Colored Motzkin Paths of Higher Order

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Abstract

Motzkin paths are integer lattice paths that use the steps $U = (1, 1)$, $L = (1, 0)$, and $D = (1, -1)$ and stay weakly above the line $y = 0$. We generalize Motzkin paths to allow for down steps with multiple slopes and for various coloring schemes on the edges of the resulting paths. These colored, higher-order Motzkin paths provide a general setting where specific coloring schemes yield sets that are in bijection with many well-studied combinatorial objects. We develop bijections between various classes of colored, higher-order Motzkin paths and certain subclasses of ℓ-ary paths, including a generalization of Fine paths, as well as certain subclasses of ℓ-ary trees. All of this utilizes the language of proper Riordan arrays, and we also include a series of results about the Riordan arrays whose entries enumerate sets of generalized Motzkin paths.

1 Introduction

Integer lattice paths are well-studied objects in combinatorics. A **(2-dimensional) lattice path** of length n is a sequence of n line segments $s_0, s_1, s_2, \ldots, s_n$ such that the end points of all segments have integer coordinates, and the terminal point for s_i is the initial point for s_{i+1}, for all i. The line segments making up these paths are called **steps**, and different types of steps can be used. If a point s_i begins at (x_0, y_0) and ends at (x_1, y_1), then s_i has a **step type** $(x_1 - x_0, y_1 - y_0)$. Different lattice paths can be produced using different variations of step types, also known as **step sets**.

A **Motzkin path** of length n and height k is an integer lattice path from $(0, 0)$ to (n, k) consisting of up, level, and down steps from the step set $\{U = (1, 1), L = (1, 0), D = (1, -1)\}$ and remaining weakly above the x-axis. The set of all Motzkin paths of length n and height k is denoted $\mathcal{M}_{n,k}$, and the cardinality of this set is denoted $|\mathcal{M}_{n,k}| = M_{n,k}$. The four Motzkin paths of length 3 and height 0 are shown in Figure 1. For more background information of Motzkin paths, see Aigner [1] or Bernhart [3].

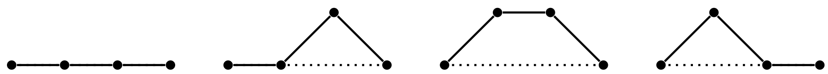


Figure 1: All paths in $\mathcal{M}_{3,0}$. Note $M_{3,0} = 4$

Motzkin paths are relatively simple combinatorial objects, so we work with more complicated generalizations. For $x, y \geq 0$, an **(x, y)-colored Motzkin path** of length n and height k is an element of $\mathcal{M}_{n,k}$ where each L step of height 0 has one of x colors and each level step of nonzero height has one of y colors. The set of all (x, y)-colored Motzkin paths of length n and height k is denoted $\mathcal{M}_{n,k}(x, y)$, and the cardinality of this set is denoted $|\mathcal{M}_{n,k}(x, y)| = M_{n,k}(x, y)$. Figure 2 shows the five $(1, 2)$-colored paths of length 3 and height 0. Although the length and height of the paths is the same as in Figure 1, the number of paths is greater because more types of level steps are allowed.

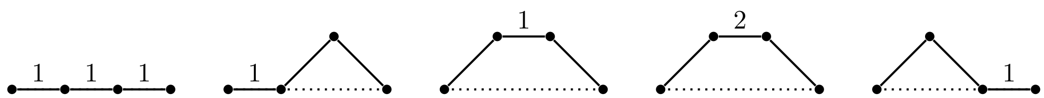


Figure 2: All paths in $\mathcal{M}_{3,0}(1, 2)$. We indicate the color of a step with a number.

Notice that when $(x, y) = (1, 1)$, there is one type of L step both on and off the x-axis. So, $\mathcal{M}_{n,k}(1, 1) = \mathcal{M}_{n,k}$. Because all three allowable steps move to the right, $M_{n,k}(x, y) = 0$ if $n < 0$. Also $M_{n,k}(x, y) = 0$ when $k < 0$, since Motzkin paths must remain weakly above $y = 0$. Finally, $M_{n,k}(x, y) = 0$ when $k > n$, since no step ascends farther than it moves to the right. It follows that $M_{n,k} \neq 0$ only if $0 \leq n \leq k$.

This makes it natural to define the **(x, y)-colored Motzkin triangle** $M(x, y)$, which is the infinite, lower-triangular array whose (n, k)-entry (for $0 \leq k \leq n$) is $M_{n,k}(x, y)$, where the top, leftmost entry corresponds to $(n, k) = (0, 0)$. The first four nonzero rows of this triangle follow.

$$\begin{array}{llll}
1 & & & \\
x & 1 & & \\
x^2 + 1 & x + y & 1 & \\
x^3 + 2x + y & x^2 + xy + y^2 + 2 & x + 2y & 1
\end{array}$$

Most entries of the (x, y)-colored Motzkin triangle depend on x and y, since these parameters affect the number of allowable steps. However, the main diagonal entries count paths consisting entirely of upsteps and thus are independent of x and y.

The following well-known proposition gives a recursion relation that can be used to compute the (x, y)-colored Motzkin triangle.

Proposition 1.1. *For all $n \geq 1$,*

$$M_{n,k}(x, y) = \begin{cases} M_{n-1,k-1}(x, y) + yM_{n-1,k}(x, y) + M_{n-1,k+1}(x, y) & \text{if } k \geq 1, \\ xM_{n-1,0}(x, y) + M_{n-1,1}(x, y) & \text{if } k = 0. \end{cases}$$

Proof. Let $P \in \mathcal{M}_{n,k}(x, y)$. If P ends in a D step, then deleting the final step of P yields a unique member of $\mathcal{M}_{n-1,k+1}(x, y)$. If P ends in U (which can only happen if $k \geq 1$), then deleting the last step yields a unique member of $\mathcal{M}_{n-1,k-1}(x,y)$. If P ends in L, then deleting the final step of a path ending in L yields a member of $\mathcal{M}_{n-1,k}(x,y)$, but in this case the correspondence is not bijective. Depending on whether or not $k=0$, there are either x or y paths in $\mathcal{M}_{n,k}(x,y)$ ending in L which are mapped to a given path in $\mathcal{M}_{n-1,k}(x,y)$. Therefore, when $k \geq 1$, $\mathcal{M}_{n,k}(x,y)$ includes $M_{n-1,k-1}(x,y)$ paths ending in U, $M_{n-1,k+1}(x,y)$ paths ending in D, and $yM_{n-1,k}(x,y)$ paths ending in L. Meanwhile, $\mathcal{M}_{n,0}(x,y)$ includes $M_{n,-1}(x,y)=0$ paths ending in U, $M_{n,1}(x,y)$ paths ending in D, and $xM_{n,0}(x,y)$ paths ending in L. ■

We are especially interested in Motzkin paths that end on the x-axis, which correspond to the leftmost columns of our (x,y)-Motzkin triangles. The recursive relations of Proposition 1.1 may be used to compute the sequences formed by the first columns of the (x,y)-colored Motzkin triangle, for various values of x and y. Observe that many of our sequences correspond to the "Catalan-like" numbers studied by Aigner [2]. Our results are displayed in Table 1.

	$\boldsymbol{y=0}$	$\boldsymbol{y=1}$	$\boldsymbol{y=2}$	$\boldsymbol{y=3}$	$\boldsymbol{y=4}$
$\boldsymbol{x=0}$	A126120	Riordan #'s	Fine #'s	A1177641	A185132
$\boldsymbol{x=1}$	A001405	Motzkin #'s	C_n	A033321	-
$\boldsymbol{x=2}$	A054341	A005773	C_{n+1}	A007317	A033543
$\boldsymbol{x=3}$	A126931	A059738	$\binom{2n+1}{n+1}$	A002212	A064613
$\boldsymbol{x=4}$	-	-	A049027	A026378	A005572

Table 1: Sequences corresponding to the first column of the (x,y)-colored Motzkin triangle for various values of x and y, where numbered entries correspond to OEIS [14] entries and a hyphen denotes the absence of an entry on OEIS. Here C_n is the nth Catalan number.

1.1 Higher-Order Motzkin Paths

Our primary results concern a further generalization of colored Motzkin paths. An **order-m Motzkin path** of length n and height k is an integer lattice path from $(0,0)$ to (n,k) that uses the step set $\{U=(1,1), L=D_0=(1,0), D_1=(1,-1), \ldots, D_m=(1,-m)\}$ and stays weakly above the x-axis. The set of all order-m Motzkin paths of length n and height k is denoted $\mathcal{M}^m_{n,k}$, and the cardinality of this set is denoted $|\mathcal{M}^m_{n,k}| = M^m_{n,k}$. Notice that when $m=1$, the step set becomes $\{(1,1),(1,0),(1,-1)\}$, meaning $\mathcal{M}^1_{n,k} = \mathcal{M}_{n,k}$.

