Stabilization of Hamiltonian Systems with Multiplicative Noise

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Abstract

Noise-induced stabilization is the phenomenon where a system of ordinary differential equations is unstable, but by adding randomness, its corresponding system of stochastic differential equations is stable. It has been proven that unstable Hamiltonian systems cannot be stabilized by adding constant noise, where global stochastic boundedness is our notion of stability. In this study, we investigate adding nonconstant noise to two classes of Hamiltonian systems to achieve noise-induced stabilization. Our method for proving noise-induced stabilization consists of constructing local Lyapunov functions on various subsets of the plane, and then smoothing them together to form a global Lyapunov function defined on the entire plane. We also pursue the minimum noise necessary for stabilization of these systems.

1 Introduction

1.1 Background & Definitions

The deterministic setting that we are considering in this paper is that of Hamiltonian systems. To begin, we give the definition for Hamiltonian systems below.

Definition 1.1 (Hamiltonian System)**.** *A **Hamiltonian system** of differential equations is a first-order, autonomous system of two differential equations of the form*

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{\partial H}{\partial y}(X_t, Y_t) \\ \dfrac{dy(t)}{dt} = -\dfrac{\partial H}{\partial x}(X_t, Y_t) \end{cases}$$

where $H(x, y)$ is an infinitely differentiable function called the Hamiltonian function.

For any Hamiltonian system, the Hamiltonian function $H(x, y)$ is constant along every solution curve. This fact is illustrated below

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0.$$

Because the Hamiltonian function is constant along every solution curve, we know the deterministic behavior of its solution curves. This is significant for our main interest of noise-induced stabilization. *Noise-induced stabilization* is the phenomena where an unstable system of ODEs, after adding noise, has a corresponding stable system of SDEs. It is impossible for a Hamiltonian system to be stabilized by noise that is constant in space. Previous work [AGKK17], has proven stabilization of Hamiltonian systems with constant noise by also adding a small deterministic perturbation that breaks the Hamiltonian structure. In this work, we achieve noise-induced stabilization by adding non-constant noise (multiplicative noise) to the unstable Hamiltonian systems. This produces a system of SDEs shown here

$$\begin{aligned}\frac{dX(t)}{dt} &= b_1(X_t, Y_t) + \sigma_1(X_t, Y_t)\frac{dB_1(t)}{dt} \\ \frac{dY(t)}{dt} &= b_2(X_t, Y_t) + \sigma_2(X_t, Y_t)\frac{dB_2(t)}{dt}\end{aligned}$$

where $b_1 = \frac{\partial H}{\partial y}$ and $b_2 = -\frac{\partial H}{\partial x}$, σ_1 and σ_2 are non-constant noise coefficients, and B_1 and B_2 are independent Brownian motions.

While there are several different types of stability, we are interested in a notion of global stability or global stochastic boundedness. We give this definition of stability below.

Definition 1.2 (Stable)**.** *$(X(t), Y(t))$ is **stable** if for all initial conditions and all $\epsilon > 0$, there exists a bound M, such that $P(|(X(t), Y(t))| \leq M) > 1 - \epsilon$ for all t.*

Then, there is a well established theorem that the existence of a Lyapunov function implies the stability of an SDE [KM11]. We give the definition for a Lyapunov function below.

Definition 1.3 (Lyapunov Function)**.** *A function $V(x, y)$ is called a **Lyapunov function** if it satisfies the following three properties*

1. $V \in C^\infty(\mathcal{R})$,

2. $\lim_{r\to\infty} \left[\inf_{(x,y)\in(\mathcal{R}\cap B_r^c)} V(x, y) \right] = \infty$,

3. $\lim_{r\to\infty} \left[\sup_{(x,y)\in(\mathcal{R}\cap B_r^c)} (\mathcal{L}V)(x, y) \right] = -\infty$

where

$$\mathcal{L} = b_1(x,y)\frac{\partial}{\partial x} + b_2(x,y)\frac{\partial}{\partial y} + \frac{1}{2}\sigma_1^2(x,y)\frac{\partial^2}{\partial x^2} + \frac{1}{2}\sigma_2^2(x,y)\frac{\partial^2}{\partial y^2}$$

is the generator corresponding to the system of SDEs.

Now we can move onto a sketch of our method for proving the existence of such functions.

1.2 Proof Method

In sections 2 and 3, we examine two different Hamiltonian systems by first giving an introduction to each. In these introductions, we include a description of the Hamiltonian function, corresponding system, and deterministic solution curves. We then move onto the main theorem for each Hamiltonian and its corresponding proof.

