mathscr{L})$ is a convex geometry. Every closed set is either in the sub-collection $\{B_j : 1 \leq j \leq 2k+1\}$, identically X, or is an intersection of these sets. Sets in the latter two cases can not be copoints. When $1 \leq j \leq 2k-1$ is odd, $B_j = X - \{j\}$ and this is clearly maximal in $X - j$. When $2 \leq j \leq 2k-2$ is even, $B_j = X - \{j-1, j, j+1\}$. Each set in R_i when $i = 1, 2, 3$ is either not contained in $X - j$ or is identically B_j. Every set in R_i with $i \geq 4$ is either not in $X - j$ or has B_j as an upper bound. Thus B_j is maximal in $X - j$. Similar reasoning applies to show that B_{2k}, B_{2k+1} are upper bounds of $X - 2k$ and are thus maximal in this set. Thus, $M(X, \mathscr{L}) = \{B_j : 1 \leq j \leq 2k+1\}$ and $\alpha(B_j) = j$ for all j.

We try to see that $\mathcal{G}(X, \mathscr{L}) = \overline{C_n}$ and use the labelling scheme in Fig. 13. There is a bijective correspondence ϕ between the A_j in Fig. 13 and the $B_j \in M(X, \mathscr{L})$ that sends B_j to A_j. Note that the range of $\phi|_{R_1 \cap M(X, \mathscr{L})}$ is given by the red vertices, the range of $\phi|_{R_2 \cap M(X, \mathscr{L})}$ is given by the green vertices, and the range of $\phi|_{R_3 \cap M(X, \mathscr{L})}$ given by the blue vertices. In $\overline{C_n}$, each red vertex is adjacent to every other red vertex and all blue vertices except for its neighbours in C_n. Furthermore, each blue vertex is adjacent to each other blue vertex and all red vertices except for its neighbours in C_n. Finally, each green vertex in $\overline{C_n}$ is adjacent to all blue vertices, all red vertices except its neighbours in C_n, and is not adjacent to the other green vertex. The goal is to demonstrate that ϕ^{-1} respects these adjacencies.

Suppose that $\{A_j, A_i\} \notin E(\overline{C_n})$ for some $i, j \in [2k+1]$. If we had that $j = i$, then since $\phi^{-1}(A_i) = B_i$, $\{\phi^{-1}(A_j), \phi^{-1}(A_i)\} = \{B_i, B_i\} \notin E(\mathcal{G}(X, \mathscr{L}))$. Otherwise, we must have $A_j = A_{i+1}$ or $A_j = A_{i-1}$. Suppose that $1 \leq i \leq 2k-1$ is odd. Then by construction, both $B_{i-1}, B_{i+1} \subset B_i$ so B_i is not adjacent to either B_{i-1} or B_{i+1}. Similarly, if $1 \leq i \leq 2k-1$ is even, then $B_i \subset B_{i-1} \cap B_{i+1}$ so B_i is not adjacent to either B_{i-1} or B_{i+1}. Now suppose that $i = 2k$. Note that $\alpha(B_{2k+1}) = \alpha(B_{2k}) \notin B_{2k}$. Furthermore, $B_{2k} \subset B_{2k-1}$ so B_{2k} is not adjacent to either B_{2k-1} or B_{2k+1}. The argument if $i = 2k+1$ is similar. We have demonstrated the implication

$$\{A_i, A_j\} \notin E(\overline{C_n}) \implies \{\phi^{-1}(A_i), \phi^{-1}(A_j)\} \notin E(\mathcal{G}(X, \mathscr{L})).$$

Now we suppose that for some $A_i, A_j \in V(\overline{C_n})$, we had $\{\phi^{-1}(A_i), \phi^{-1}(A_j)\} \notin E(\mathcal{G}(X, \mathscr{L}))$. Note that we can not have both $\phi^{-1}(A_i), \phi^{-1}(A_j) \in R_1$ or both $\phi^{-1}(A_i), \phi^{-1}(A_j) \in R_3$. Suppose without loss of generality that $\phi^{-1}(A_i) = B_i \in R_1$. Then, there are two cases for $\phi^{-1}(A_j) = B_j$. Suppose $B_j \in R_2$, then either $B_j = B_{2k}$ or B_{2k+1}. Note that B_{2k} and B_{2k+1} are adjacent to each copoint in r_1 except for B_{2k-1} and B_1 respectively. Thus, if $B_j = B_{2k}$ then we must have had $B_i = B_{2k-1}$. But then $A_j = A_{2k}$ and $A_i = A_{2k-1}$ which are not adjacent in $\overline{C_n}$. If $B_j = B_{2k+1}$ then $B_i = B_1$. But then $A_j = A_{2k+1}$ and $A_i = A_1$, which are not adjacent in $\overline{C_n}$.

Now suppose that $B_j \in R_3$. Note that B_j is adjacent to every copoint in R_1 except for B_{j-1}, B_{j+1}. Thus we must have had $i = j-1$ or $i = j+1$. In either case we have that

A_i and A_j are not adjacent in $\overline{C_n}$.

The case if $B_i \in R_2$ and $B_j \in R_1, R_3$ is symmetric to what we just showed. Thus, there is only one case to check, which is if $B_i, B_j \in R_2$. Now we could have that $i = j$, in which case A_i would not be adjacent to A_j in $\overline{C_n}$. Otherwise, without loss of generality $B_i = B_{2k}$ and $B_j = B_{2k+1}$. Thus $A_i = A_{2k}$ and $A_j = A_{2k+1}$. Then, A_i and A_j are not adjacent in $\overline{C_n}$. We have demonstrated the implication

$$\{A_i, A_j\} \notin E(\overline{C_n}) \impliedby \{\phi^{-1}(A_i), \phi^{-1}(A_j)\} \notin E(\mathcal{G}(X, \mathscr{L})).$$

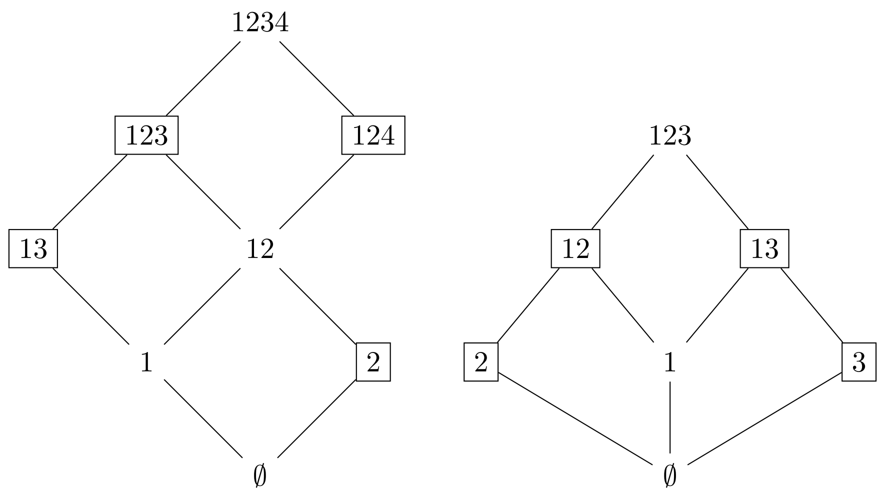
Taking both implications together we have for all $A_i, A_j \in \overline{C_n}$,

$$\{A_i, A_j\} \in E(\overline{C_n}) \iff \{\phi^{-1}(A_i), \phi^{-1}(A_j)\} \in E(\mathcal{G}(X, \mathscr{L})).$$

Therefore ϕ^{-1} is a graph isomorphism from $\overline{C}_n$ to $\mathcal{G}(X, \mathscr{L})$ and it follows that ϕ is a graph isomorphism from $\mathcal{G}(X, \mathscr{L})$ to $\overline{C}_n$. $\square$

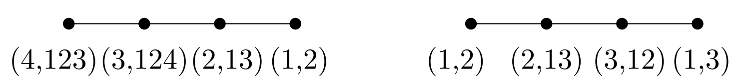
5. Uniqueness of the Hasse structure for paths arising as copoint graphs

Let P_n be the path on n vertices oriented parallel to the x axis in the plane. Label the points traversed from left to right by $n, n-1, ..., 1$. The technical report from Ciruli et. al [CEK] shows that for the convex geometry $([n], \mathscr{L}_p)$ with co-points $M([n], \mathscr{L}_p) = \{[n-1], [n-2] \cup \{n\}, [n-3] \cup \{n\}, [n-i] \cup \{n-(i-2)\}$ for $4 \leq i \leq n-1$, we have $\mathcal{G}([n], \mathscr{L}_p) = P_n$. Furthermore, as we traverse this copoint graph from left to right, the sequence of point attachments coincides with the above labeling. We would like to show that up to permutation of labels, this is the only way to obtain the path as a co-point graph. Note that this is not necessarily true in the case $n = 4$ as we see in Fig. 14.