We also allow colorings of higher-order Motzkin paths. For $\vec{x}=(x_0,x_1,...,x_{m-1})$ and $\vec{y}=(y_0,y_1,...,y_{m-1})$, an **$(\vec{x},\vec{y})$-colored, order-$m$ Motzkin path** is an element of $\mathcal{M}^m_{n,k}$ where D_i steps that end on the x-axis have one of x_i colors, and D_i steps that end above the x-axis have one of y_i colors for all $0 \leq i < m$. Hence, the only steps which are not colorable are U steps and maximal down steps D_m. Figure 3 shows the twelve order-2 paths of length 3, height 0, and coloring $\vec{x}=(1,2)$, $\vec{y}=(3,3)$.

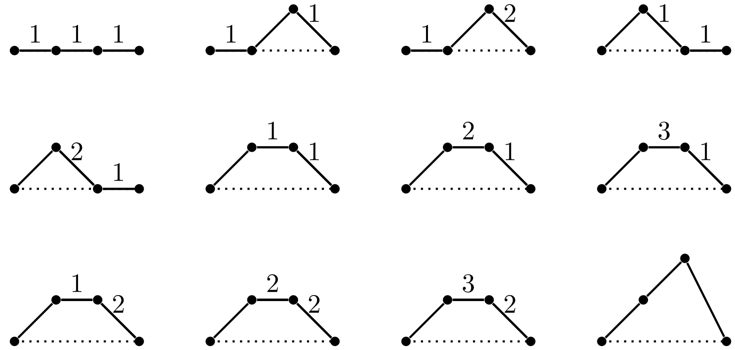


Figure 3: All paths in $\mathcal{M}^2_{3,0}(\vec{x},\vec{y})$ with $\vec{x}=(1,2)$, $\vec{y}=(3,3)$.

Observe that our higher-order Motzkin numbers are distinct from the "higher-rank" Motzkin numbers studied by Mansour, Schork and Sun [8], or Sapounakis and Tsikouras [11]. In particular, we don't allow for multiple types of up steps as in Mansour, Schork and Sun.

As with order-1 Motzkin paths, $M^m_{n,k}(\vec{x},\vec{y})=0$ unless $0\le n\le k$. Thus we can also define triangles for colored, higher-order Motzkin paths. The **$(\vec{x},\vec{y})$-colored, order-m Motzkin triangle** $M^m(\vec{x},\vec{y})$ is the infinite, lower-triangular array whose (n,k) entry is $M^m_{n,k}(\vec{x},\vec{y})$. The entries of this triangle can be calculated recursively, via a method that directly generalizes from Proposition 1.1.

Proposition 1.2. *For all $n\ge 1$,*

$$M^m_{n,k}(\vec{x},\vec{y})=\begin{cases} M^m_{n-1,k-1}(\vec{x},\vec{y})+y_0M^m_{n-1,k}(\vec{x},\vec{y})+\ldots+y_{m-1}M^m_{n-1,k+m-1}(\vec{x},\vec{y})+M^m_{n-1,k+m}(\vec{x},\vec{y}) & k\ge 1,\\ x_0M^m_{n-1,0}(\vec{x},\vec{y})+x_1M^m_{n-1,1}(\vec{x},\vec{y})+\ldots+x_{m-1}M^m_{n-1,m-1}(\vec{x},\vec{y})+M^m_{n-1,m}(\vec{x},\vec{y}) & k=0.\end{cases}$$

Proof. This proof proceeds similarly to that of Proposition 1.1. Let $P\in\mathcal{M}^m_{n,k}(x,y)$. If P ends in D_m, then deleting the final step of P yields a unique member of $\mathcal{M}^m_{n-1,k+m}(x,y)$. If P ends in U (which again can only occur if $k\ge 1$), the deleting the final step yields a unique member of $\mathcal{M}^m_{n-1,k-1}(x,y)$. If P ends in D_i for any $0\le i<m$, then deleting the final step yields a member of $\mathcal{M}^m_{n-1,k+i}(x,y)$, but again, this is not a bijective procedure. Depending on whether or not $k=0$, there are either x_i or y_i paths in $\mathcal{M}^m_{n,k}(x,y)$ ending in D_i which are mapped to a given path in $\mathcal{M}^m_{n-1,k+i}(x,y)$. When $k\ge 1$, $\mathcal{M}^m_{n,k}(x,y)$ consists of $M^m_{n-1,k-1}(x,y)$ paths ending in U, $M^m_{n-1,k+m}(x,y)$ paths ending in D_m, and $y_iM^m_{n-1,k+i}(x,y)$ paths ending in D_i for $0\le i<m$. Meanwhile $\mathcal{M}_{n,0}(x,y)$ consists of $M^m_{n,-1}(x,y)=0$ paths ending in U, $M^m_{n,m}(x,y)$ paths ending in D_m, and $x_iM^m_{n,0}(x,y)$ paths ending in D_i for $0\le i<m$. ■

Proposition 1.2 can be used to quickly generate elements of the $M^m(\vec{x},\vec{y})$ Motzkin triangle. We utilized the recursive relations of Proposition 1.2 to compute the sequences formed by the first columns of the $(\vec{x},\vec{y})$-colored, order-m Motzkin triangle for various values of $\vec{x}$ and $\vec{y}$. Our results are displayed in Tables 3 through 6, located in Appendix A.

1.2 Proper Riordan Arrays

Many of our results require the language of proper Riordan arrays to describe the recursive relations within infinite, lower triangular arrays such as $M^m(\vec{x}, \vec{y})$. A **proper Riordan array** is an infinite, lower triangular array defined by a pair of formal power series, $(d(t), h(t))$, where $d(0) \neq 0$, $h(0) = 0$, and $h'(0) \neq 0$ such that the (n, k)-entry is $d_{n,k} = [t^n]d(t)(h(t))^k$. By convention, $[t^n]p(t)$ denotes the coefficient of the nth-degree term in the power series $p(t)$. See Rogers [10] or Merlini, Rogers, Sprugnoli, and Verri [9] for background information on Riordan arrays. For specific Riordan arrays of a type similar to what we study, see the Catalan triangle of Shapiro [12] or the "Catalan-like" triangles of Aigner [2].

For a proper Riordan array $(d(t), h(t))$, we define the **A-sequence** and **Z-sequence** to be the sequences whose generating functions $A(t)$ and $Z(t)$ satisfy

$$h(t) = tA(h(t)) \tag{1}$$

$$d(t) = \frac{d(0)}{1 - tZ(h(t))}. \tag{2}$$

The A-sequence of a proper Riordan array is determined solely by the power series $h(t)$, and vice versa. The following well-known result relates proper Riordan arrays to the recursion relations in Propositions 1.1 and 1.2.

Proposition 1.3. *A proper Riordan array whose (n, k)-entry is $d_{n,k}$ has A- and Z-sequences $A(t) = a_0 + a_1t + a_2t^2 + \dots$ and $Z(t) = z_0 + z_1t + z_2t^2 + \dots$ if and only if the values $d_{n,k}$ follow the recursion relation*

$$d_{n,k} = \begin{cases} a_0 d_{n-1,k-1} + a_1 d_{n-1,k} + a_2 d_{n-1,k+1} + \dots & k \geq 1 \\ z_0 d_{n-1,0} + z_1 d_{n-1,1} + z_2 d_{n-1,2} + \dots & k = 0. \end{cases}$$

The following corollary is a quick consequence of Propositions 1.2 and 1.3.

Corollary 1.4. *Let $m \in \mathbb{N}$, and let $\vec{x} = \langle x_0, x_1, x_2, \dots, x_{m-1} \rangle$ and $\vec{y} = \langle y_0, y_1, y_2, \dots, y_{m-1} \rangle$ be vectors with nonnegative integer components. Then $M^m(\vec{x}, \vec{y})$ is a Riordan array with A- and Z-sequences*

$$A(t) = 1 + y_0t + y_1t^2 + \dots + y_{m-1}t^m + t^{m+1},$$

$$Z(t) = x_0 + x_1t + x_2t^2 + \dots + x_{m-1}t^{m-1} + t^m.$$

It's important to note that if two proper Riordan arrays have identical A- and Z-sequences and the same $(0, 0)$-entry, then they generate identical triangles. So, if two different sets of lattice paths correspond to infinite, lower triangular arrays which share an A- and Z-sequence and have the same $(0, 0)$-entry, then the cardinality of the two sets of (n, k)-paths is the same for any (n, k).

1.3 Outline of Paper

The remaining sections of this paper proceed as follows. Section 2 contains some basic results about colored, higher-order Motzkin paths. Included is a theorem regarding the binomial transformation of sequences counted by colored, higher-order Motzkin paths. In Section 3, we prove that the number of generalized ℓ-ary paths create a Riordan array similar to Motzkin paths. From this, we show bijective correspondences between certain sets of colored, higher-order Motzkin paths and sets of generalized ℓ-ary paths. In Section 4, we prove that similar correspondences exist between generalized Motzkin paths and a generalization of Fine paths. Section 5 discusses a new group of ℓ-ary paths such that peaks can only exist at certain heights. And finally, in section 6, we prove a bijection between Motzkin paths and trees, a commonly studied combinatorial object.