We use the same proof method for both theorems, which is based upon the meta-algorithm described in [AKM$^+$12]. To begin, we section the plane into regions. In each of these regions, we have an associated local Lyapunov function, which is simply a function that meets the three criteria from Definition 1.3 on a subset of $\mathbb{R}^2$. On the sections of the plane that are non-overlapping, we define the function to be the local Lyapunov function that works only on that part of the plane. However, for the overlapping regions we describe the function as the convex combination of the two local Lyapunov functions that work on that region. We want the transition through the overlapping area to be smooth and so we use a mollifier function $\phi(t)$, defined below

$$\phi(t) = \frac{\int_{-\infty}^{t}\psi(s)ds}{\int_{-\infty}^{\infty}\psi(s)ds} \quad \text{where} \quad \psi(t) = \begin{cases} \exp\left(\frac{-1}{1-(2t-1)^2}\right) & \text{for } 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using $\phi(t)$ we patch the local Lyapunov regions together so that we can move smoothly from one local region to another, which allows us to construct our global Lyapunov function. Once this is complete, we can declare that the system is stable, completing our proof.

2 Hamiltonian System with $H(x,y) = cx^m y^n$

We consider the Hamiltonian function

$$H(x,y) = cx^m y^n$$

where c is some nonzero constant and m and n are integers greater than or equal to 2. The corresponding Hamiltonian system (x_t, y_t) is the solution to the following two-dimensional system of ODEs:

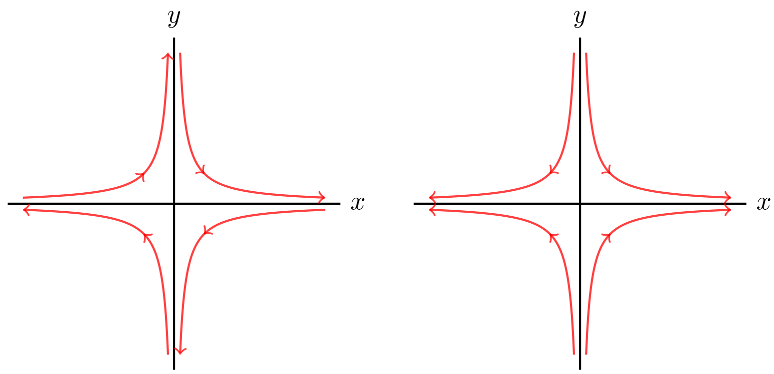

Figure 1: Phase portraits of the deterministic Hamiltonian system with $H(x, y) = cx^m y^n$, $c > 0$, and $m = n$ where m, n is even (left) and m, n is odd (right).

$$\frac{dx_t}{dt} = \frac{\partial H}{\partial y} = cnx_t^m y_t^{n-1}$$
$$\frac{dy_t}{dt} = -\frac{\partial H}{\partial x} = -cmx_t^{m-1} y_t^n.$$

We observe that both the x-axis and the y axis consist of a continuum of equilibrium points. Since the Hamiltonian function is constant along all solution curves, the solution has the property that

$$y_t = y_0 \left(\frac{x_t}{x_0}\right)^{-m/n}.$$

Some of the solution curves for this system are illustrated in Figure 1.

Theorem 2.1. *The Hamiltonian system corresponding to $H(x, y) = cx^m y^n$, where $c \neq 0$ and integers $m, n \geq 2$, is stabilized by noise coefficients*

$$\sigma_1 = \begin{cases} \epsilon_1 |x|^r & \text{for } |x| \geq 1 \\ \epsilon_1 & \text{for } |x| < 1 \end{cases} \text{ and } \sigma_2 = \begin{cases} \epsilon_2 |y|^s & \text{for } |y| \geq 1 \\ \epsilon_2 & \text{for } |y| < 1. \end{cases}$$

Figure 2 shows two simulations corresponding to the SDE referenced in Theorem 2.1, with $c = \epsilon_1 = \epsilon_2 = 1$ and two different values of $m = n$. The figure shown has been zoomed in about the origin so more detail could be shown. With an initial condition of $(1, 1)$, the process for both of these simulations are bounded, so the process was stable in these simulations. Due to the non-degenerate noise, there was a quasi-periodic motion of the process as the solution curve traveled throughout all quadrants.

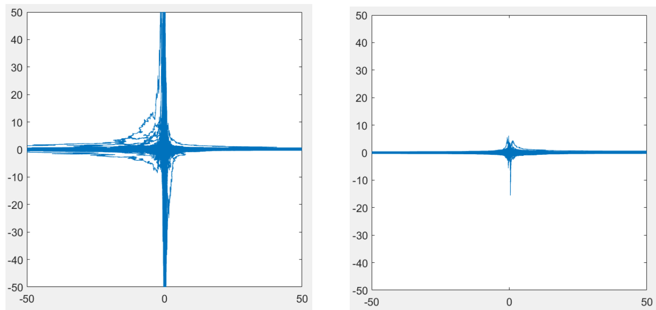

Figure 2: Simulation of SDE from Theorem 2.1 with $m = n = 2$ (left) and $m = n = 3$ (right).

We prove this theorem with the following lemmas and the resulting global Lyapunov function. We begin with local Lyapunov functions that cover different portions of the plane, and later patch those areas together after defining their overlap regions.