(A) $(\{1,2,3,4\}, \mathscr{L}_1)$ (B) $(\{1,2,3\}, \mathscr{L}_2)$

Figure 14. Hasse diagrams of distinct convex geometries giving rise to P_4.



(A) $\mathcal{G}(\{1,2,3,4\}, \mathscr{L}_1)$ (B) $\mathcal{G}(\{1,2,3\}, \mathscr{L}_2)$

Figure 15. End-point vs. non-end point behaviour of extreme copoints.

Lemma 5.1. *If $(X, \mathscr{L})$ is a convex geometry such that $\mathcal{G}(X, \mathscr{L}) = P_n$ and $n \geq 5$, then one of the endpoints in P_n is an extreme copoint in $\mathscr{L}$.*

Proof. No ground set X with $|X| = 3, 4$ can have $\mathcal{G}(X, \mathscr{L}) = P_n$ if $n \geq 5$. Suppose for the sake of a contradiction that neither end-point in P_n is a copoint attached to an extreme point. Note that since P_n is connected and contains no cycles, there are exactly two co-points A and B attached to extreme points $\alpha(A)$ and $\alpha(B)$. Furthermore, $\{A, B\} \in E(P_n)$.

Suppose we did not have a copoint of size $|X| - 2$ in $\mathscr{L}$. Then $|R_2| = 1$. Note that the other $n-2 \geq 3$ copoints in $\mathscr{L}$, are contained in $\bigcup_{j=3}^{|X|-1} \subset A \cap B$. These copoints can not contain $\alpha(A)$ or $\alpha(B)$, and so A, B are not adjacent to any of the other $n-2$ copoints. Then $\mathcal{G}(X, \mathscr{L})$ is disconnected, but this is a contradiction since we said $\mathcal{G}(X, \mathscr{L}) = P_n$.

Note that we can not have two copoints C and D each contained inside one of A or B. If we did, supposing without loss of generality that $C, D \subset A$, then both $\{C, B\} \in E(P_n)$ and $\{D, B\} \in E(P_n)$ and thus B has degree three, a contradiction since $\mathcal{G}(X, \mathscr{L}) = P_n$.

Furthermore, if we had two copoints in R_2, then they must be attached to the same point, or else we get a 4-cycle.

Suppose first that $|R_2| \geq 3$. Thus, we must have had two copoints in R_2. For some $y \neq \alpha(A), \alpha(B)$ the following are copoints in $\mathscr{L}$,

$$A = X - \{\alpha(A)\},$$
$$B = X - \{\alpha(B)\},$$
$$C = X - \{y, \alpha(B)\},$$
$$D = X - \{y, \alpha(A)\}.$$

Now since $n > 4$, we must have a copoint in R_3 by the same connectedness argument earlier. Call this copoint E. We can't have $E \subset C$ or $E \subset D$, since in the former we have $\{E, A\} \in E(P_n)$ and in the latter we have $\{E, B\} \in E(P_n)$. In either case we end up with a vertex of degree three, a contradiction. Thus we must have

$$E \subset A \cap B = X - \{\alpha(A), \alpha(B)\}.$$

Thus,

$$E = X - \{x, \alpha(A), \alpha(B)\}$$

for some x. We can not have $x = y$ since then $E = C \cap D$ and E would not be a copoint. Thus $x \neq y$ and we have both $\{E, C\}, \{E, D\} \in E(P_n)$ which creates a cycle. This is a contradiction. Thus we can not have two copoints in R_2

Thus $|R_2| = 2$ and there is a copoint $C \in R_2$. Without loss of generality suppose that $C \subset B$. Then $\{A, C\} \in E(P_n)$ and since we have supposed that B is not an endpoint, there must be a co-point $D \in R_3$ such that $\alpha(B) \in D$. But this can not be possible since both $\alpha(B) \notin C$ and $\alpha(B) \notin A \cap B$. Thus we have a contradiction to the assumption that no extreme copoint was an endpoint of P_n.
□

Theorem 5.2. *Suppose there exists a convex geometry* $([m], \mathscr{L})$ *such that* $\mathcal{G}([m], \mathscr{L}) = P_n$ *with* $n \geq 5$. *Then* $m = n$ *and there exists a permutation* $\sigma : [n] \to [n]$ *such that* $([n], \mathscr{L}_p) = (\sigma([n]), \sigma(\mathscr{L}))$ *where* $\sigma(\mathscr{L}) := \{\sigma(C) : C \in \mathscr{L}\}$.

Proof. Suppose the hypotheses of the theorem. There are two co-points $A, B \in \mathscr{L}$ attached to extreme points $\alpha(A), \alpha(B) \in [m]$. By Lemma 5.1, one of these copoints, say A, must be an end-point. Note that $\{A, B\} \in E(P_n)$. Furthermore, $A \cap B = [m] - \{\alpha(A), \alpha(B)\} \in R_2$.

We try to see that there is exactly one copoint in R_2 and that furthermore, this copoint is contained in A. By connectivity and the fact that $n > 2$ we must get a copoint of size $m-2$. Let C_2 be this copoint. Suppose that $C_2 \subset B$. Further, $\alpha(A) \neq \alpha(C_2)$ because otherwise $C_2 = A \cap B$ which is not a copoint. Thus, $\alpha(C_2) \in A$ and also $\alpha(A) \in C_2$. Therefore $\{C_2, A\} \in E(P_n)$, but this is a contradiction, since we said that A was an endpoint. Therefore, we must have $C_2 \subset A$ as desired. Furthermore, $\alpha(B) \in C_2$ and $\alpha(C_2) \in B$ by the previous argument.

Now we try to show that there is only one copoint in R_2. Suppose $D_2 \neq C_2$ is a copoint in R_2. We already saw that we can not have $D_2 \subset B$ since this contradicts that A is an endpoint. Thus, $D_2 \subset A$ and so $\alpha(D_2) \neq \alpha(C_2)$. We see that $\{C_2, D_2\} \in E(P_n)$, but also $\alpha(D_2) \in B$, $\alpha(B) \in D_2$, so $\{D_2, B\} \in E(P_n)$, but this creates a 3-cycle, which is a contradiction, so such a D_2 could not have existed.

There is a copoint in R_3 by the same connectedness argument that we have been using. Call this copoint C_3. We show that we must have $C_3 \subset A \cap B$. Suppose $C_3 \subset C_2$. Then $\alpha(B) \in C_3$ and $\alpha(C_3) \in B$, so $\{C_3, B\} \in E(P_n)$. Thus, B has degree 3, a contradiction. We must have had $C_3 \subset A \cap B$. It follows that C_3 has point attachment unique from $\alpha(A), \alpha(B), \alpha(C_2)$ and furthermore that $\{C_2, C_3\} \in E(P_n)$. Now if we had an additional copoint, say D_3 in R_3, it also must be contained in $A \cap B$ and thus it would follow that we have the 3-cycle C_2, C_3, D_3, C_2, which is a contradiction.

This argument can be continued inductively. If we have a copoint in some R_ℓ, we call it C_ℓ in keeping with our notation from before. Suppose that for some $3 \leq \ell \leq m-2$ we have that for all $2 \leq k \leq \ell$, there is exactly one copoint C_k and one other closed set in R_k. This closed set is given by $A \cap B$ in the case of R_2 and $A \cap B \cap C_2 \cap C_3 \cap ... \cap C_{k-1}$ in the case of R_k when $k > 2$. We have only guaranteed $\ell + 1 < m \leq n$ copoints, and thus by connectivity, there must be a copoint $C_{\ell+1} \in R_{\ell+1}$. By the arguments given to go from R_2 to R_3 we know that this copoint is unique in $R_{\ell+1}$ and contained in $A \cap B \cap C_2 \cap C_3 \cap ... \cap C_\ell$. It follows that $\{C_{\ell+1}, C_\ell\} \in E(P_n)$. Thus by strong induction we have that for all $\ell = 3, 4..., m-1$: $|R_\ell| = 2$; there is exactly one copoint in R_ℓ; this copoint is contained in the non-copoint of $R_{\ell-1}$. Furthermore, we see that the empty set can not be a copoint because there are two sets in R_{m-1}. We have $m+1$ rows in total. There are no copoints in R_0 or R_m and 2 copoints in R_1. Now since we must have n copoints in total, we have that $m = n$.