2 Generalized Motzkin Number Identities

Before proceeding to our main results about specific coloring schemes, we prove several identities which hold for more generic colorings of higher-order Motzkin paths. Our first result in this category concerns the row-sums of the $(\vec{x}, \vec{x})$-colored, order-m Motzkin triangle.

Theorem 2.1. *Let $n, k \geq 0$, $m \geq 1$, and $\vec{x} = \langle x_0, x_1, \dots, x_m \rangle$ have nonnegative integer components. Then the sum of the entries in the n^{th} row of the $(\vec{x}, \vec{x})$-colored, order-m Motzkin triangle is $M^m_{n,0}(\langle x_0 + 1, x_1, \dots, x_m \rangle, \vec{x})$. That is,*

$$\sum_{k=0}^{n} M^m_{n,k}(\vec{x}, \vec{x}) = M^m_{n,0}(\langle x_0 + 1, x_1, \dots, x_m \rangle, \vec{x}). \tag{3}$$

Proof. We construct a bijection between the set S of $(\vec{x}, \vec{x})$-colored, order-m Motzkin paths of length n (and unspecified height), which has cardinality $\sum_{k=0}^{n} M^m_{n,k}(\vec{x}, \vec{x})$, and $\mathcal{M}^m_{n,0}(\langle x_0+1, x_1, \dots, x_m \rangle, \vec{x})$. Paths in $\mathcal{M}^m_{n,0}(\langle x_0 + 1, x_1, \dots, x_m \rangle, \vec{x})$ have one additional color available for level steps on the x-axis. Let $P \in S$ have height k. Then P contains exactly k up steps which are "visible" from the right, meaning that horizontal rays extending from any of these up steps in the positive x-direction do not intersect another step of the path. See Figure 4 for an example. Replacing these visible up steps with level steps of the new color yields a unique $P' \in \mathcal{M}^m_{n,0}(\langle x_0 + 1, x_1, \dots, x_m \rangle, \vec{x})$.

Note that this process is invertible. Given $P' \in \mathcal{M}^m_{n,0}(\langle x_0+1, x_1, \dots, x_m \rangle, \vec{x})$ with k level steps steps of the final color on the x-axis, replacing all such level steps with U steps yields a unique $P \in \mathcal{M}^m_{n,k}(\vec{x}, \vec{x})$. ■

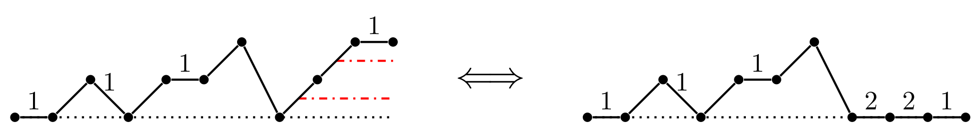


Figure 4: An example of the bijection in the proof of Theorem 2.1. A member of $\mathcal{M}^2_{10,2}(\langle 1, 2\rangle, \langle 1, 2\rangle)$ corresponds to a member of $\mathcal{M}^2_{10,0}(\langle 2, 2\rangle, \langle 1, 2\rangle)$ which has 2 2-colored L steps on the x-axis.

Our next result concerns the binomial transforms of the sequences formed by the columns of colored, higher-order Motzkin triangles. Recall that the **binomial transform** of a sequence $\{a_0, a_1, a_2, \dots\}$ is the unique sequence $\{b_0, b_1, b_2, \ldots\}$ which satisfies

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.$$

The order-1 case of the following result is well-known. We generalize the argument to account for paths of any order $m \geq 1$.

Theorem 2.2. *Let $m \geq 1$ and $k \geq 0$, and let $\vec{x} = \langle x_0, x_1, x_2, \ldots, x_m \rangle$ and $\vec{y} = \langle y_0, y_1, y_2, \ldots, y_m \rangle$ be vectors with nonnegative integer components. The binomial transform of the k^{th} column of $M^m_{n,k}(\vec{x}, \vec{y})$ is the k^{th} column of $M^m_{n,k}(\langle x_0+1, x_1, \ldots x_{m-1}\rangle, \langle y_0+1, y_1, \ldots y_{m-1}\rangle)$. That is,*

$$\sum_{\alpha=0}^{n} \binom{n}{\alpha} M^m_{i,k}(\vec{x}, \vec{y}) = M^m_{n,k}(\langle x_0+1, x_1, \ldots, x_{m-1}\rangle, \langle y_0+1, y_1, \ldots, y_{m-1}\rangle). \tag{4}$$

Proof. Partition $\mathcal{M}^m_{n,k}(\langle x_0+1, x_1, \ldots, x_{m-1}\rangle, \langle y_0+1, y_1, \ldots, y_{m-1}\rangle)$ into subsets S_1, S_2, $\ldots$, S_{n-k} such that $P \in M^m_{n,k}(\langle x_0+1, x_1, \ldots, x_{m-1}\rangle, \langle y_0+1, y_1, \ldots, y_{m-1}\rangle)$ is in S_α if and only if P has exactly α D_0 steps of the final color. For each $0 \leq \alpha \leq n-k$, we define the map $\phi_\alpha : S_\alpha \to \mathcal{M}^m_{n-\alpha,k}(\vec{x}, \vec{y})$ which simply deletes these D_0 steps. Clearly ϕ_α is surjective, but not injective. There are exactly $\binom{n}{\alpha}$ paths in S_α which ϕ_α maps to any particular $Q \in \mathcal{M}^m_{n-\alpha,k}(\vec{x}, \vec{y})$. See Figure 5 for an example. It follows that $M^m_{n,k}(\langle x_0+1, x_1, \ldots, x_{m-1}\rangle, \langle y_0+1, y_1, \ldots, y_{m-1}\rangle) = \sum_{\alpha=0}^{n-k} |S_\alpha| = \sum_{\alpha=0}^{n-k} \binom{n}{\alpha} M^m_{n-\alpha,k}(\vec{x}, \vec{y}) = \sum_{\alpha=0}^{n} \binom{n}{\alpha} M^m_{\alpha,k}(\vec{x}, \vec{y})$. ■

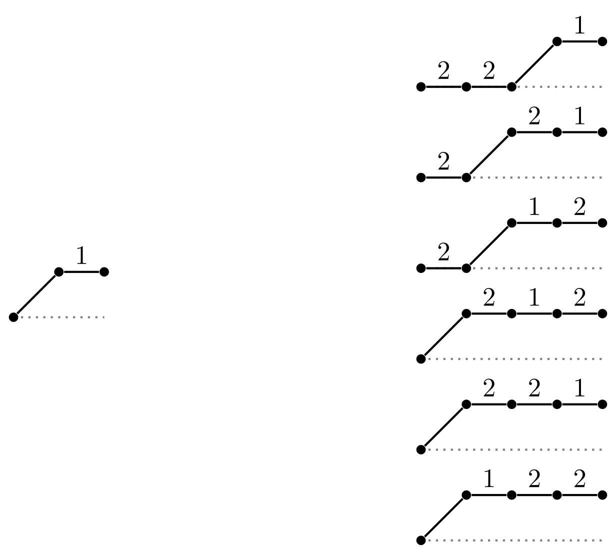
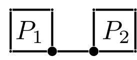


Figure 5: An example of the map from the proof of Theorem 2.2. A member of $\mathcal{M}_{2,1}(1,1)$ is shown on the left; the $\binom{4}{2} = 6$ paths from $\mathcal{M}_{4,1}(2,2)$ which are sent to it under ϕ_2 are shown on the right.

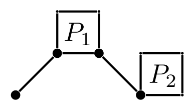
Unlike the previous two theorems, the theorem below does not appear to generalize from order-1 Motzkin paths to higher-order paths. However, we include it here as the identity does not seem to be noted anywhere else in the literature.

Theorem 2.3. *For any integer $x, n \geq 0$, $M_{n,0}(x,x) = M_{n,0}(0,x) + xM_{n+1,0}(0,x)$.*

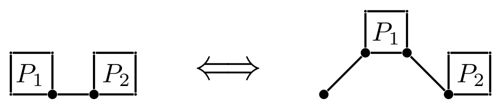
Proof. Since $\mathcal{M}_{n,0}(0,x) \subseteq \mathcal{M}_{n,0}(x,x)$, it suffices to show that $S = \mathcal{M}_{n,0}(x,x) - \mathcal{M}_{n,0}(0,x)$ has cardinality $xM_{n+1,0}(0,x)$. Every path in S has at least one L step on the x-axis, and is therefore of the form P_1LP_2, where P_1 is a generic subpath and P_2 is a subpath with no L steps on the x-axis.