Lemma 2.2. *There exists $a_1 > 0$ such that $V_1 = |x|^{mp}$ is a local Lyapunov function on the region $\mathcal{R}_1 = \{(x, y) : |y| \leq 2a$ and $|x| \geq 1\}$ where $0 < p < min\{\frac{1}{m}, \frac{1}{n}\}$ and $0 < a < a_1$.*

Proof. First, note that V_1 meets the first and second conditions for a Lyapunov function in $\mathcal{R}_1$, depicted in Figure 3. For the third Lyapunov condition we apply the generator in to get

$$
\begin{aligned}
\mathcal{L}V_1 &= cnmp|x|^{mp-1}\text{sgn}(x)x^m y^{n-1} + \frac{1}{2}\epsilon_1^2|x|^{2r}mp(mp-1)|x|^{mp-2} \\
&= cnmp|x|^{mp}x^{m-1}y^{n-1} + \frac{1}{2}\epsilon_1^2|x|^{2r-2}mp(mp-1)|x|^{mp} \\
&= mp|x|^{mp}\left(cnx^{m-1}y^{n-1} + \frac{1}{2}\epsilon_1^2|x|^{2r-2}(mp-1)\right) \\
&\leq mp|x|^{mp}\left(n|c||x|^{m-1}(2a)^{n-1} + \frac{1}{2}\epsilon_1^2|x|^{2r-2}(mp-1)\right).
\end{aligned}
$$

Thus, $\mathcal{L}V_1 \to -\infty$ as $|x| \to \infty$ whenever $2r - 2 > m - 1 \implies r > \frac{m+1}{2}$ since $mp - 1 < 0$. When $r = \frac{m+1}{2}$, and hence $2r - 2 = m - 1$, $\mathcal{L}V_1$ still converges to $-\infty$ as $|x| \to \infty$ as long as a is sufficiently small. Therefore, there exists $a_1 > 0$ such that $\mathcal{L}V_1$ is a local Lypaunov function on $\mathcal{R}_1$ for any $a < a_1$. $\square$

Lemma 2.3. *There exists $a_2 > 0$ such that $V_2 = |y|^{np}$ is a local Lyapunov function on the region $\mathcal{R}_2 = \{(x, y) : |x| \leq 2a$ and $|y| \geq 1\}$ where $0 < p < min\{\frac{1}{m}, \frac{1}{n}\}$ and $0 < a < a_2$.*

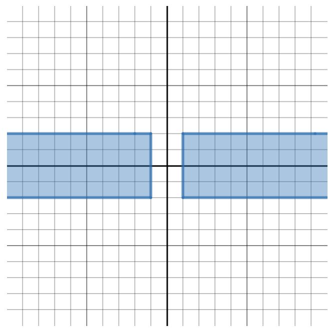

Figure 3: The region, $\mathcal{R}_1$, where V_1 works as a local Lyapunov function

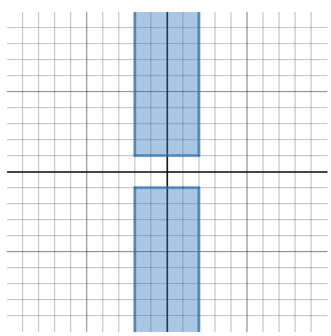

Figure 4: The region, $\mathcal{R}_2$, where V_2 works as a local Lyapunov function

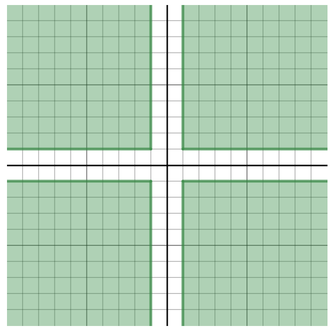

Figure 5: The region, $\mathcal{R}_3$, where V_3 works as a local Lyapunov function

Proof. First, note that V_2 meets the first and second conditions for a Lyapunov function in $\mathcal{R}_2$, depicted in Figure 4. For the third Lyapunov condition we apply the generator to get

$$\begin{aligned}\mathcal{L}V_2 &= -cmnp|y|^{np-1}\text{sgn}(y)x^{m-1}y^n + \frac{1}{2}\epsilon_2^2|y|^{2s}np(np-1)|y|^{np-2} \\ &= -cmnp|y|^{np}x^{m-1}y^{n-1} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}np(np-1)|y|^{np} \\ &= np|y|^{np}(-cmx^{m-1}y^{n-1} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(np-1)) \\ &\leq np|y|^{np}(m|c|(2a)^{m-1}|y|^{n-1} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(np-1))\end{aligned}$$

Thus, $\mathcal{L}V_2 \to -\infty$ as $|y| \to \infty$ whenever $2s-2 > n-1 \implies s > \frac{n+1}{2}$ since $np-1<0$. When $s = \frac{n+1}{2}$, and hence $2s-2 = n-1$, $\mathcal{L}V_2$ still converges to $-\infty$ as $|y| \to \infty$ as long as a is sufficiently small. Therefore, there exists $a_2 > 0$ such that $\mathcal{L}V_2$ is a local Lypaunov function on $\mathcal{R}_2$ for any $a < a_2$. □