Thus since $[m] = [n]$ we may relabel each $\alpha(C_k) \in [m]$ using a permutation. Put $\sigma(\alpha(A)) = n$, $\sigma(\alpha(B)) = n-1$ and $\sigma(\alpha(C_\ell)) = n - \ell$ for $2 \leq \ell \leq n-1$. We see that $([n], \mathscr{L}_p) = (\sigma([n]), \sigma(\mathscr{L}))$ as desired. $\square$

6. Hasse diagram structure of trees as copoint graphs

Let $(X, \mathscr{L})$ be a convex geometry. Then $R_k := \{C \in \mathscr{L} : |C| = |X| - k\}$ as before and define γ_k to be the number of copoints in each R_k. Note that by the greedy property of convex geometries, R_k is nonempty for all $0 \leq k \leq |X|$. Further, let $V_k := \{A \in R_j : \text{A is a copoint}, 1 \leq j \leq k\}$ and $E_k := \{\{A, B\} : A, B \in V_k \text{ and } \{A, B\} \in E(\mathcal{G}(X, \mathscr{L})\}$ and then define $\mathcal{G}_k(X, \mathscr{L}) := (V_k, E_k)$.

Theorem 6.1. *Suppose that $(X, \mathscr{L})$ is a convex geometry with $\mathcal{G}(X, \mathscr{L})$ a tree. For every $2 \leq k \leq |X| - 1$,*

(1) $2 \leq |R_k| \leq 3$;
(2) $1 \leq \gamma_k \leq 2$; *Furthermore, if $\gamma_k = 2$ then the two copoints are attached to identical points;*
(3) *For all distinct $C, D \in R_k$, $|C \Delta D| = 2$, where Δ denotes the symmetric difference set operation. Furthermore, when $|R_k| = 3$, for all distinct $C, D, H \in R_k$, we have $C \cap D = C \cap H = D \cap H = C \cap D \cap H$.*
(4) *The subgraph $\mathcal{G}_k(X, \mathscr{L}) \subset \mathcal{G}(X, \mathscr{L}))$ is a tree.*

Proof. We have shown that the claims hold in the case that $k = 2$ in the start of the proof of Lemma 5.1. The assumption that any of the claims 1) through 4) did not hold in R_k gave rise to a cycle. Now if $|X| = 3$ then R_3 contains only the empty set, so we suppose $|X| \geq 4$. Proceeding by induction, we suppose that for some $|X| - 2 \geq n \geq 4$ we have seen that all of 1) through 4) hold for each r_k with $4 \leq k \leq n$. We will show that each of 1) through 4) holds for R_{n+1}. We suppose here that $n \neq |X| - 1$ because then the induction hypothesis would be equivalent to the statement of the lemma. If we can show this implication, then by strong induction we will have that the lemma holds.

Case 1) Suppose that $\gamma_n = 1$ and that C_n is a copoint in R_n. By hypothesis, there is a unique non-copoint $I_n \in R_n$. Note that we can not have $\gamma_{n+1} = 3$ and have 3 copoints contained in either of I_n or C_n because then we have a 3 cycle and there would be a contradiction. Furthermore, $I_n = (C_n - y) \sqcup \alpha(C_n)$ by the hypothesis that $|C_n \Delta I_n| = 2$. Thus $C_n \cap I_n = C_n - y = I_n - \alpha(C_n)$. Now suppose that we had 3 copoints $C_{n+1}, C'_{n+1}, C''_{n+1}$ in R_{n+1}. We seek to contradict this statement in all possible cases.

Suppose without loss of generality that $C_{n+1}, C'_{n+1} \subset C_n$ and $C''_{n+1} \subset I_n$. Note that by hypothesis 4) there is a copoint $H \in \bigcup_{j=1}^{n-1} R_j$ such that H and C_n are adjacent. Furthermore, I_n is contained in all sets in this union. Thus $y = \alpha(H)$ and we could not have had $\alpha(C_{n+1}), \alpha(C_{n'+1}) = y$ or else neither would be a copoint. Thus $\alpha(H) \in C_{n+1}, C'_{n+1}$. Furthermore, $\alpha(C_{n+1}), \alpha(C'_{n+1}) \in I_n \subset H$. Thus we have the edges $\{C_{n+1}, C'_{n+1}\}, \{C'_{n+1}, H\}$, and $\{H, C_{n+1}\}$ which are a 3 cycle. This contradicts that $\mathcal{G}(X, \mathscr{L})$ is a tree.

The case with $C_{n+1}, C'_{n+1} \subset I_n$ is similar, but instead we get the 3 cycle $\{C_{n+1}, C'_{n+1}\}$, $\{C'_{n+1}, C_n\}, \{C_n, C_{n+1}\}$, which is again a contradiction. We see that we could not have had 3 or more copoints in R_{n+1}.

Now we try to see that if we had two copoints $C_{n+1}, C'_{n+1} \in R_{n+1}$, then $\alpha(C_{n+1}) = \alpha(C'_{n+1})$. Suppose otherwise for the sake of a contradiction. Note that $\alpha(C_{n+1}), \alpha(C'_{n+1}) \in$

$C_n \cap I_n$. Suppose without loss of generality that $C_{n+1} \subset C_n$ and $C'_{n+1} \subset I_n$. By hypothesis 4) again there exists an $H \in \bigcup_{j=1}^{n-1} R_j$ such that H and C_n are adjacent. For such an H we have $\alpha(C_{n+1}) \in C_n \cap I_n \subset H$, and $\alpha(H) \in C_{n+1}$. Thus we have the cycle $\{C'_{n+1}, C_n\}, \{C_n, H\}, \{H, C_{n+1}\}, \{C_{n+1}, C'_{n+1}\}$. This is a contradiction. We conclude that 2) holds in R_{n+1}.

Now that we have verified 2), one can verify that 3) holds in R_{n+1} in either the case with $\gamma_{n+1} = 1$ or $\gamma_{n+1} = 2$ and both copoints attached to the same point. Finally, for the statement of 4), note that we have already supposed that $\mathcal{G}(X, \mathscr{L})$ is a tree, so $\mathcal{G}_{n+1}(X, \mathscr{L}) \subset \mathcal{G}(X, \mathscr{L})$ can not contain any cycles. We just need to verify that $\mathcal{G}_{n+1}(X, \mathscr{L})$ is connected.

Case 1i) Suppose that $\gamma_{n+1} = 1$ and C_{n+1}, the copoint in R_{n+1} is such that $C_{n+1} \subset I_n$. Then C_{n+1} and C_n are adjacent in $\mathcal{G}_{n+1}(X, \mathscr{L})$ and the claim is proved.

Case 1ii) Suppose that $\gamma_{n+1} = 1$ and C_{n+1}, the copoint in R_{n+1} is such that $C_{n+1} \subset C_n$. By hypothesis 4) there exists a copoint that is adjacent to C_n and by the same reasoning as before, this copoint will be adjacent to C_{n+1} too.

Case 1iii) Suppose that $\gamma_{n+1} = 2$ and that the two copoints in R_{n+1} are attached to the same point. For the copoint contained in C_n the same argument as in case 1ii) holds. The copoint contained in I_n will be adjacent to C_n. We see that in case 1), all of the claims 1)-4) hold in R_{n+1}.

Case 2) Suppose that $\gamma_n = 2$ with two copoints C_n, C'_n and that $\alpha(C_n) = \alpha(C'_n)$. Again by hypothesis 2 there is a unique non-copoint $I_n \in R_n$. It follows that

$$C'_n \cap C_n = C_n \cap C'_n \cap I_n = C'_n \cap I_n = C_n \cap I_n := I_{n+1},$$

by the induction hypothesis. We again must have $\gamma_{n+1} \geq 1$ by the connectedness hypothesis. If we had any copoint in R_{n+1} contained in I_n, then it would be adjacent to both C_n and C'_n. By the hypothesis that $\mathcal{G}_n(X, \mathscr{L})$ is connected, there is a path in $\mathcal{G}_n(X, \mathscr{L})$ from C_n to C'_n. This produces a cycle. In general, if there are two copoints in R_n which are attached to the same point, as well as a non-copoint in R_n, then a copoint in R_{n+1} can not be contained in the non-copoint. This remark will be important later in the paper.