Every path in $\mathcal{M}_{n+1,0}(0,x)$ is of the form UP_1DP_2, where P_1 is an (x,x)-colored Motzkin path and P_2 is a $(0,x)$-colored Motzkin path.



This yields a natural correspondence between S and $\mathcal{M}_{n+1,0}(0,x)$, which associates P_1LP_2 with UP_1DP_2.



This correspondence is x-to-1, since the L step in P_1LP_2 can be any of x colors. This means S has cardinality $xM_{n+1,0}(0,x)$. Therefore $M_{n,0}(x,x) = M_{n,0}(0,x) + xM_{n+1,0}(0,x)$. ■

3 ℓ-ary Paths

Our first major result is about another common class of lattice paths called ℓ-ary paths. Because they have a smaller step set and are not colored, ℓ-ary paths are simpler than generalized Motzkin paths. For any $\ell \geq 2$, an ***ℓ*-ary path** is a lattice path which starts at (0,0), uses the step set $\{U = (1,1), D_{\ell-1} = (1-\ell)\}$ and remains weakly above the x-axis. 2-ary paths, more commonly known as Dyck paths, are particularly well-studied. See Hilton and Pedersen [7] and Heubach, Li and Mansour [6] for more information on ℓ-ary paths, including bijections between ℓ-ary paths and other combinatorial objects. See Figure 6 for examples of 2-ary paths and 3-ary paths. Note that an ℓ-ary path is also a $(\vec{0},\vec{0})$-colored, order-$(\ell-1)$ Motzkin path.

For $\ell \geq 2$, a **generalized *ℓ*-ary** path is a lattice path starting at (0,0), using the step set $\{U = (1,1), D_{\ell-1} = (1-\ell)\}$, and remaining weakly above the line $y = -a$ for some $a \geq 0$. Note that a normal ℓ-ary path is a generalized ℓ-ary path where $a = 0$. The following proposition states a known property of ℓ-ary paths which is shared by generalized ℓ-ary paths.

Proposition 3.1. *If (n,k) is a point on a generalized ℓ-ary path, then $n \equiv k \mod \ell$.*

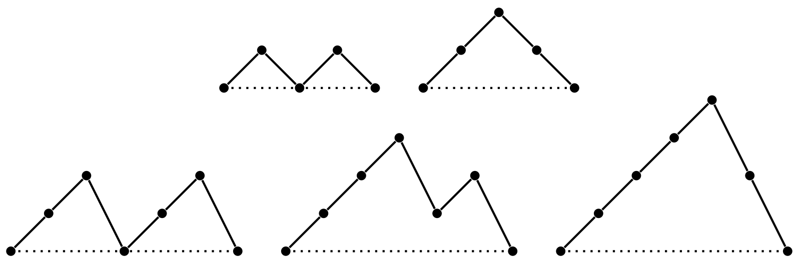


Figure 6: Top row: the two 2-ary (Dyck) paths of length 4 and height 0. Bottom row: the three 3-ary paths of length 6 and height 0. Note that all of these paths have *semilength* 2 and *semiheight* 0.

Proof. Let u be the number of up steps (1,1) before the point (n,k), and let d be the number of down steps $(1,1-\ell)$ before the point (n,k). Then $n = u + d$ and

$$\begin{aligned} k &= u + d(1-\ell) \\ &= u + d - d\ell \\ &= n - d\ell. \end{aligned}$$

Since n and k differ by a multiple of ℓ, clearly $n \equiv k \mod \ell$. ■

As a result of Proposition 3.1, we are interested exclusively in generalized ℓ-ary paths with lengths and heights which are multiples of ℓ. The set of generalized ℓ-ary paths of length ℓn and height ℓk remaining weakly above $y = -a$ is denoted by $\mathcal{D}^{\ell,a}_{n,k}$, and the cardinality of this set is denoted $|D^{\ell,a}_{n,k}| = D^{\ell,a}_{n,k}$. We say that such paths have **semilength** n and **semiheight** k. See Figure 7 for an example.

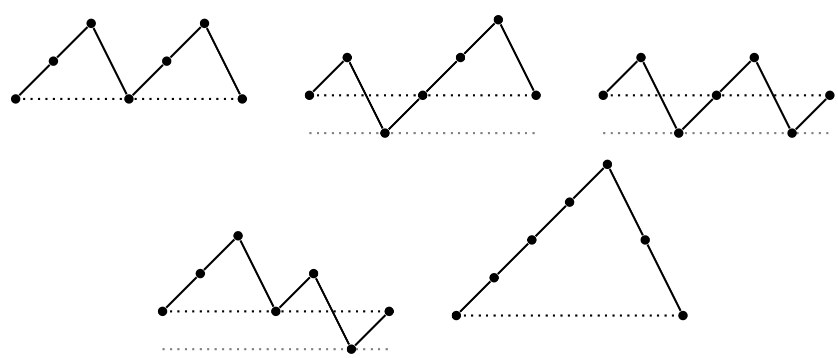


Figure 7: The 5 paths in $\mathcal{D}^{3,1}_{2,0}$. These paths have semilength 2, but length 6.

It is known that for $\ell \geq 2$ and $n \geq 0$, $D^{\ell,0}_{n,0}$ is the n^{th} ℓ-Catalan number $C^\ell_n = \frac{1}{(\ell-1)n-1}\binom{\ell n}{n}$. We henceforth denote the generating function for these numbers by $C_\ell(t) = \sum_{i=0}^{\infty} C^\ell_i t^i$. This function satisfies a fundamental identity which we will rely on for the proofs of Corollaries 3.5 and 4.4:

$$C_\ell(t) = 1 + tC_\ell(t)^\ell. \tag{5}$$

Similar to the (x,y)-colored Motzkin triangle, we define an infinite, lower-triangular array $D^{\ell,a}$ whose (n,k)-entry (for $0 \leq k \leq n$) is $D^{\ell,a}_{n,k}$. Note that the (n,k)-entry of this array is the number of paths with *semilength* n and *semiheight* k, not length n and height k.

In the following theorem, we show that $D^{\ell,a}$ is a proper Riordan array by finding its A- and Z-sequences.

Theorem 3.2. *For all integers $\ell \geq 2$ and $0 \leq a < \ell$, $D^{\ell,a}_{n,k}$ is a proper Riordan array with A- and Z-sequences*

$$A(t) = (1+t)^{\ell},$$

$$Z(t) = \frac{(1+t)^{\ell} - (1+t)^{\ell-a-1}}{t}.$$

Proof. Let n and k be integers with $n > 0$ and $0 \leq k \leq n$. Any member of $\mathcal{D}^{\ell,a}_{n,k}$ has the form P_1P_2, where $P_1 \in \mathcal{D}^{\ell,a}_{n-1,j}$ with some semiheight $j \geq 0$ satisfying $k-1 \leq j \leq k+\ell-1$, and P_2 is a final subpath of length ℓ which contains exactly $j-k+1$ down steps. See Figure 8 for an example.

We'll first consider the case where $k \geq 1$. We know that P_1 can be any of $D^{\ell,a}_{n-1,j}$ possible paths. Since P_2 ends at positive height, the up steps and down steps in P_2 may appear in any order while remaining weakly above the $y = -a$. Therefore P_2 can be any of the $\binom{\ell}{j-k+1}$ subpaths of length ℓ which contain $j-k+1$ down steps. This implies

$$D^{\ell,a}_{n,k} = \sum_{j=k-1}^{k+\ell-1} \binom{\ell}{j-k+1} D^{\ell,a}_{n-1,j}$$

$$= \binom{\ell}{0} D^{\ell,a}_{n-1,k-1} + \binom{\ell}{1} D^{\ell,a}_{n-1,k} + \ldots + \binom{\ell}{\ell} D^{\ell,a}_{n-1,k+\ell-1},$$

giving $D^{\ell,a}$ the A-sequence $A(t) = \binom{\ell}{0} + \binom{\ell}{1}t + \ldots + \binom{\ell}{\ell}t^{\ell} = (1+t)^{\ell}$.