Lemma 2.4. *$V_3 = |x|^{mp}|y|^{np}$ is a local Lyapunov function on the region $\mathcal{R}_3 = \{(x,y) : |y| \geq a$ and $|x| \geq a\}$ where $0 < p < min\{\frac{1}{m}, \frac{1}{n}\}$ and $0 < a < 1$.*

Proof. First, note that V_3 meets the first and second conditions for a Lyapunov function in $\mathcal{R}_3$, depicted in Figure 5. For the third Lyapunov condition we apply the generator to get

$$
\begin{aligned}
\mathcal{L}v_3 &= cnmpx^m y^{n-1}|x|^{mp-1}|y|^{np}\text{sgn}(x) - cnpmx^{m-1}y^n|x|^{mp}|y|^{np-1}\text{sgn}(y) \\
&+ \frac{1}{2}\sigma_1^2 mp(mp-1)|x|^{mp-2}|y|^{np} + \frac{1}{2}\sigma_2^2 np(np-1)|x|^{mp}|y|^{np-2} \\
&= |x|^{mp}|y|^{np}\left(cnmpx^{m-1}y^{n-1} - cnpmx^{m-1}y^{n-1}\right) \\
&+ |x|^{mp}|y|^{np}\left(\frac{1}{2}\sigma_1^2 mp(mp-1)\frac{1}{x^2} + \frac{1}{2}\sigma_2^2 np(np-1)\frac{1}{y^2}\right) \\
&= |x|^{mp}|y|^{np}\left(\frac{1}{2}\sigma_1^2 mp(mp-1)\frac{1}{x^2} + \frac{1}{2}\sigma_2^2 np(np-1)\frac{1}{y^2}\right)
\end{aligned}
$$

We now consider 3 cases of different noise coefficients that will vary over the region $\mathcal{R}_3$.

Case 1: $\{(x,y) : |x| \geq 1, |y| \geq 1\} \implies \sigma_1 = \epsilon_1|x|^r,\ \sigma_2 = \epsilon_2|y|^s$

$$
\begin{aligned}
\mathcal{L}V_3 &= |x|^{mp}|y|^{np}\left(\frac{1}{2}\sigma_1^2 mp(mp-1)\frac{1}{x^2} + \frac{1}{2}\sigma_2^2 np(np-1)\frac{1}{y^2}\right) \\
&= |x|^{mp}|y|^{np}\left(\frac{1}{2}\epsilon_1^2 mp(mp-1)|x|^{2r-2} + \frac{1}{2}\epsilon_2^2 np(np-1)|y|^{2s-2}\right)
\end{aligned}
$$

This will go to negative infinity as $|(x,y)| \to \infty$ in the region $\{(x,y) : |x| \geq 1, |y| \geq 1\}$.

Case 2: $\{(x,y) : a \leq |x| < 1, |y| \geq 1\} \implies \sigma_1 = \epsilon_1,\ \sigma_2 = \epsilon_2|y|^s$

$$
\begin{aligned}
\mathcal{L}V_3 &= |x|^{mp}|y|^{np}\left(\frac{1}{2}\sigma_1^2 mp(mp-1)\frac{1}{x^2} + \frac{1}{2}\sigma_2^2 np(np-1)\frac{1}{y^2}\right) \\
&= |x|^{mp}|y|^{np}\left(\frac{1}{2}\epsilon_1^2 mp(mp-1)\frac{1}{x^2} + \frac{1}{2}\epsilon_2^2 np(np-1)|y|^{2s-2}\right) \\
&\leq |y|^{np}\left(\frac{1}{2}\epsilon_1^2 mp(mp-1)a^{mp-2} + \frac{1}{2}\epsilon_2^2 np(np-1)(2a)^{mp}|y|^{2s-2}\right)
\end{aligned}
$$

This will go to negative infinity as $|(x,y)| \to \infty$ in the region $\{(x,y) : a \leq |x| < 1, |y| \geq 1\}$.

Case 3: $\{(x,y) : |x| \geq 1, a \leq |y| < 1\} \implies \sigma_1 = \epsilon_1|x|^r,\ \sigma_2 = \epsilon_2$

$$
\begin{aligned}
\mathcal{L}V_3 &= |x|^{mp}|y|^{np}\left(\frac{1}{2}\sigma_1^2 mp(mp-1)\frac{1}{x^2} + \frac{1}{2}\sigma_2^2 np(np-1)\frac{1}{y^2}\right) \\
&= |x|^{mp}|y|^{np}\left(\frac{1}{2}\epsilon_1^2 mp(mp-1)|x|^{2r-2} + \frac{1}{2}\epsilon_2^2 np(np-1)\frac{1}{y^2}\right) \\
&\leq |x|^{mp}\left(\frac{1}{2}\epsilon_1^2 mp(mp-1)|x|^{2r-2}(2a)^{np} + \frac{1}{2}\epsilon_2^2 np(np-1)a^{np-2}\right)
\end{aligned}
$$