Therefore, in the case that $\gamma_{n+1} = 2$ for copoints $C_{n+1}, C'_{n+1} \in R_{n+1}$, we then have $C_{n+1} \subset C_n$ and $C'_{n+1} \subset C_n$. We suppose for the sake of contradiction that $\alpha(C_{n+1}) \neq \alpha(C'_{n+1})$. By hypothesis 2) $\alpha(C_{n+1}) \in C'_{n+1}$ and $\alpha(C'_{n+1}) \in C_{n+1}$. Thus $\{C_{n+1}, C'_{n+1}\} \in E(\mathcal{G}(X, \mathscr{L}))$. Furthermore, we must have had that $\alpha(C_{n+1}), \alpha(C_{n+1}) \in I_{n+1}$ or else neither would have been a copoint. Let $H \neq C'_n$ be a copoint in $\mathcal{G}_n(X, \mathscr{L})$ that is adjacent to C_n. We know that one exists by hypothesis 4. Let $H' \neq C_n$ be the same for C'_n. Note that $\alpha(H), \alpha(H') \notin I_n$ since $I_n \subset H, H'$. Thus $\alpha(C_{n+1}), \alpha(C'_{n+1}) \neq \alpha(H), \alpha(H')$ and so $\alpha(H) \in C_n \implies \alpha(H) \in C_{n+1}$. Similarly $\alpha(H') \in C'_n \implies \alpha(H') \in C'_{n+1}$. Furthermore, $\alpha(C_{n+1}), \alpha(C'_{n+1}) \in I_{n+1} \subset H, H'$. We see that C_{n+1} is adjacent to H and C'_{n+1} is adjacent to H'. There is a unique path from H to H' in $\mathcal{G}_n(X, \mathscr{L})$ and since there is an edge between C_{n+1} and C'_{n+1}, we have a cycle. This is a contradiction, so $\alpha(C_{n+1}) = \alpha(C'_{n+1})$.

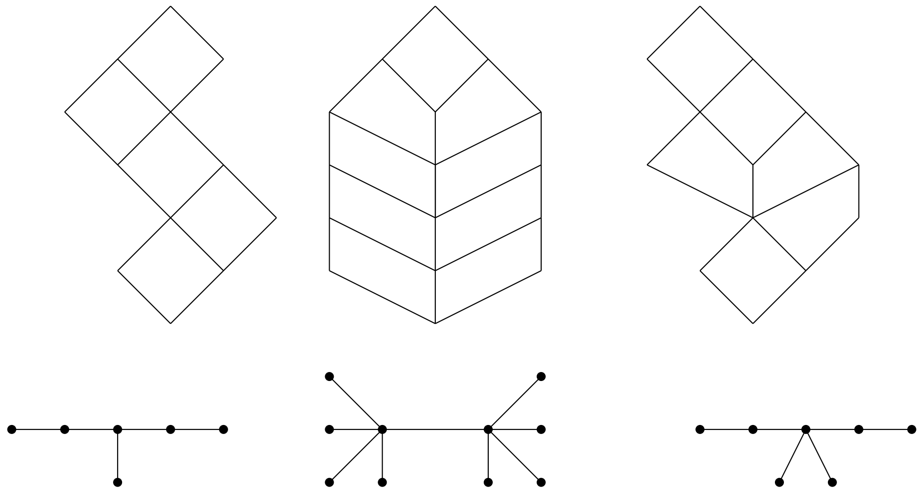


Figure 16. Examples of rhomboidal Hasse diagram structures and their corresponding copoint graphs below

Note that if we had 3 copoints in R_{n+1}, they must have been contained in C_n, C'_n and 2 of them would have been attached to different points. In case 2, we already showed that this could not happen even supposing the existence of only two copoints in R_{n+1}

We conclude that 2) holds in R_{n+1}. The same arguments from case 1 show that this implies 1),3), and 4) also hold in case 2.

We have verified that 1) through 4) hold for R_{n+1} in all possible cases for R_n. We are done by strong induction on n. □

Remark 6.2. Theorem 6.1 is particularly of use to us because it provides many restrictions about as to what types of convex geometries will have copoint graphs that are trees. Suppose that $(X, \mathscr{L})$ is a convex geometry with $\mathcal{G}(X, \mathscr{L})$ a tree. The Hasse diagram of $(X, \mathscr{L})$ can be made planar. When this is done in the natural way, the diagram has a nice 'rhomboidal shape'. See Fig. 16 for some examples.

7. Characterizations of some copoint trees

Define $\wp(n_1, n_2, ..., n_k)$ to be the graph that is obtained by gluing all left endpoints of the paths $P_{n_1}, P_{n_2}, ..., P_{n_k}$. We denote the central vertex by v^*. Refer to Fig. 17 to see $\wp(3, 3, 3, 3)$ and $\wp(3, 3, 3)$.

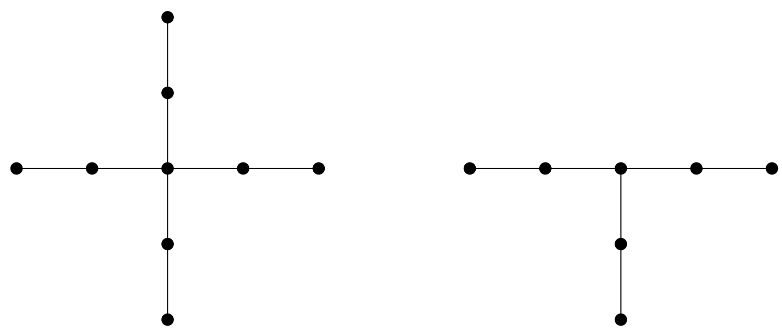


Figure 17. $\wp(3, 3, 3, 3)$; 4 paths glued (left) and $\wp(3, 3, 3)$; 3 paths glued (right)

Theorem 7.1. *Let $(n_1, n_2, ..., n_k)$ be such that $k \geq 3$ and $n_i \geq 3$ for all $1 \leq i \leq k$. There does not exist a convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L}) = \wp(n_1, n_2, ..., n_k)$.*

Proof. First we suppose for the sake of contradiction that there does exist a convex geometry $(X, \mathscr{L})$ with $\mathcal{G}(X, \mathscr{L}) = \wp(n_1, n_2, ..., n_k)$ where $k \geq 3$ and $n_i > 2$ for all $1 \leq i \leq k$. In particular, this means that the central vertex v^* has no **leaves** or degree-one vertices, adjacent to it. Note that $d(v^*) = k$.

Since $\mathcal{G}(X, \mathscr{L})$ is a tree, there are two extreme copoints $A, B \in R_1$. If we had that $|X| \leq 4$, there could be at most 6 copoints in $\mathscr{L}$, but by hypothesis, there are at least 7 vertices in $\mathcal{G}(X, \mathscr{L})$, so $|X| \geq 5$.

Using Theorem 6.1, we consider the two possible cases for R_2. Either $\gamma_2 = 1$ and without loss of generality $C_2 \subset A$ for the unique copoint $C_2 \in R_2$; or $\gamma_2 = 2$ and we have $C_2 \subset A, D_2 \subset B$ and $\alpha(C_2) = \alpha(D_2)$ for copoints $C_2, D_2 \in R_2$.

Case 1) $\gamma_2 = 2$ and $C_2 \subset A$, $D_2 \subset B$ and $\alpha(C_2) = \alpha(D_2)$. By Remark 6.2, we can not have a copoint in R_3 which is contained in $A \cap B$. It follows that all copoints in R_j, with $j \geq 2$ can not contain $\alpha(C_2) = \alpha(D_2)$. Thus C_2 and D_2 are end-points of $\mathcal{G}(X, \mathscr{L})$. Furthermore, there can be no copoint that contains either both of $\alpha(A)$ and $\alpha(C_2)$ or both $\alpha(B)$ and $\alpha(D_2)$. Thus, the graph $\mathcal{G}_2(X, \mathscr{L}) = P_4 \subset \mathcal{G}(X, \mathscr{L})$ has each of its endpoints also as endpoints of $\mathcal{G}(X, \mathscr{L})$. This is a contradiction even in the minimal case of $\mathcal{G}(X, \mathscr{L}) = \wp(3, 3, 3)$.