Next we consider the case where $k = 0$. There are still $D^{\ell,a}_{n-1,j}$ possibilities for P_1, but now P_2 must have a down step in the last $a+1$ steps to remain weakly above $y = -a$. This reduces the number of possibilities for P_2 to $\binom{\ell}{j+1} - \binom{\ell-a-1}{j+1}$. So, we have

$$D^{\ell,a}_{n,0} = \sum_{j=0}^{\ell-1} \left(\binom{\ell}{j+1} - \binom{\ell-a-1}{j+1} \right) D^{\ell,a}_{n-1,j}$$

$$= \left(\binom{\ell}{1} - \binom{\ell-a-1}{1} \right) D^{\ell,a}_{n-1,0} + \ldots + \left(\binom{\ell}{\ell} - \binom{\ell-a-1}{\ell} \right) D^{\ell,a}_{n-1,\ell-1}.$$

This gives $D^{\ell,a}$ the Z-sequence

$$Z(t) = \left(\binom{\ell}{1} - \binom{\ell-a-1}{1} \right) + \left(\binom{\ell}{2} - \binom{\ell-a-1}{2} \right) t + \ldots + \left(\binom{\ell}{\ell} - \binom{\ell-a-1}{\ell} \right) t^{\ell-1}$$

$$= \frac{(1+t)^{\ell} - (1+t)^{\ell-a-1}}{t}.$$

■

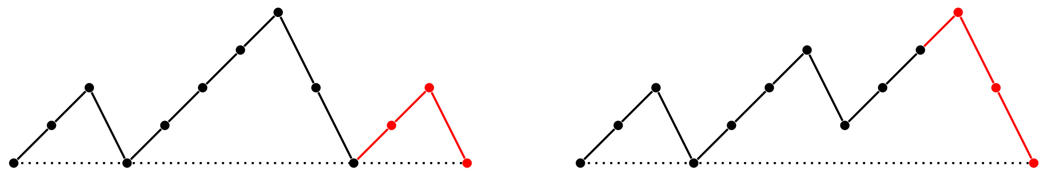


Figure 8: An example of the decomposition described in the proof of Theorem 3.2. Two paths in $\mathcal{D}^{3,0}_{4,0}$ are decomposed into two subpaths, P_1 (black) and P_2 (red). On the left, P_1 has a semiheight $j = 0$, so $P_1 \in \mathcal{D}^{3,0}_{3,0}$ and P_2 contains 1 down step. On the right, P_1 has a semiheight $j = 1$, so $P_1 \in \mathcal{D}^{3,0}_{3,1}$ and P_2 contains 2 down steps.

We can use this knowledge of the structure of $D^{\ell,a}$ to prove a bijective correspondence between generalized ℓ-ary paths and colored, higher-order Motzkin paths.

Corollary 3.3. *Let ℓ and a be integers with $\ell \geq 2$ and $a \geq 0$. Let $\vec{x} = \langle x_0, x_1, x_2, \ldots, x_{\ell-2}\rangle$ and $\vec{y} = \langle y_0, y_1, y_2, \ldots, y_{\ell-2}\rangle$ where*

$$x_i = \binom{\ell}{i+1} - \binom{\ell-a-1}{i+1},$$

$$y_i = \binom{\ell}{i+1}.$$

Then $D^{\ell,a}_{n,k} = M^{\ell-1}_{n,k}(\vec{x}, \vec{y})$ for all $0 \leq k \leq n$.

Proof. By Theorem 3.2, $D^{\ell,a}$ is a proper Riordan array with A- and Z-sequences $A(t) = (1+t)^{\ell}$ and $Z(t) = \frac{(1+t)^{\ell}-(1+t)^{\ell-a-1}}{t}$. By Corollary 1.4, the proper Riordan array $M^{\ell-1}(\vec{x}, \vec{y})$ has these same A- and Z-sequences. Also, $D^{\ell,a}_{0,0} = M^{\ell-1}_{0,0}(\vec{x}, \vec{y}) = 1$. It follows that $D^{\ell,a}$ and $M^{\ell-1}(\vec{x}, \vec{y})$ are identical arrays and therefore that $D^{\ell,a}_{n,k} = M^{\ell-1}_{n,k}(\vec{x}, \vec{y})$ for all $0 \leq k \leq n$. ∎

Example 3.4. *The table below shows the consequences of Corollary 3.3 for low values of ℓ. The (ℓ, a)-entry is the pair of ordered $(\ell-1)$-tuples $(\vec{x}, \vec{y})$ such that $D^{\ell,a} = M^{\ell-1}(\vec{x}, \vec{y})$. Note that $D^{\ell,a}$ is only identical to a colored, order-$(\ell-1)$ Motzkin triangle when $0 \leq a < \ell$. The OEIS [14] entries corresponding to the first columns of these triangles are shown in Tables 3, 4, and 5 in the appendix.*

	$\boldsymbol{a=0}$	$\boldsymbol{a=1}$	$\boldsymbol{a=2}$	$\boldsymbol{a=3}$
$\boldsymbol{\ell=2}$	$(1,2)$	$(2,2)$	-	-
$\boldsymbol{\ell=3}$	$\big(\langle 1,2\rangle, \langle 3,3\rangle\big)$	$\big(\langle 2,3\rangle, \langle 3,3\rangle\big)$	$\big(\langle 3,3\rangle, \langle 3,3\rangle\big)$	-
$\boldsymbol{\ell=4}$	$\big(\langle 1,3,3\rangle, \langle 4,6,4\rangle\big)$	$\big(\langle 2,5,4\rangle, \langle 4,6,4\rangle\big)$	$\big(\langle 3,6,4\rangle, \langle 4,6,4\rangle\big)$	$\big(\langle 4,6,4\rangle, \langle 4,6,4\rangle\big)$

Finally, we use the proper Riordan array identities (1) and (2) to deduce the power series $d(t)$ and $h(t)$ which define $D^{\ell,a}$. This gives generating functions for the values $D^{\ell,a}_{n,k}$.

Corollary 3.5. *Let a, ℓ, n, and k be nonnegative integers with $\ell \geq 2$ and $a < \ell$. Then $D^{\ell,a}_{n,k} = [t^n]C_{\ell}(t)^{a+1}(tC_{\ell}(t)^{\ell})^k$.*

Proof. By Theorem 3.2, we know $D^{\ell,a}$ is a proper Riordan array with $A(t) = (1+t)^\ell$ and $Z(t) = \frac{(1+t)^\ell - (1+t)^{\ell-a-1}}{t}$. Since $D^{\ell,a}$ is a proper Riordan array, there must exist power series $d(t)$ and $h(t)$ such that $D^{\ell,a}_{n,k} = [t^n]d(t)(h(t))^k$. These are the unique power series satisfying the identities $h(t) = tA(h(t))$ and $d(t) = \frac{d(0)}{1-tZ(h(t))}$. They are $d(t) = C_\ell(t)^{a+1}$ and $h(t) = tC_\ell(t)^\ell$, since, using Equation (5),

$$tA(h(t)) \;=\; t(1+tC_\ell(t)^\ell)^\ell \;=\; t(C_\ell(t))^\ell \;=\; h(t),$$

and also

$$\frac{d(0)}{1-tZ(h(t))} \;=\; \frac{C_\ell(t)^{a+1}(0)}{1-t\frac{(1+tC_\ell(t)^\ell)^\ell-(1+tC_\ell(t)^\ell)^{\ell-a-1}}{tC_\ell(t)^\ell}} \;=\; \frac{1}{1-\frac{C_\ell(t)^\ell - C_\ell(t)^{\ell-a-1}}{C_\ell(t)^\ell}}$$

$$=\; \frac{1}{\frac{C_\ell(t)^\ell - C_\ell(t)^\ell + C_\ell(t)^{\ell-a-1}}{C_\ell(t)^\ell}} \;=\; \frac{1}{\frac{C_\ell(t)^{\ell-a-1}}{C_\ell(t)^\ell}} \;=\; \frac{C_\ell(t)^\ell}{C_\ell(t)^{\ell-a-1}} \;=\; C_\ell(t)^{a+1} \;=\; d(t).$$

■

4 Generalized Fine Paths

In the previous section, we relied on the recursive structure of the proper Riordan arrays defined by our generalized ℓ-ary paths to find a bijective correspondence to colored, higher-order Motzkin paths. In this section, we will prove a similar result about a generalization of Fine paths rather than ℓ-ary paths. A **Fine** path is a 2-ary (Dyck) path without a subpath of the form UD_1 ending on the x-axis. Sometimes such subpaths are called **hills**. See Figure 9 for an example. For additional results regarding Fine paths, see Deutsch and Shapiro [5].

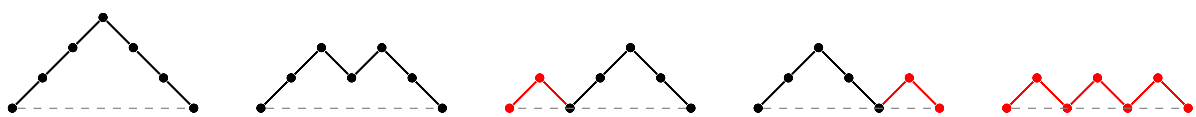


Figure 9: The two paths on the left are Fine paths. The three paths on the right are not Fine paths.

In the literature, Fine paths are strictly defined as 2-ary paths. We extend the principle to the generalized ℓ-ary paths we define in the previous section, which use steeper down steps and also remain weakly above lines other than the x-axis. As shown in Figure 10, when $\ell > 2$, we can give more or less strict definitions for what subpaths constitute generalized "hills" and are therefore not allowable in generalized Fine paths.