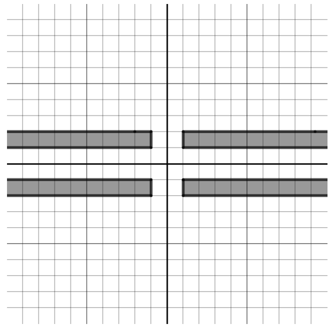

Figure 6: The region, $\mathcal{R}_1 \cap \mathcal{R}_3$, where V_{13} works as a local Lyapunov function

This will go to negative infinity as $|(x, y)| \to \infty$ in the region $\{(x, y) : |x| \geq 1, a \leq |y| < 1\}$.
From the above cases, we know that $\mathcal{L}V_3$ will go to negative infinity over the parts of $\mathcal{R}_3$ that go to infinity □

We utilize a local Lyapunov function that works over the overlap regions (shown in Figure 6) between the regions shown in Lemma 2.2 and Lemma 2.4. This local Lyapunov function is a convex combination of the local Lyapunov functions of Lemma 2.2 and Lemma 2.4. This convex combination allows us to smoothly transition from one local region to another.

Lemma 2.5. *There exists* $0 < a_{13} < 1$ *such that* $V_{13} = \phi V_1 + (1 - \phi)V_3$ *is a local Lyapunov function on* $\mathcal{R}_1 \cap \mathcal{R}_3$ *where* $0 < a < a_{13}$*,* $\phi = \phi(\omega_{13}(x, y))$ *is a smooth function from* $[0, 1]$*, and*

$$\omega_{13}(x, y) = \frac{2a - |y|}{a}.$$

Proof. Since the first two Lyapunov conditions hold for V_1 and V_3, the first two conditions hold for V_{13}. Then, applying the generator we get

$$\begin{aligned}
\mathcal{L}V_{13} &= \mathcal{L}[\phi V_1 + (1 - \phi)V_3] \\
&= \phi \mathcal{L}V_1 + (1 - \phi)\mathcal{L}V_3 + (V_1 - V_3)\mathcal{L}\phi \\
&+ \sigma_1^2 \frac{\partial \phi}{\partial x} \frac{\partial (V_1 - V_3)}{\partial x} + \sigma_2^2 \frac{\partial \phi}{\partial y} \frac{\partial (V_1 - V_3)}{\partial y}
\end{aligned}$$

For the first two terms we have

$$\begin{aligned}\phi\mathcal{L}v_1 + (1-\phi)\mathcal{L}v_3 &= \phi|x|^{mp}(cmnpy^{n-1}x^{m-1} + \frac{1}{2}\epsilon_1^2|x|^{2r-2}mp(mp-1)) \\ &+ (1-\phi)\frac{1}{2}|x|^{mp}(\epsilon_1^2 mp(mp-1)|x|^{2r-2}|y|^{np} + \epsilon_2^2 np(np-1)\frac{1}{y^2}|y|^{np}) \\ &\le \phi|x|^{mp}\left(|c|mnp|x|^{m-1}|y|^{n-1} + \frac{1}{2}\epsilon_1^2|x|^{2r-2}mp(mp-1)\right) \\ &+ (1-\phi)\frac{1}{2}|x|^{mp}(\epsilon_1^2 mp(mp-1)|x|^{2r-2}|y|^{np} + \epsilon_2^2 np(np-1)|y|^{np-2})\end{aligned}$$

Since $a \le |y| \le 2a$, we get

$$\begin{aligned}&\le \phi|x|^{mp}\left(|c|mnp|x|^{m-1}(2a)^{n-1} + \frac{1}{2}\epsilon_1^2|x|^{2r-2}mp(mp-1)\right) \\ &+ (1-\phi)\frac{1}{2}|x|^{mp}(\epsilon_1^2 mp(mp-1)|x|^{2r-2}a^{np} + \epsilon_2^2 np(np-1)|2a|^{np-2}) \\ &\le |x|^{mp}(|c|mnp|x|^{m-1}(2a)^{n-1}\phi + \frac{1}{2}\epsilon_1^2|x|^{2r-2}mp(mp-1)(\phi + a^{np}(1-\phi))\end{aligned}$$

Then for $(V_1 - V_3)\mathcal{L}\phi_{13}$ we have

$$(v_1 - v_3)\mathcal{L}\phi_{13} = (|x|^{mp} - |x|^{mp}|y|^{np})\left(-cmx^{m-1}y^n\phi'\frac{-\text{sgn}(y)}{a} + \frac{1}{2}\epsilon_2^2\phi''\left(\frac{-1}{a}\right)^2\right)$$

We can say that ϕ', $\phi'' \le M$ for some positive real number M.