We must then have that $\gamma_2 = 1$ and without loss of generality $C_2 \subset A$ for some copoint $C_2 \in R_2$. We see that A is an endpoint of $\mathcal{G}(X, \mathscr{L})$. Using the hypothesis that $\mathcal{G}(X, \mathscr{L}) = \wp(n_1, n_2, ..., n_k)$ and that $n_i \geq 3$, we can say without loss of generality that A is the endpoint of the path $P_{n_1} = \mathcal{G}_{n_1-1}(X, \mathscr{L}) \subset \mathcal{G}(X, \mathscr{L})$. The equality follows from the proof of Theorem 5.2, in particular the fact that we know A is an endpoint. We move

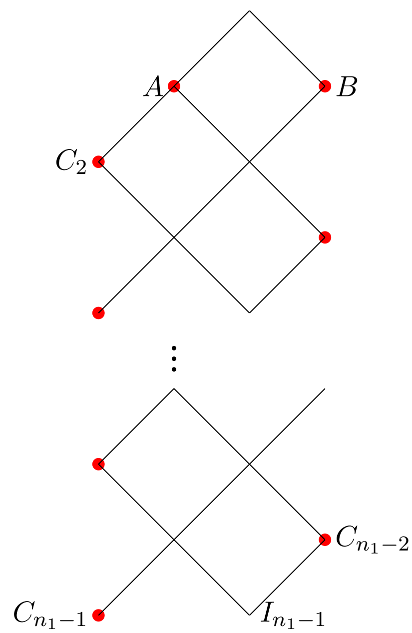


FIGURE 18. Necessary partial Hasse diagram (copoints in red)

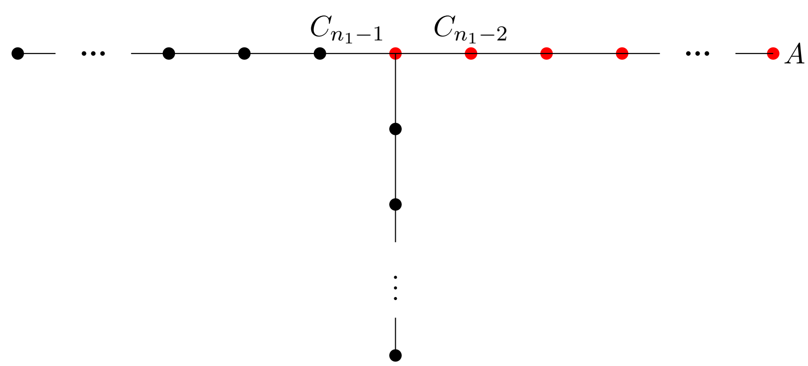


FIGURE 19. $\wp(n_1, n_2, n_3) \subset \mathcal{G}(X, \mathscr{L})$. Subgraph $\mathcal{G}_{n_1-1}(X, \mathscr{L})$ highlighted in red.

down the Hasse diagram and use the adjacencies in $\wp(n_1, n_2, ..., n_k)$ to justify that $(X, \mathscr{L})$ has the Hasse structure of the path graph up through row $n_1 - 1$. This is shown in Fig. 18. In Fig. 19 we show the subgraph $\wp(n_1, n_2, n_3)$ that is guaranteed by the hypothesis $k \geq 3$. Note that there may be other paths stemming from C_{n_1-1}, but this is the minimal case.

We now reason as to what the remaining structure of the Hasse diagram for $(X, \mathscr{L})$ should be. This is shown in Fig. 20. There is a copoint in R_{n_1} since C_{n_1-1} is not an endpoint of $\mathcal{G}(X, \mathscr{L})$. There can not be a copoint $C_{n_1} \in R_{n_1}$ with $C_{n_1} \subset C_{n_1-1}$, since such a copoint would be adjacent to C_{n_1-2}. Thus by Theorem 6.1, $\gamma_{n_1} = 1$ and there is a copoint $C_{n_1} \subset I_{n_1-1}$. Thus, $|R_{n_1}| = 2$ and $C_{n_1-1} \cap I_{n_1-1} := I_{n_1}$, $C_{n_1} \in R_{n_1}$. Note that $d(C_{n_1-1}) = k \geq 3$ and C_{n_1} is the only set in R_{n_1} that contains $\alpha(C_{n_1-1})$, so there is a copoint $C_{n_1+1} \in R_{n_1+1}$. Furthermore, C_{n_1} can not be a degree 1 vertex and I_{n_1} is

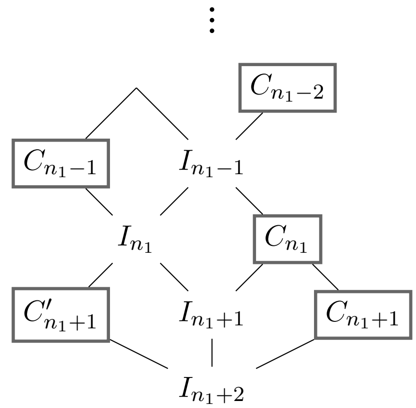


FIGURE 20. Necessary Hasse diagram through R_{i+2}. Note that $I_{n_1} := C_{n_1-1} \cap I_{n_1-1}$, $I_{n_1+1} := I_{n_1} \cap C_{n_1}$, $I_{n_1+2} := C'_{n_1+1} \cap C_{n_1+1}$.

the only set in R_{n_1} that contains $\alpha(C_{n_1})$. Thus there is a copoint $C'_{n_1+1} \in R_{n_1+1}$ with $C'_{n_1+1} \subset I_{n_1}$. There can be no other sets in R_{n_1+1} by Theorem 6.1. Note that C_{n_1+1} is not a degree 1 vertex. Thus, there must be a copoint $C \in \bigcup_{j=n_1+2}^{|X|-1} R_j$ that contains $\alpha(C_{n_1+1})$. However, by the remark at the end of Theorem 6.1, there can not be a copoint in R_{n_1+2} that is contained in I_{n_1+1} as we have $\gamma_{n_1+1} = 2$ and $\alpha(C'_{n_1+1}) = \alpha(C_{n_1+1})$. Note that $\alpha(C_{n_1+1}) \notin C'_{n_1+1}C_{n_1+1}$, Thus C can not be contained in any of the sets in R_{n_1+1}. This is a contradiction. We conclude that when we assume $(X, \mathscr{L})$ is a convex geometry with $\mathcal{G}(X, \mathscr{L}) = \wp(n_1, n_2, ..., n_k)$ meeting the hypotheses of the theorem, we reach a contradiction. Thus, no such convex geometry could have existed to begin with. □

Theorem 7.2. *For all $n \geq 3$ there is a convex geometry $(X, \mathscr{L})$ with*

$$\mathcal{G}(X, \mathscr{L}) = \wp\underbrace{(3, ..., 3, 2, ..., 2)}_{2,3\,repeated\,n-times}.$$

Proof. We claim that there is a convex geometry with Hasse diagram structure given by Fig. 22. Let $X = \{1, 2, ..., 2n+1\}$. Let $C_1 = X - \{1\}$ and define inductively for $1 \leq i \leq 2n-1$, $C_{i+1} = C_i - \{i+1\}$. Let $A = X - \{2n+1\}$ and $I_2 = A \cap C_1 = X - \{1, 2n+1\}$. Define $D_2 = (C_2 - \{2n+1\}) \cup \{1\}$ and in general $D_k = (C_k - \{2n+1\}) \cup \{k-1\}$. Thus $D_k \cap C_k = C_k - \{2n+1\}$ for all $k = 2, 4, .., 2n$. Note that

$$C_2 \cap I_2 \cap D_2 = C_2 \cap D_2 = I_2 \cap D_2 = C_2 \cap I_2,$$

so define this set to be I_3 and define $I_{i+1} = C_i \cap I_i$ for all $3 \leq i \leq 2n-1$. In general we have that $I_i = \{i, i+1, ..., 2n\}$. Thus, for $i \leq 2n$ even we have $C_i \cap I_i = I_{i+1} = C_i - \{2n+1\}$ but also

$$D_i \cap I_i = (C_i - \{2n+1\}) \cup \{i-1\}) \cap I_i = C_i - \{2n+1\} = C_i \cap I_i = C_i \cap I_i \cap D_i = C_i \cap D_i = I_{i+1}.$$

Define the collection $\mathscr{L} := \{X, \emptyset, A, C_i : 1 \leq i \leq 2n\} \cup \{I_i : 2 \leq i \leq 2n\} \cup \{D_i : 2 \leq i \leq 2n,\, i\,\text{even}\}$. Note that $\mathscr{L}$ has precisely the Hasse diagram given by Fig. 22. Furthermore, $\mathscr{L}$ is an allignment from the inclusion of the I_i. By construction, $\mathscr{L}$ has the greedy property. To see this, reason by cases with some set $C \in \mathscr{L} - X$ with $C = \emptyset, C_i, D_i$ or

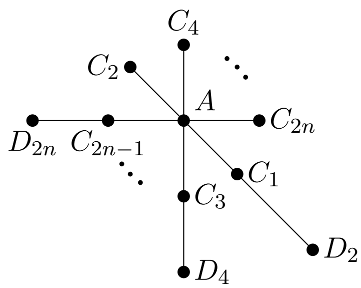


FIGURE 21. $\mathcal{G}(X, \mathscr{L}) = \wp\underbrace{(3, ..., 3, 2, ..., 2)}_{2,3\,\text{repeated}\,n-\text{times}}$.

I_i for some i and note that in the above calculations it is shown that a single element can be added to any of the sets of this form to give another set in $\mathscr{L}$. Thus $(X, \mathscr{L})$ is a convex geometry. Furthermore, $M(X, \mathscr{L}) = \{A, C_i, D_k : 1 \leq i \leq 2n, k = 2, 4, ..., 2n\}$, $\alpha(A) = 2n + 1$, $\alpha(C_i) = i$ and for all $k = 2, 4, ..., 2n$ we have $\alpha(C_k) = \alpha(D_k) = k$.