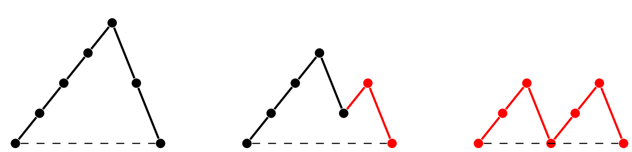


Figure 10: The left path has no subpath which can reasonably be classified as a generalized "hill." The middle path has a subpath which may or may not be considered a hill, depending on how strictly a hill is defined. The right path contains two hills even under the strictest possible definition of a hill.

The appearance of these different types of hills at higher values of ℓ motivates the following extension of the Fine condition to generalized ℓ-ary paths. For $1 \le r < \ell$, a generalized ℓ-ary path is ***r*-Fine** if it does not contain a subpath of the form $U^r D_{\ell-1}$ ending on the x-axis. Figure 11 shows two 4-ary paths, of which one is 3-Fine and the other is not. Note that a traditional Fine path is a 1-Fine 2-ary path.

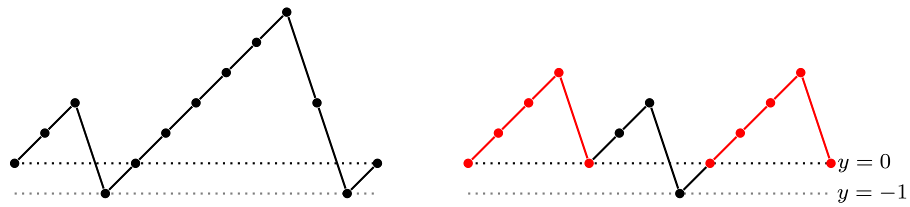


Figure 11: Left: 3-Fine path from $\mathcal{F}^{4,1,3}_{3,0}$, right: not a 3-Fine path.

We will denote the r-Fine subset of $\mathcal{D}^{\ell,a}_{n,k}$ by $\mathcal{F}^{\ell,a,r}_{n,k}$, and its cardinality by $|\mathcal{F}^{\ell,a,r}_{n,k}| = F^{\ell,a,r}_{n,k}$. The infinite, lower-triangular array whose (n,k)-entry is $F^{\ell,a,r}_{n,k}$ will be denoted by $F^{\ell,a,r}$.

Theorem 4.1. *Let ℓ, a, and r be integers with $\ell \geq 2$, $0 \leq a < \ell$, and $1 \leq r < \ell$. Then $F^{\ell,a,r}$ is a Riordan array with the following A- and Z-sequences:*

$$A(t) = (1+t)^\ell,$$

$$Z(t) = \frac{(1+t)^\ell - (1+t)^{\ell-a-1}}{t} - (1+t)^{\ell-r-1}.$$

Proof. This proof proceeds similarly to that of Theorem 3.2. Let n and k be integers where $n > 0$ and $0 \leq k \leq n$. As with generalized ℓ-ary paths, any member of $F^{\ell,a,r}_{n,k}$ has the form $P_1 P_2$, where $P_1 \in \mathcal{F}^{\ell,a}_{n-1,j}$ with some semiheight $j \geq 0$ satisfying $k-1 \leq j \leq k+\ell-1$ and P_2 is a final subpath of length ℓ which contains exactly $j-k+1$ down steps. Figure 12 shows an example.

Again, when $k \geq 1$, all orderings of the steps in P_2 are legal, so the A-sequence is $A(t) = \binom{\ell}{0} + \binom{\ell}{1}t + \ldots + \binom{\ell}{\ell}t^\ell = (1+t)^\ell$.

Only the $k=0$ case differs significantly from the analogous case for Theorem 3.2. We know that F is one of $F^{\ell,a,r}_{n,j}$ possible paths. The proof of Theorem 3.2 established that excluding the subpaths which pass below $y=-a$ leaves $\binom{\ell}{j+1} - \binom{\ell-a-1}{j+1}$ possibilities for P_2. Now we must also exclude the subpaths which contain a return to the x-axis immediately preceded by the steps $U^r D_{\ell-1}$. By Proposition 3.1, such a return must occur at the end of P_2, since the x-coordinate of the point of return must be congruent to 0 modulo ℓ. Therefore the illegal P_2 subpaths that violate the r-Fine condition are the $\binom{\ell-r-1}{j}$ subpaths of length ℓ which end in the steps $U^r D_{\ell-1}$. Note that since all of these subpaths remain weakly above $y=-a$, we have not already excluded any of them. Thus only $\binom{\ell}{j+1} - \binom{\ell-a-1}{j+1} - \binom{\ell-r-1}{j}$ possibilities exist for P_2. It follows that

$$F^{\ell,a,r}_{n,0} = \sum_{j=0}^{\ell-1} \left(\binom{\ell}{j+1} - \binom{\ell-a-1}{j+1} - \binom{\ell-r-1}{j} \right) D^{\ell,a}_{n-1,j},$$

which yields the Z-sequence

$$Z(t) = \left(\binom{\ell}{1} - \binom{\ell-a-1}{1} - \binom{\ell-r-1}{0} \right) + \ldots + \left(\binom{\ell}{\ell} - \binom{\ell-a-1}{\ell} - \binom{\ell-r-1}{\ell-1} \right) t^{\ell-1}$$

$$= \frac{(1+t)^{\ell} - (1+t)^{\ell-a-1}}{t} - (1+t)^{\ell-r-1}.$$

■

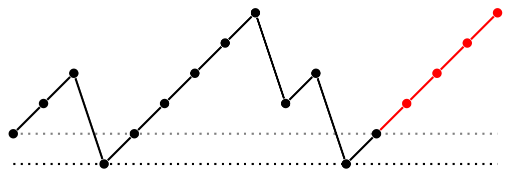


Figure 12: An example of the decomposition from the proof of Theorem 4.1. A member of $\mathcal{F}_{4,1}^{4,1,1}$, with subpath $P_1 \in \mathcal{F}_{3,0}^{4,1,1}$ shown in black and subpath P_2 shown in red. Since $a = 1$, this path is allowed to pass below the x-axis as long as it remains weakly above $y = -1$. Also note that with $r = 1$, no returns to the x-axis can be preceded by $UD_{\ell-1}$.

We can now prove a bijective correspondence between r-Fine paths and colored, higher-order Motzkin paths, similar to the correspondence shown in Corollary 3.3.

Corollary 4.2. *Let ℓ, a, and r be integers with $\ell \geq 2$, $0 \leq a < \ell$, and $1 \leq r < \ell$. Let $\vec{x} = \langle x_0, x_1, x_2, \ldots, x_{\ell-2} \rangle$ and $\vec{y} = \langle y_0, y_1, y_2, \ldots, y_{\ell-2} \rangle$ where*

$$x_i = \binom{\ell}{i+1} - \binom{\ell-a-1}{i+1} - \binom{\ell-r-1}{i},$$

$$y_i = \binom{\ell}{i+1}.$$

Then $F_{n,k}^{\ell,a,r} = M_{n,k}^{\ell-1}(\vec{x}, \vec{y})$ for all $0 \leq k \leq n$.

Proof. This proof echoes the proof of Corollary 3.3. By Theorem 4.1 and Corollary 1.4, we know that $F^{\ell,a,r}$ and $M^{\ell-1}(\vec{x}, \vec{y})$ have the same A- and Z-sequences. We also know that $F_{0,0}^{\ell,a,r} = M_{0,0}^{\ell-1}(\vec{x}, \vec{y}) = 1$. Therefore the two proper Riordan arrays are identical and have equivalent (n,k)-entries for all $0 \leq k \leq n$. ■

Example 4.3. *The table below shows the consequences of Corollary 4.2 for low values of ℓ and $a = 0$. The (ℓ, r)-entry is the pair of ordered $(\ell-1)$-tuples $(\vec{x}, \vec{y})$ such that $F^{\ell,0,r} = M^{\ell-1}(\vec{x}, \vec{y})$. Note that $F^{\ell,0,r}$ is only identical to a colored, order-$(\ell-1)$ Motzkin triangle when $0 < r < \ell$. (For any a, $F^{\ell,a,r}$ is only equivalent to a colored, order-$(\ell-1)$ Motzkin triangle when $0 \leq a < \ell$ and $0 < r < \ell$.) The OEIS [14] entries corresponding to the first columns of these triangles are shown in Tables 3, 4, and 5 in the appendix.*

	$r=1$	$r=2$	$r=3$
$\ell=2$	$(0,2)$	-	-
$\ell=3$	$\big(\langle 0,1\rangle,\langle 3,3\rangle\big)$	$\big(\langle 0,2\rangle,\langle 3,3\rangle\big)$	-
$\ell=4$	$\big(\langle 0,1,2\rangle,\langle 4,6,4\rangle\big)$	$\big(\langle 0,2,3\rangle,\langle 4,6,4\rangle\big)$	$\big(\langle 0,3,3\rangle,\langle 4,6,4\rangle\big)$

As in the previous section, we can apply identities (1) and (2) to find generating functions for the values $F^{\ell,a,r}_{n,k}$.