$$\begin{aligned}&\le |x|^{mp}(m|c||x|^{m-1}|y|^n\frac{M}{a} + \frac{1}{2}\epsilon_2^2\frac{M}{a^2}) + |x|^{mp}(m|c||x|^{m-1}|y|^{n+np}\frac{M}{a} - \frac{1}{2}\epsilon_2^2\frac{M}{a^2}|y|^{np}) \\ &\le |x|^{mp}(m|c||x|^{m-1}|2a|^n\frac{M}{a} + \frac{1}{2}\epsilon_2^2\frac{M}{a^2}) + |x|^{mp}(m|c||x|^{m-1}|a|^{n+np}\frac{M}{a} - \frac{1}{2}\epsilon_2^2|a|^{np}\frac{M}{a^2}) \\ &= |x|^{mp}(m|c||x|^{m-1}\frac{M}{a}((2a)^n + a^{n+np}) + \frac{1}{2}\epsilon_2^2\frac{M}{a^2}(1 - a^{np}))\end{aligned}$$

Now, for the final two terms we have

$$\begin{aligned}&\sigma_1^2\frac{\partial\phi}{\partial x}\frac{\partial(V_1 - V_3)}{\partial x} + \sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_1 - V_3)}{\partial y} \\ &= -\epsilon_2^2\phi'\frac{d}{dy}np|x|^{mp}|y|^{np-1}\text{sgn}(y) \\ &= -\epsilon_2^2\phi'(\frac{-\text{sgn}(y)}{a})np|x|^{mp}|y|^{np-1}\text{sgn}(y) \\ &\le \epsilon_2^2\phi'(\frac{np}{a})|x|^{mp}(2a)^{np-1}\end{aligned}$$

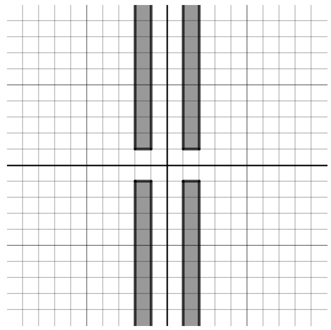

Figure 7: The region, $\mathcal{R}_2 \cap \mathcal{R}_3$, where V_{23} works as a local Lyapunov function

When $2r-2 > m-1$, the magnitude of the first two terms in $\mathcal{L}V_{13}$ go to infinity at a faster rate than all the other terms. Since these first two terms are negative, it follows that V_{13} goes to negative infinity when the generator is applied.

When $2r-2 = m-1$, $\mathcal{L}V_{13}$ is less than or equal to a function which is asymptotic to

$$|x|^{mp+m-1}(|c|mnp(2a)^{n-1}\phi + \frac{1}{2}\epsilon_1^2 mp(mp-1)(\phi + a^{np}(1-\phi)) + m|c|\frac{M}{a}((2a)^n + a^{n+np})).$$

Because the positive terms in the above expression contain the constant a to the $n-1$ power or higher, we can choose a sufficiently small so that that magnitude of the positive terms is always smaller than the magnitude of the negative term. Thus, there exists $0 < a_{13} < 1$ such that V_{13} meets the third condition for a Lyapunov function for any $0 < a < a_{13}$. □

We utilize another local Lyapunov function that works over the overlap regions (shown in Figure 7) between the regions shown in Lemma 2.3 and Lemma 2.4. This local Lyapunov function is a convex combination of the local Lyapunov functions of Lemma 2.3 and Lemma 2.4.

Lemma 2.6. *There exists* $0 < a_0 < 1$ *s.t.* $V_{23} = \phi V_2 + (1-\phi)V_3$ *is a local Lyapunov function on* $\mathcal{R}_2 \cap \mathcal{R}_3$ *where* $0 < a < 2a < a_0$*,* $\phi = \phi(\omega_{23}(x,y))$ *is a smooth function from* $[0,1]$*, and*

$$\omega_{23}(x,y) = \frac{2a - |y|}{a}$$

.

Proof. Since the first two Lyapunov conditions hold for V_2 and V_3, the first two conditions hold for V_{23}. Then, applying the generator we get

$$\begin{aligned}\mathcal{L}V_{23} &= \mathcal{L}[\phi V_2 + (1-\phi)V_3] \\ &= \phi\mathcal{L}V_2 + (1-\phi)\mathcal{L}V_3 + (V_2 - V_3)\mathcal{L}\phi \\ &+ \sigma_1^2 \frac{\partial\phi}{\partial x}\frac{\partial(V_2 - V_3)}{\partial x} + \sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_2 - V_3)}{\partial y}\end{aligned}$$

For the first two terms we have

$$\begin{aligned}\phi\mathcal{L}v_2 + (1-\phi)\mathcal{L}v_3 &= \phi|y|^{np}\left(-cmnpy^{n-1}x^{m-1} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}np(np-1)\right) \\ &+ (1-\phi)\frac{1}{2}|y|^{np}(\epsilon_1^2 mp(mp-1)|x|^{mp}\frac{1}{x^2} + \epsilon_2^2 np(np-1)|x|^{mp}|y|^{2s-2}) \\ &\leq \phi|y|^{np}\left(|c|mnp|y|^{n-1}|x|^{m-1} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}np(np-1)\right) \\ &+ (1-\phi)\frac{1}{2}|y|^{np}(\epsilon_1^2 mp(mp-1)|x|^{mp}\frac{1}{x^2} + \epsilon_2^2 np(np-1)|x|^{mp}|y|^{2s-2})\end{aligned}$$