Note that $\alpha(D_k) = \alpha(C_k) \in C_{k-1}$ while $k - 1 = \alpha(C_{k-1}) \in D_k$ by construction. Thus D_k is adjacent to C_{k-1} in $\mathcal{G}(X, \mathscr{L})$ for all $k = 2, 4, ..., 2n$. Note also that D_k is an endpoint in $\mathcal{G}(X, \mathscr{L})$. Clearly C_i is not adjacent to any other C_j since $\alpha(C_j) \notin C_i$ if $j \geq i$. Furthermore, for i even, C_i is not adjacent to D_i since $\alpha(C_i) = \alpha(D_i)$. For i odd, C_i is not adjacent to D_j for $j < i$ since $\alpha(D_j) = \alpha(C_j)$ and $C_i \subset C_j$. For $j > i + 1$ even note that $D_j \subset I_{j-1} \subset C_i$. Thus for C_i with i odd, C_i is adjacent to D_j if and only if $j = i + 1$. Note that C_i is adjacent to A for all i.

We have then shown that for all $1 \leq i \leq 2n - 1$, C_i is degree two and adjacent to D_{i+1} if and only if i is odd. Furthermore, $D_j \subset A, D_k$ whenever $j < k$, so D_j, D_k are not adjacent for any j, k and D_j is not adjacent to A for any j. Finally, all of the C_i are adjacent to A, and there are $2n$ total C_i. We see that

$$\mathcal{G}(X, \mathscr{L}) = \wp\underbrace{(3, ..., 3, 2, ..., 2)}_{2,3\,\text{repeated}\,n-\text{times}},$$

from Fig. 21. ☐

Remark 7.3. Note that the 'exploding n-star' is an induced subgraph of

$$\wp\underbrace{(3, ..., 3, 2, ..., 2)}_{2,3\,\text{repeated}\,n-\text{times}}.$$

We denote the k-tuple where 3 appears n times and 2 appears $m := k - n$ times by $(n3, m2)$. We have seen in this section that

$$0 < \mathcal{M}(n) := \min_m\{\exists(X, \mathscr{L})\text{ a convex geometry with }\mathcal{G}(X, \mathscr{L}) = \wp((n3, m2))\} \leq n.$$

Note that $\mathcal{M}(3) = 1$ by the construction in Fig. 23. It is unknown whether this construction can be generalized to show that $M(n) < n$ for all $n \geq 4$.

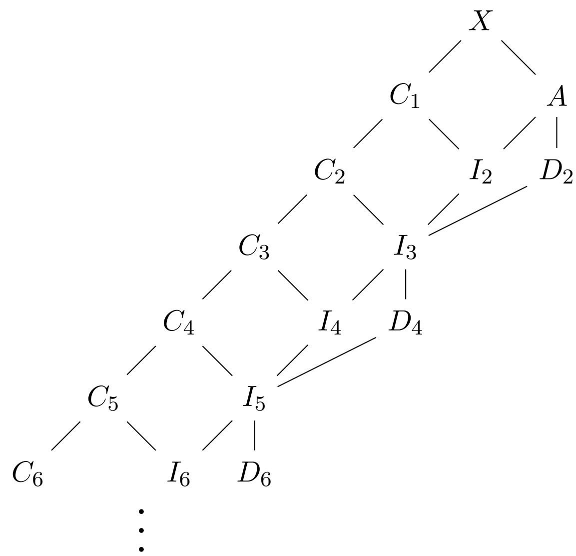


FIGURE 22. Hasse diagram of $(X, \mathscr{L})$ constructed in Theorem 7.2.

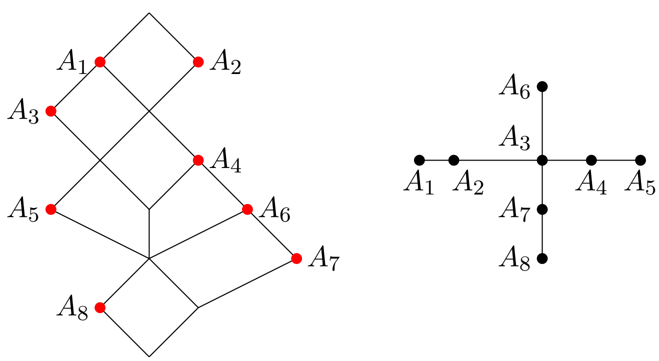


FIGURE 23. Hasse diagram of $(X, \mathscr{L})$ (left) and copoint graph $\mathcal{G}(X, \mathscr{L}) = \wp(3, 3, 3, 2)$ (right).

8. Counting convex geometries whose copoint graphs are trees

Let $T_n := \{([n], \mathscr{L}) : ([n], \mathscr{L})$ is a convex geometry and $\mathcal{G}([n], \mathscr{L})$ is a tree$\}$. We define an equivalence relation $\simeq$ on T_n as follows. Say $([n], \mathscr{L}_1) \simeq ([n], \mathscr{L}_2)$ if there is an order preserving bijection $\phi : \mathscr{L}_1 \to \mathscr{L}_2$. Note that ϕ^{-1} must also be an order preserving bijection.

Theorem 8.1. *Writing $T_n/\simeq$ to be the set of equivalence classes of T_n under $\simeq$, the recursion $|T_{n+1}/\simeq| = 3|T_n/\simeq| - 1$ holds for all $n \geq 2$.*

Proof. In order to count $|T_n \simeq|$, we first want to find some properties of order preserving bijections so as to better understand these equivalence classes we are counting.
We try to see that $|A| = |\phi(A)|$ for all $A \in \mathscr{L}_1$. This is clearly the case for $A = \emptyset, X$. Suppose that there was $A_0 \in \mathscr{L}_1 - \{X, \emptyset\}$ such that without loss of generality $|A_0| < |\phi(A_0)|$. Write $k = |X| - |A_0| - 1$. Now let $A_0 \subsetneq A_1 \subsetneq ... \subsetneq A_k \subsetneq X$ be a chain of length k

in $\mathscr{L}_1$. We then have $\phi(A_0) \subsetneq \phi(A_1) \subsetneq ... \subsetneq \phi(A_k) \subsetneq X$. However, since we supposed that $|\phi(A_0)| > |A_0|$ and the fact that the image chain is also of length k, there exists a $0 \leq j \leq k$ such that $\phi(A_j) = X$ and then we have the contradiction $X \subsetneq X$. Thus, ϕ preserves the cardinality of elements in $\mathscr{L}$.

Now we try to see that copoints map to copoints under ϕ. Let $A \in \mathscr{L}_1$ be a copoint. Suppose that there was $\phi(A) \cup q, \phi(A) \cup r \in \mathscr{L}_2$ such that $q \neq r$. But then, since $\phi(A) \subset \phi(A) \cup r, \phi(A) \cup q$, we have $A \subset \phi^{-1}(\phi(A) \cup q) \neq \phi^{-1}(\phi(A) \cup r)$, but this is a contradiction since A is a copoint and can only be contained in one set of size $|A|+1$. We conclude that $q = r$ and that $\phi(A)$ is a copoint in $\mathscr{L}_2$.

Now we try to see that given an element $([n], \mathscr{L}) \in T_n$, we can 'append' sets onto the bottom of its Hasse diagram to obtain $([n+1], \mathscr{L}') \in T_{n+1}$ for some $\mathscr{L}'$. We will also show that given any $([n+1], \mathscr{L}) \in T_{n+1}$, there exists a $([n], \mathscr{L}') \in T_n$ such that appending sets to this gives $([n+1], \mathscr{L})$.

Suppose that $([n], \mathscr{L})$ is a convex geometry such that $\mathcal{G}([n], \mathscr{L})$ is a tree. By Theorem 6.1 there are two possibilities for R_{n-1}. Either there is one copoint, say A_{n-1} and a non-copoint I_{n-1} or two copoints A_{n-1}, A'_{n-1} and a non-copoint I_{n-1}. Using properties 2) and 3) of Theorem 6.1 applied to R_{n-2}, we have $\{\alpha(A_{n-1})\} = \{\alpha(A'_{n-1})\} = I_{n-1}$. Now consider the collection $\mathscr{L}_0 := \{C \cup \{n+1\}, \emptyset : C \in \mathscr{L}\}$. Note that $([n+1], \mathscr{L}_0)$ is a convex geometry and for all copoints $A \in \mathscr{L}$, $A \cup \{n+1\}$ is a copoint in $\mathscr{L}_0$. However, we also have that $\emptyset$ is a copoint in $\mathscr{L}_0$. Thus $\mathcal{G}([n+1], \mathscr{L}_0)$ is $\mathcal{G}([n], \mathscr{L})$ together with an isolated vertex.