Corollary 4.4. *Let ℓ, a, r, n, and k be nonnegative integers with $\ell \geq 2$ and $a, r < \ell$. Then* $F^{\ell,a,r}_{n,k} = [t^n]\frac{C_\ell(t)^\ell}{C_\ell(t)^{\ell-a-1}+tC_\ell(t)^{2\ell-r-1}}(tC_\ell(t)^\ell)^k$.

Proof. Theorem 4.1 tells us that $F^{\ell,a,r}$ is a proper Riordan array with $A(t) = (1+t)^\ell$ and $Z(t) = \frac{(1+t)^\ell-(1+t)^{\ell-a-1}}{t} - (1+t)^{\ell-r-1}$. In proving Corollary 3.5 we already showed that $h(t) = tA(h(t))$ is satisfied by $h(t) = tC_\ell(t)^\ell$. Also, note that when $d(t) = \frac{C_\ell(t)^\ell}{C_\ell(t)^{\ell-a-1}+tC_\ell(t)^{2\ell-r-1}}$ we have

$$\frac{d(0)}{1-tZ(h(t))} = \frac{\frac{C_\ell^\ell(0)}{C_\ell^{\ell-a-1}(0)+0C_\ell^{2\ell-r-1}(0)}}{1-t\left(\frac{(1+tC_\ell(t)^\ell)^\ell-(1+tC_\ell(t)^\ell)^{\ell-a-1}}{tC_\ell(t)^\ell}-(1+tC_\ell(t)^\ell)^{\ell-r-1}\right)}$$

$$= \frac{1}{1-t\left(\frac{(1+tC_\ell(t)^\ell)^\ell-(1+tC_\ell(t)^\ell)^{\ell-a-1}}{tC_\ell(t)^\ell}-(1+tC_\ell(t)^\ell)^{\ell-r-1}\right)} = \frac{1}{1-t\left(\frac{C_\ell(t)^\ell-C_\ell(t)^{\ell-a-1}}{tC_\ell(t)^\ell}-C_\ell(t)^{\ell-r-1}\right)}$$

$$= \frac{1}{1-\frac{C_\ell(t)^\ell-C_\ell(t)^{\ell-a-1}}{C_\ell(t)^\ell}+tC_\ell(t)^{\ell-r-1}} = \frac{1}{1-1+\frac{C_\ell(t)^{\ell-a-1}}{C_\ell(t)^\ell}+tC_\ell(t)^{\ell-r-1}}$$

$$= \frac{1}{\frac{C_\ell(t)^{\ell-a-1}+tC_\ell(t)^{2\ell-r-1}}{C_\ell(t)^\ell}} = \frac{C_\ell(t)^\ell}{C_\ell(t)^{\ell-a-1}+tC_\ell(t)^{2\ell-r-1}} = d(t).$$

■

Notice that a simpler generating function $F_{\ell,r}(t)$ for the sequence $(F^{\ell,0,r}_{n,0})_{n=0,1,2,\ldots}$ which counts regular ℓ-ary paths of height 0 which satisfy the r-Fine condition can be derived by performing the substitution $a=0$ above:

$$F_{\ell,r}(t) = \frac{C_\ell(t)^\ell}{C_\ell(t)^{\ell-0-1}+tC_\ell(t)^{2\ell-r-1}} = \frac{C_\ell(t)}{1+tC_\ell(t)^{\ell-r}}.$$

5 ℓ-ary Paths with Peaks at Particular Heights

In this section we generalize the following identities noted by Callan [4]:

1. Dyck paths of semilength n with all peaks at even height are counted by the Riordan number $R_n = M_{n,0}(0,1)$.

2. Dyck paths of semilength n with all peaks at odd height are counted by the Motzkin number M_{n-1}.

Naturally, a **peak** is a subpath of the form UD_i. The height of the peak is the height at which the U step ends.

Lemma 5.1. *Let ℓ, a, n, and k be integers with $\ell \geq 2$, $0 \leq a < \ell$, and $n, k \geq 0$. Let $\vec{x} = \langle x_0, x_1, x_2, \ldots, x_{\ell-2} \rangle$ where*

$$x_i = \begin{cases} 1 & \text{if } i < a, \\ 0 & \text{if } i \geq a, \end{cases}$$

and let $\vec{1} = \langle 1, 1, \ldots, 1 \rangle$. Then members of $\mathcal{D}_{n,k}^{\ell,a}$ whose only peaks occur at heights congruent to 0 modulo ℓ are counted by $M_{n,k}^{\ell-1}(\vec{x}, \vec{1})$.

Proof. We describe a bijection ψ which maps an ℓ-ary path $P \in \mathcal{D}_{n,k}^{\ell,a}$ whose only peaks occur at heights congruent to 0 modulo ℓ to a member of $\mathcal{M}_{n,k}^{\ell-1}(\vec{x}, \vec{1})$. Consider P as a sequence of blocks of ℓ steps. By Proposition 3.1, any U step followed by a $D_{\ell-1}$ step in the middle of a block would constitute a peak at a height which is not a multiple of ℓ. Therefore every block is a subpath of the form $D_{\ell-1}^i U^{\ell-i}$ for some $0 \leq i \leq \ell$. To obtain $\psi(P)$, replace each U^ℓ block with U, and each $D_{\ell-1}^i U^{\ell-i}$ block ($0 < i \leq \ell$) with D_{i-1}. An example is shown in Figure 13.

Note that $\psi(P)$ moves exactly 1 unit up, down, or sideways for every ℓ units moved in that direction by P, and thus must remain weakly above the x-axis and end at the point (n, k). Furthermore, since P remains weakly above the $y = -a$, P cannot contain a $D_{\ell-1}^i U^{\ell-i}$ block ending at height 0 for any $i < a$. Therefore $\psi(P)$ cannot contain any D_i step ending on the x-axis for $i < a$, so $\psi(P) \in \mathcal{M}_{n,k}^{\ell-1}(\vec{x}, \vec{1})$.

To prove ψ is a bijection, we'll define its inverse ψ^{-1}. Given a colored, higher-order Motzkin path $R \in \mathcal{M}_{n,k}^{\ell-1}(\vec{x}, \vec{1})$, obtain $\psi^{-1}(R)$ by replacing each U with a U^ℓ block, and each D_i with a $D_{\ell-1}^{i+1} U^{\ell-i-1}$ block. Observe that $\psi^{-1}(R)$ moves ℓ units in any direction for each 1 unit moved by R in that direction, so $\psi^{-1}(R)$ ends at $(\ell n, \ell k)$. Since R contains no U steps or D_i steps with $i < a$ which end on the x-axis, $\psi^{-1}(R)$ does not contain any block which ends in more than a U steps to the x-axis. Therefore $\phi_0^{-1}(R)$ remains weakly above $y = -a$. Finally, since no peaks occur in the middle of the $D_{\ell-1}^i U^{\ell-i}$ blocks of $\psi^{-1}(R)$, the only peaks in $\psi^{-1}(R)$ must occur at lengths which are multiples of ℓ, and thus, by Proposition 3.1, heights which are congruent to 0 modulo ℓ.

We have shown that both ψ and ψ^{-1} are well-defined functions. It is obvious for any ℓ-ary path P of semilength n and semiheight k whose only peaks occur at heights congruent to 0 modulo ℓ that $\psi(\psi^{-1}(P)) = P$. Therefore ψ and ψ^{-1} are bijections. ■

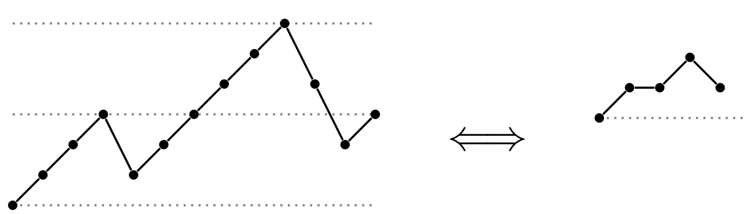


Figure 13: An example of the bijection ψ from in Lemma 5.1. Note that the only peaks in the 3-ary path on the right occur at heights which are multiples of 3.

Theorem 5.2. *Let ℓ, n, h be integers with $\ell \geq 2$, $n \geq 1$, and $0 \leq h < \ell$. Let $\vec{x} = \langle x_0, x_1, x_2, \dots, x_{\ell-2}\rangle$ where*

$$x_i = \begin{cases} 1 & \text{if } i < h, \\ 0 & \text{if } i \geq h. \end{cases}$$

Then members of $\mathcal{D}^{\ell,0}_{n,0}$ whose peaks occur at heights congruent to h modulo ℓ are counted by $M^{\ell-1}_{n-1,\ell-h-1}(\vec{x}, \vec{1})$.