We know $a \leq |x| \leq 2a$

$$\begin{aligned}&\leq |y|^{np}(|c|mnpa^{m-1}|y|^{n-1}\phi + \frac{1}{2}\epsilon_2^2|y|^{2s-2}np(np-1)) \\ &+ (1-\phi)|y|^{np}(\frac{1}{2}\epsilon_1^2(2a)^{mp-2}mp(mp-1) + \frac{1}{2}\epsilon_2^2|y|^{2s-2}a^{mp}np(np-1)) \\ &\leq |y|^{np}(|c|mnpa^{m-1}|y|^{n-1}\phi + \frac{1}{2}\epsilon_2^2|y|^{2s-2}np(np-1)(\phi + a^{mp}(1-\phi))) \\ &\simeq \frac{1}{2}\epsilon_2^2 np(np-1)|y|^{np}|y|^{2s-2}(\phi + a^{mp}(1-\phi))\end{aligned}$$

Then for $(V_2 - V_3)\mathcal{L}\phi_{23}$ we have

$$(v_2 - v_3)\mathcal{L}\phi_{23} = (|y|^{np} - |x|^{mp}|y|^{np})\left(cnx^m y^{n-1}\phi'\frac{-\text{sgn}(x)}{a} + \frac{1}{2}\epsilon_1^2\phi''\left(\frac{-1}{a}\right)^2\right)$$

we know ϕ' and ϕ'' are bounded, let them be bounded by M

$$\begin{aligned}&\leq |y|^{np}(n|c||x|^m|y|^{n-1}\frac{M}{a} + \frac{1}{2}\epsilon_1^2\frac{M}{a^2}) \\ &+ |y|^{np}(n|c||x|^{m+mp}|y|^{n-1}\frac{M}{a} - \frac{1}{2}\epsilon_1^2|x|^{mp}\frac{M}{a^2}) \\ &\leq |y|^{np}(n|c|a^m|y|^{n-1}\frac{M}{a} + \frac{1}{2}\epsilon_1^2\frac{M}{a^2}) \\ &+ |y|^{np}(n|c|(2a)^{m+mp}|y|^{n-1}\frac{M}{a} - \frac{1}{2}\epsilon_1^2 a^{mp}\frac{M}{a^2}) \\ &= |y|^{np}(n|c||y|^{n-1}\frac{M}{a}(a^m + (2a)^{m+mp}) + \frac{1}{2}\epsilon_1^2\frac{M}{a^2}(1-a^{mp}))\end{aligned}$$

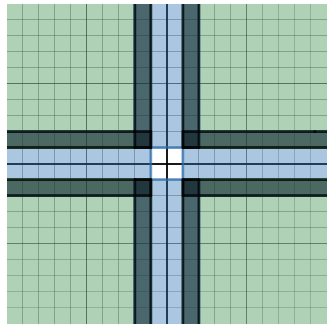

Figure 8: The region, $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, which depicts the region where $\overline{V}(x, y)$ is defined

Now, for the final two terms we have

$$\begin{aligned}
&\epsilon_1^2\sigma_1^2\frac{\partial\phi}{\partial x}\frac{\partial(V_2 - V_3)}{\partial x} + \epsilon_2^2\sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_2 - V_3)}{\partial y} \\
&= \epsilon_1^2\phi'\frac{dr}{dx}(-mp|x|^{mp-1}\text{sgn}(x)|y|^{np}) \\
&= \epsilon_1^2\phi'(\frac{-\text{sgn}(x)}{a})(-mp|x|^{mp-1}\text{sgn}(x)|y|^{np}) \\
&\leq \epsilon_1^2\phi'(\frac{mp}{a})|2a|^{mp-1}|y|^{np}
\end{aligned}$$

When $2s - 2 > n - 1$, the magnitude of the first two terms in $\mathcal{L}V_{23}$ go to infinity at a faster rate than all the other terms. Since these first two terms are negative, it follows that V_{23} goes to negative infinity when the generator is applied.

When $2s - 2 = n - 1$, $\mathcal{L}V_{23}$ is less than or equal to a function which is asymptotic to

$$|y|^{np+n-1}(|c|mnp(2a)^{m-1}\phi + \frac{1}{2}\epsilon_2^2 np(np-1)(\phi + a^{mp}(1-\phi)) + n|c|\frac{M}{a}((2a)^m + a^{m+mp})).$$

Because the positive terms in the above expression contain the constant a to the $m-1$ power or higher, we can choose a sufficiently small so that that magnitude of the positive terms is always smaller than the magnitude of the negative term. Thus, there exists $0 < a_{23} < 1$ such that V_{23} meets the third condition for a Lyapunov function for any $0 < a < a_{23}$. □