We consider the collection $\mathscr{L}' = \mathscr{L}_0 \cup \{A_{n-1}\}$. We say that a collection obtained in this way is $\mathscr{L}$ with sets 'appended' to its Hasse diagram. Note that $([n+1], \mathscr{L}')$ is a convex geometry and that the copoints of $\mathscr{L}'$ are given by $\{A \cup \{n+1\}, A_{n-1} : A \text{ is a copoint in } ([n], \mathscr{L})\}$. We claim that $\mathcal{G}([n+1], \mathscr{L}')$ is a tree.

Note that $\{n+1\} \in \mathscr{L}'$ and that $\alpha(A_{n-1}) = n+1$. Beagley [Bea13] shows that $\mathcal{G}([n+1] - \{n+1\}, \mathscr{L}'/\{n+1\}) = \mathcal{G}([n+1], \mathscr{L}')|_{n+1}$ up to graph isomorphism, where the graph on the right is the copoint graph of $([n+1], \mathscr{L}')$ restricted to the copoints that contain $\{n+1\}$. Every copoint of $([n+1], \mathscr{L}')$ contains $n+1$ except for A_{n-1}. Removing A_{n-1} from the collection of copoints in $([n+1], \mathscr{L}')$ leaves us with the copoints $A \in M([n], \mathscr{L})$ up to the bijection $A \cup \{n+1\} \mapsto A$. We see that $\mathcal{G}([n], \mathscr{L}) = \mathcal{G}([n+1], \mathscr{L}')|_{n+1} = \mathcal{G}([n+1] - \{n+1\}, \mathscr{L}'/\{n+1\})$ up to graph isomorphism. Thus the graph on the right of this equality is a tree. Note that $\mathcal{G}([n+1], \mathscr{L}') = \mathcal{G}([n+1], \mathscr{L}')|_{n+1} \cup (A_{n-1}, E_{n-1})$ up to graph isomorphism, where E_{n-1} is the set of edges containing A_{n-1}.

Now we examine E_{n-1} to conclude that adding this back in gives us a tree. Since every copoint in $([n+1], \mathscr{L}')$ except for A_{n-1} contains $n+1$, A_{n-1} will be adjacent to $A \cup \{n+1\} \in M([n+1], \mathscr{L}')$ for all $A \in M([n], \mathscr{L})$ such that $\{A, A_{n-1}\} \in E(\mathcal{G}([n], \mathscr{L}))$. Thus A_{n-1} is adjacent to some vertex in $\mathcal{G}([n+1], \mathscr{L}')$. Furthermore, this can not create a cycle in $\mathcal{G}([n+1], \mathscr{L}')$ or else we would have had a cycle in $\mathcal{G}([n], \mathscr{L})$. Thus $\mathcal{G}([n+1], \mathscr{L}')$ has no cycles and is connected so it is a tree. Note that we are not claiming this is the only way we could have constructed $\mathscr{L}'$ such that $\mathcal{G}([n+1], \mathscr{L}')$ is a tree.

If we start with some $([n+1], \mathscr{L}) \in T_{n+1}$, we can consider the unique non-copoint $I_n \in R_n$ that is guaranteed by Theorem 6.1. We show that $([n+1] - I_n, \mathscr{L}/I_n) \in T_n$. We note that any copoint of size 1 in $\mathscr{L}$ is attached to the element contained in I_n. Since I_n is an intersection set, it is contained in every set of $\mathscr{L}$ except for the copoints in R_n and the empty set. Thus $\mathcal{G}([n+1], \mathscr{L})|_{I_n} = \mathcal{G}_{n-1}([n+1], \mathscr{L})$ which is a tree by Theorem 6.1. Furthermore, $\mathcal{G}([n+1], \mathscr{L})|_{I_n} = \mathcal{G}([n+1] - I_n, \mathscr{L}/I_n)$ up to graph isomorphism [Bea13]. We conclude that $([n+1] - I_n, \mathscr{L}/I_n) \in T_n$. Note also that $\mathscr{L} = \{C \cup I_n, A : C \in \mathscr{L}/I_n, A \,\text{is a copoint of size 1 in}\, \mathscr{L}\}$. Thus $([n+1], \mathscr{L})$ can be obtained by 'appending' sets onto the bottom of the Hasse diagram of some convex geometry with a ground set of size n. Furthermore, for a given $([n+1], \mathscr{L}) \in T_{n+1}$, $([n+1] - I_n, \mathscr{L}/I_n)$ is the only element of T_n with the property that appending sets gives $([n+1], \mathscr{L})$.

Now with a formal notion of appending rows to a convex geometry, we try to determine the size of $T_n/\simeq$. Note that $|T_2/\simeq| = 1$. We claim that

$$|T_{n+1}/\simeq| = 3|T_n/\simeq| - 1.$$

Consider two convex geometries $([n+1], \mathscr{L}_1), ([n+1], \mathscr{L}_2) \in T_{n+1}$ with intersection sets $I_n \in \mathscr{L}_1$ and $I'_n \in \mathscr{L}_2$ each of size 1. Recall that $([n+1]-I_n, \mathscr{L}_1/I_n), ([n+1]-I'_n, \mathscr{L}_2/I'_n) \in T_n$. We would like to see the number of possible cases for appending sets to each convex geometry in which we have $([n+1] - I_n, \mathscr{L}_1/I_n) \simeq ([n+1] - I'_m, \mathscr{L}_2/I'_n)$ but $([n+1], \mathscr{L}_1) \not\simeq ([n+1], \mathscr{L}_2)$.

Note that if two convex geometries do not have the same number of copoints of a given size, then they are not in the same equivalence class.

First suppose that each of $([n+1], \mathscr{L}_1), ([n+1], \mathscr{L}_2)$ has two copoints in R_n, say $C_n, D_n \in \mathscr{L}_1$ and $C'_n, D'_n \mathscr{L}_2$ where $\alpha(C_n) = \alpha(D_n)$, $\alpha(C'_n) = \alpha(D'_n)$. If we had that $([n+1] - I_n, \mathscr{L}_1/I_n) \simeq ([n+1] - I'_n, \mathscr{L}_2/I'_n)$ under the map $\phi : \mathscr{L}_1/I_n \to \mathscr{L}_2/I'_n$, then we can see that $([n+1], \mathscr{L}_1) \simeq ([n+1], \mathscr{L}_2)$ under the map $\psi : \mathscr{L}_1 \to \mathscr{L}_2$ defined by

$$\psi(C) = \begin{cases} \phi(C - I_n) \cup I'_n & \text{if } C \neq I_n, C_n, D_n \\ I'_n & \text{if } C = I_n \\ C'_n & \text{if } C = C_n \\ D'_n & \text{if } C = D_n \end{cases}$$

Therefore, for every equivalence class $[([n], \mathscr{L})] \in T_n/\simeq$, there is one equivalence class in $T_{n+1}/\simeq$ whose representative is obtained by appending two copoints to $([n], \mathscr{L})$.

By Theorem 6.1, the only other possibility for R_n is if $([n+1], \mathscr{L}_1), ([n+1], \mathscr{L}_2)$ both have one copoint in R_n, say $C_n \in \mathscr{L}_1$ and $C'_n \in \mathscr{L}_2$. Now we suppose again that $([n+1] - I_n, \mathscr{L}_1/I_n) \simeq ([n+1] - I'_n, \mathscr{L}_2/I'_n)$ under the map ϕ. It follows that $\mathscr{L}_1$ and $\mathscr{L}_2$ have the same number of copoints of size $k \geq 2$.

Case 1: Both $\mathscr{L}_1$ and $\mathscr{L}_2$ have one copoint of size 2, say $C_{n-1} \in \mathscr{L}_1$ and $C'_{n-1} \in \mathscr{L}_2$. We also have intersection sets $I_{n-1} \in \mathscr{L}_1$ and $I'_{n-1} \in \mathscr{L}_2$ of size 2. If $C_n \subset C_{n-1}$ and

$C'_n \subset C'_{n-1}$ or $C_n \subset I_{n-1}$ and $C'_n \subset I'_{n-1}$, then $([n+1], \mathscr{L}_1) \simeq ([n+1], \mathscr{L}_2)$ under the auxillary map ψ defined in the same way as above

Supposing that $([n+1], \mathscr{L}_1) \simeq ([n+1], \mathscr{L}_2)$, then the converse implication holds by the observation that copoints map to copoints under an order preserving bijection. Thus, $([n+1], \mathscr{L}_1) \not\simeq ([n+1], \mathscr{L}_2)$ if and only if one of C_n, C'_n is contained in a copoint of R_{n-1} while the other is not.