Proof. Every path $P \in \mathcal{D}^{\ell,0}_{n,0}$ whose only peaks occur at heights congruent to h modulo ℓ is of the form $U^h Q D^{\ell-h}_{\ell-1}$, where Q is a member of $\mathcal{D}^{\ell,h}_{n,k}$ whose only peaks occur at heights congruent to 0 modulo ℓ. By Lemma 5.1, Q could be any of $M^{\ell-1}_{n-1,\ell-h-1}(\vec{x}, \vec{1})$ possible paths. Hence there are also $M^{\ell-1}_{n-1,\ell-h-1}(\vec{x}, \vec{1})$ possibilities for P. ■

6 ℓ-ary Trees

Previously, we have proven relations between higher-order Motzkin paths and other lattice paths. Now, we prove a relation between higher-order Motzkin paths and another well-studied class of combinatorial objects, ℓ-ary trees. A **rooted tree** is a planar tree with a single distinguished **root** vertex. If an edge exists between two vertices, the vertex closer to the root vertex is called the **parent**, and the vertex farther from the root vertex is called the **child**. The **outdegree** $|v|$ of a vertex v is the number of children v has.

For any $\ell \in \mathbb{N}$, an ***ℓ*-ary tree** is a rooted tree in which every vertex has an outdegree of at most ℓ, and each vertex's children are ordered from "left" to "right." The set of all ℓ-ary trees with n edges is denoted $\mathcal{T}^\ell_n$, with cardinality $|\mathcal{T}^\ell_n| = T^\ell_n$. An ℓ-ary tree is called **complete** if every vertex has outdegree 0 or ℓ. The set of all complete ℓ-ary trees with n edges is denoted $\mathcal{K}^\ell_n$, with cardinality $|\mathcal{K}^\ell_n| = K^\ell_n$. An example of 2-ary (or more commonly, "binary") trees is shown in Figure 14.

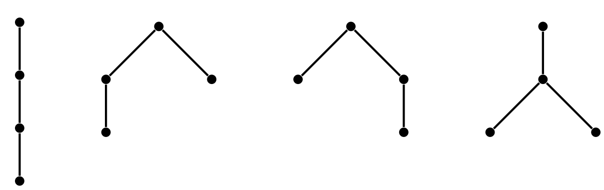


Figure 14: All 2-ary, or binary, trees with 3 edges. We draw rooted trees with their roots at the top.

The following two results are well-established. See Aigner [1] and Hilton and Pedersen [7] for proofs of these standard results.

Proposition 6.1. *Let $\ell \geq 2$ and $n \geq 0$. Then $T^2_n = M_{n,0}$ and $K^\ell_n = C^\ell_n$.*

In our final theorem we show injective maps between ℓ-ary trees for any $\ell \geq 2$ and order-$(\ell-1)$ Motzkin paths.

Theorem 6.2. *Let $\ell \geq 2$ and $S \subseteq \{0, 1, 2, \dots, \ell\}$ be a set of allowable outdegrees for vertex in ℓ-ary trees* *with* $0, \ell \in S$*. Let* $\vec{x} = \langle x_0, x_1, \ldots, x_{\ell-2} \rangle$ *such that*

$$x_i = \begin{cases} 1 & \text{if } \; i+1 \in S, \\ 0 & \text{if } \; i+1 \notin S. \end{cases}$$

Then $M_{n,0}^{\ell-1}(\vec{x}, \vec{x})$ *counts* ℓ*-ary trees with* n *edges and every vertex of outdegree in* S*.*

Proof. Let Y be the subset of $\mathcal{T}_{n+1}^{\ell}$ whose members' vertices have outdegrees in S. We describe an injection $\sigma : Y \to \mathcal{M}_{n,0}^{\ell-1}(\vec{x}, \vec{x})$. For a tree $G \in Y$, start at the end of the Motzkin path $(n, 0)$ and perform a pre-order traversal of G. At each vertex v visited, prepend a U step to the path if $|v| = 0$, or a $D_{|v|-1}$ step if $|v| > 0$. Stop when the path reaches $(0, 0)$ at the last pre-order vertex of G. See Figure 15 for an example. Note that since every vertex in G has outdegree in S, the only D_i steps in $\sigma(G)$ will be for $i + 1 \in S$. Therefore $\sigma(G) \in M_{n,0}^{\ell-1}(\vec{x}, \vec{x})$. Since any two distinct trees will have corresponding vertices of different outdegrees at some point in a pre-order traversal, σ is clearly injective.

We also describe an injection $\rho : Y \to \mathcal{T}_{n+1}^{\ell}$. Given $P \in Y$, execute the following procedure to obtain the tree $\rho(P)$:

1. Begin at the first step of P. Create a root vertex.
2. If the end of P has been reached, then $\rho(P)$ is fully constructed, and the procedure ends here. Otherwise, consider the next step of P. Assign the current vertex v an outdegree

$$|v| := \begin{cases} 0 & \text{if next step} = U, \\ i+1 & \text{if next step} = D_i. \end{cases}$$

   This is the number of children that v will have when the tree is fully constructed.
3. Backtrack through the tree to the nearest vertex that has fewer children than its assigned outdegree. (This could be the current vertex.)
4. Add a child to the vertex backtracked to. Return to step 2 starting at this child.

Since D_i cannot occur in P unless $i + 1 \in S$, every vertex in $\rho(P)$ will have an outdegree in S. And since ρ starts by creating a root vertex and then adding one addition vertex for every step in P, clearly $\rho(P)$ has $n + 1$ vertices and therefore n edges. ■

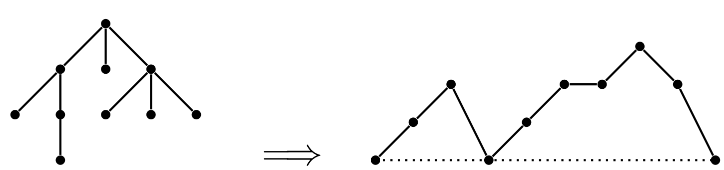


Figure 15: An example of the map σ from Theorem 6.2.

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A Tables of Values

We developed a Java program that calculates the $(\vec{x}, \vec{y})$-colored, higher-order Motzkin triangle for user-specified values of $\vec{x}$ and $\vec{y}$ using the recursive relation in Proposition 1.2. The following tables display the OEIS [14] entries that correspond the main column sequences of the triangles calculated. Our code is available upon request.

Table 2: The sequences formed by the of row sums of the (x, y)-colored, order-1 Motzkin triangle.

	$y=0$	$y=1$	$y=2$	$y=3$	$y=4$	$y=5$
$x=0$	$\binom{n}{\lfloor \frac{n}{2} \rfloor}$	A002426	A026641	A126952	-	-
$x=1$	A000079	A005773	A000984	A126568	A227081	-
$x=2$	A127358	A000244	$\binom{2n+1}{n+1}$	A026375	A133158	-
$x=3$	A127359	A126932	A000302	A026378	A081671	-
$x=4$	A127360	-	A141223	-	A005573	A098409
$x=5$	-	-	-	-	A000400	A122898

Table 3: Main column sequences of the $(\langle x_0, 0\rangle, \langle y_0, 0\rangle)$-colored, order-2 Motzkin triangle.

	$y_0=0$	$y_0=1$	$y_0=2$
$x_0=0$	-	-	-
$x_0=1$	A076227	A071879	-
$x_0=2$	-	-	-

Table 4: Main column sequences of the $(\langle x_0, 1\rangle, \langle y_0, 1\rangle)$-colored, order-2 Motzkin triangle.

	$y_0=0$	$y_0=1$	$y_0=2$
$x_0=0$	A001005	-	A303730
$x_0=1$	-	A036765	-
$x_0=2$	-	A159772	-

Table 5: Main column sequences of the $(\langle x_0, x_1\rangle, \langle 3, 3\rangle)$-colored, order-2 Motzkin triangle. Recall C_n^3 is the n^{th} 3-Catalan number.

	$\boldsymbol{x_1 = 0}$	$\boldsymbol{x_1 = 1}$	$\boldsymbol{x_1 = 2}$	$\boldsymbol{x_1 = 3}$	$\boldsymbol{x_1 = 4}$
$\boldsymbol{x_0 = 0}$	-	A089354	A023053	-	-
$\boldsymbol{x_0 = 1}$	-	-	C_n^3	A121545	-
$\boldsymbol{x_0 = 2}$	-	-	A098746	A006013	-
$\boldsymbol{x_0 = 3}$	-	-	-	C_{n+1}^3	-
$\boldsymbol{x_0 = 4}$	-	-	-	A047099	-

Table 6: Main column sequences of the $(\langle 1, x_1, x_2\rangle, \langle 4, 6, 4\rangle)$-colored, order-3 Motzkin triangle.

	$\boldsymbol{x_2 = 2}$	$\boldsymbol{x_2 = 3}$	$\boldsymbol{x_2 = 4}$	$\boldsymbol{x_2 = 5}$
$\boldsymbol{x_1 = 2}$	-	-	-	-
$\boldsymbol{x_1 = 3}$	-	A002293	-	-
$\boldsymbol{x_1 = 4}$	-	-	-	-
$\boldsymbol{x_1 = 5}$	-	-	-	-