Case 2: Both $\mathscr{L}_1$ and $\mathscr{L}_2$ have two copoints of size 2, say $C_{n-1}, D_{n-1} \in \mathscr{L}_1$ and $C'_{n-1}, D'_{n-1} \in \mathscr{L}_2$. Note that $C_{n-1} - I_n, D_{n-1} - I_n$ are copoints in $\mathscr{L}_1/I_n$ and $C'_{n-1} - I'_n, D'_{n-1} - I'_n$ are copoints in $\mathscr{L}_2/I'_n$. Furthermore, without loss of generality, we have that $\phi(C_{n-1} - I_n) = C'_{n-1} - I'_n$ and $\phi(D_{n-1} - I_n) = D'_{n-1} - I'_n$ since we supposed that $([n+1] - I_n, \mathscr{L}_1/I_n) \simeq ([n+1] - I'_n, \mathscr{L}_2/I'_n)$. Recall by Theorem 6.1 that we can not have a copoint contained in the unique intersection set of size 2, else there is a cycle. Thus we have one of the following,

(1) $C_n \subset C_{n-1}$ and $C'_n \subset C'_{n-1}$
(2) $C_n \subset D_{n-1}$ and $C'_n \subset C'_{n-1}$
(3) $C_n \subset C_{n-1}$ and $C'_n \subset D'_{n-1}$
(4) $C_n \subset D_{n-1}$ and $C'_n \subset D'_{n-1}$.

We claim that we have $([n+1], \mathscr{L}_1) \simeq ([n+1], \mathscr{L}_2)$ if and only if 1) or 4) are true, unless it is the case that each of $([n+1], \mathscr{L}_1)$ and $([n+1], \mathscr{L}_2)$ have two copoints of every size $k \leq n$.

Now if we had that 1) or 4) was true, then $([n+1], \mathscr{L}_1) \simeq ([n+1], \mathscr{L}_2)$ by constructing an order preserving bijection ψ using the ϕ that we know exists, in the same way as we did previously.

Suppose without loss of generality that 2) is true, $([n+1], \mathscr{L}_1) \simeq ([n+1], \mathscr{L}_2)$ and $([n+1], \mathscr{L}_1), ([n+1], \mathscr{L}_2)$ do not have two copoints of every size $k \leq n$. Let ψ be the order preserving bijection between $([n+1], \mathscr{L}_1)$,and $([n+1], \mathscr{L}_2)$. Let $1 \leq j < n$ be the maximum row index such that $([n+1], \mathscr{L}_1), ([n+1], \mathscr{L}_2)$ each has one copoint in R_j. Then, there are copoints $D_{n-1} \subsetneq D_{n-2} \subsetneq ... \subsetneq D_{j+1}$ and $C_{n-1} \subsetneq C_{n-2} \subsetneq ... \subsetneq C_{j+1}$ in $\mathscr{L}_1$. Similarly, there are copoints $C'_{n-1} \subsetneq C'_{n-2} \subsetneq ... \subsetneq C'_{j+1}$ and $D'_{n-1} \subsetneq D'_{n-2} \subsetneq ... \subsetneq D'_{j+1}$ in $\mathscr{L}_2$. We must have that $\psi(C_n) = C'_n$ and that $\psi(D_k) = C'_k$, $\psi(C_k) = D'_k$ for all $j+1 \leq k \leq n-1$. Supposing that we have the solid lines case as in Fig. 24, note that the chain of D_k's in $\mathscr{L}_1$ maps to the chain of C'_k's in $\mathscr{L}_2$, but C'_{j+1} is contained in a non-copoint, while D_{j+1} is contained in a copoint. Similarly in the dashed lines case, the chain of C_k's in $\mathscr{L}_1$, maps to the chain of D'_k's in $\mathscr{L}_2$, but D'_{j+1} is contained in a non-copoint, while C_{j+1} is contained in a copoint. This contradicts the assumption that there was an order preserving bijection between $([n+1], \mathscr{L}_1)$ and $([n+1], \mathscr{L}_2)$ in the case where $C_n \subset D_{n-1}$ and $C'_n \subset C'_{n-1}$.

Note that if we did have that $([n+1], \mathscr{L}_1), ([n+1], \mathscr{L}_2)$ each had two copoints of every size $2 \leq k \leq n$ and $([n+1] - I_n, \mathscr{L}_1/I_n) \simeq ([n+1] - I'_n, \mathscr{L}_1/I_n)$, then in any of the cases 1),2),3) or 4), mapping C_n to C'_n together with the map $C \mapsto \phi(C - I_n) \cup I'_n$ gives an order preserving bijection as before. Note that all convex geometries with this property in T_n are in the same equivalence class of $T_n/\simeq$.

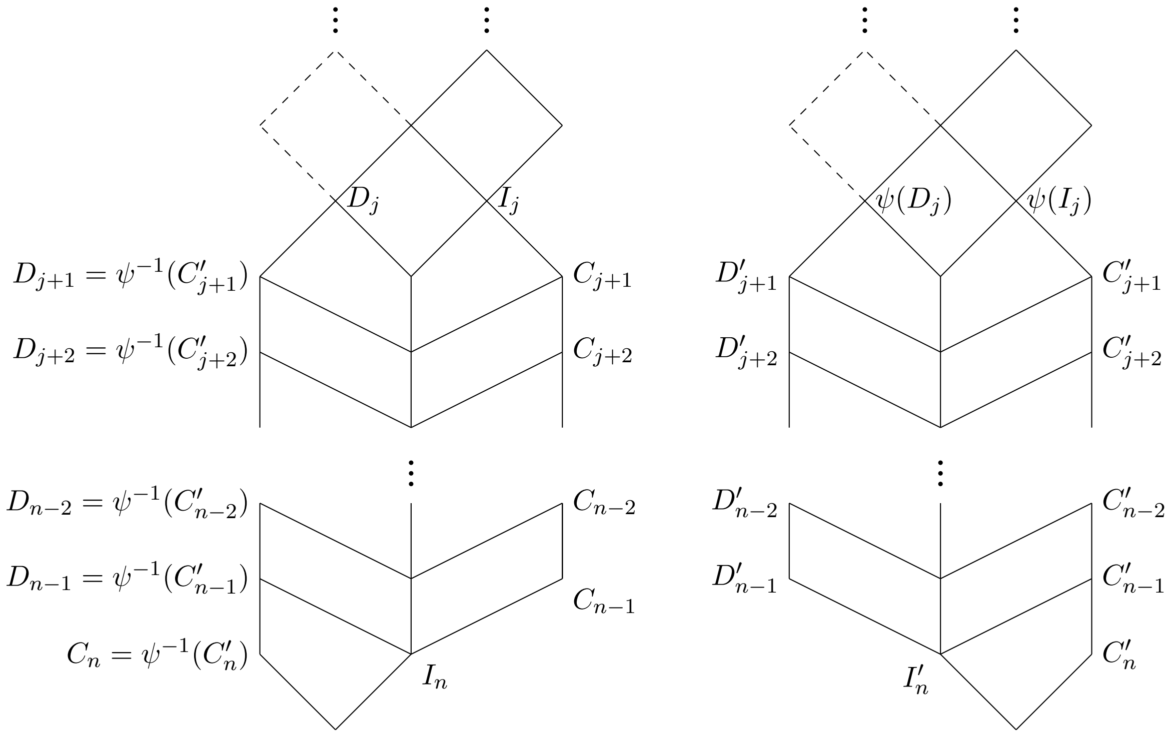


FIGURE 24. $\mathscr{L}_1$ on the left and $\mathscr{L}_2$ on the right under the hypothesis we seek to contradict; Another possibility for R_j is given by the dashed lines.

Thus, combining the results of case 1 and 2, we have (except for in the case of one equivalence class) that for every equivalence class in $T_n/\simeq$ there are two equivalence classes in $T_{n+1}/\simeq$ whose representatives can be obtained by appending a single copoint to a representative of the equivalence class in $T_n/\simeq$.

Essentially we see that two convex geometries are in the same equivalence class of $T_n/\simeq$ if and only if their Hasse diagrams are identical up to a horizontal reflection.

Now for a given $[([n], \mathscr{L})] \in T_n/\simeq$, we see that there are 3 possible equivalence classes of convex geometries $[([n+1], \mathscr{L}')] \in T_{n+1}/\simeq$ such that $[([n+1] - I_n, \mathscr{L}'/I_n)] = [([n], \mathscr{L})]$. This is true with the exception of one equivalence class in $T_n/\simeq$. Every equivalence class of $T_{n+1}/\simeq$ can be constructed by appending sets to some representative of an equivalence class in $T_n/\simeq$. We conclude that $|T_{n+1}/\simeq| = 3(|T_n/\simeq| - 1) + 2 = 3|T_n/\simeq| - 1$. $\square$

Corollary 8.2. $|T_2/\simeq| = 1$ *and*

$$|T_n/\simeq| = \frac{3^{n-2}+1}{2}.$$

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