From the above lemmas, we construct a global Lyapunov function $V(x, y)$ that works over the entire plane; i.e., it satisfies all three criteria for being a Lyapunov function. For the final choice of a in the regions depicted in Figure 8, we choose $a < \min\{a_1, a_2, a_{13}, a_{23}\}$. We then define

$$V(x,y) = \begin{cases} \overline{V}(x,y) & \text{for } x^2 + y^2 > 4 \\ \text{arbitrary positive and smooth} & \text{for } x^2 + y^2 \leq 4 \end{cases}$$

where $\overline{V}(x,y)$ is a piecewise Lyapunov function that works over the parts of the plane, depicted in Figure 8, that go to infinity, given by

$$\overline{V}(x,y) = \begin{cases} V_1(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3^c \\ V_2(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \cap \mathcal{R}_3^c \\ V_3(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2^c \cap \mathcal{R}_3 \\ V_{13}(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3 \\ V_{23}(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \cap \mathcal{R}_3. \end{cases}$$

3 Hamiltonian System with $H(x,y) = cy^n e^{\alpha x^m}$

We consider the Hamiltonian function

$$H(x,y) = cy^n e^{\alpha x^m}$$

where c is some nonzero constant, α is positive, m and n are integers greater than 2, and m is even. The corresponding Hamiltonian system (x_t, y_t) is the solution to the following two-dimensional system of ODEs:

$$\frac{dx_t}{dt} = \frac{\partial H}{\partial y} = cny_t^{n-1} e^{\alpha x_t^m}$$
$$\frac{dy_t}{dt} = -\frac{\partial H}{\partial x} = -\alpha cm x_t^{m-1} y_t^n e^{\alpha x_t^m}.$$

We observe that the x-axis consists of a continuum of equilibium points. Since the Hamiltonian function is constant along all solution curves, the solution has the property that

$$y_t = y_0 e^{-\frac{\alpha x_t^m}{n}}.$$

Some of the solution curves for this system are illustrated in Figure 9.

Theorem 3.1. *The Hamiltonian system corresponding to $H(x,y) = cy^n e^{\alpha x^m}$, where $c \neq 0$, $\alpha > 0$, integers $m, n \geq 2$, and m is even, is stabilized by noise coefficients*

$$\sigma_1 = \epsilon_1 e^{\beta x^m} \text{ with } \beta > \frac{\alpha}{2} \tag{1}$$

$$\sigma_2 = \begin{cases} \epsilon_2 |y|^s & \text{for } |y| \geq 1 \text{ with } s > \frac{(n-1)\alpha}{2(2\beta-\alpha)} + \frac{n+1}{2} \\ \epsilon_2 & \text{for } |y| < 1 \end{cases} \tag{2}$$

for any $\epsilon_1, \epsilon_2 \neq 0$.

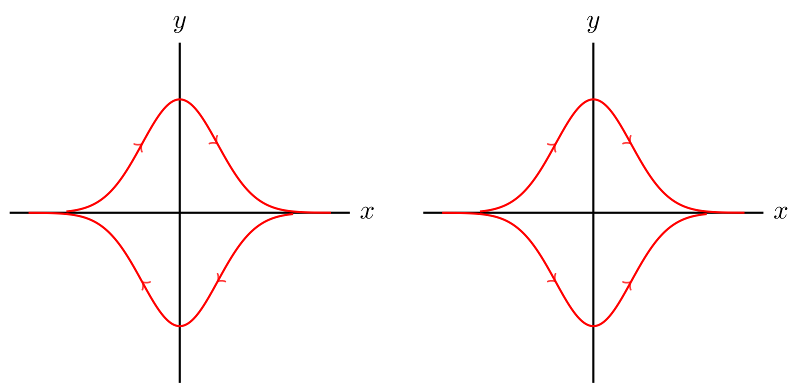

Figure 9: Phase portraits of the deterministic Hamiltonian system with $H(x,y) = cy^n e^{\alpha x^m}$, $c > 0$, and m even where n is even (left) and n is odd (right).

We prove this theorem with the following lemmas and the resulting global Lyapunov function. We begin with local Lyapunov functions that cover different portions of the plane and patch those areas together after defining their overlap regions.

Lemma 3.2. *$V_1 = |x|^p$ is a local Lyapunov function on the region $\mathcal{R}_1 = \{(x,y) : |y| \leq 2a$ and $|x| \geq 1\}$ where $0 < p < 1$ and $a > 0$.*

Proof. First, note that V_1 meets the first and second Lyapunov conditions on $\mathcal{R}_1$, depicted in Figure 3. For the third Lyapunov condition we apply the generator to get

$$\begin{aligned}
\mathcal{L}V_1 &= cnp|x|^{p-1}\text{sgn}(x)y^{n-1}e^{\alpha x^m} + \frac{1}{2}\epsilon_1^2 e^{2\beta x^m} p(p-1)|x|^{p-2} \\
&= cnp|x|^p \frac{y^{n-1}}{x} e^{\alpha x^m} + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m} p(p-1)|x|^p \\
&= p|x|^p \big(cn\frac{y^{n-1}}{x} e^{\alpha x^m} + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)\big) \\
&\leq p|x|^p \big(|c|n\frac{(2a)^{n-1}}{x} e^{\alpha x^m} + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)\big)
\end{aligned}$$

Thus, $\mathcal{L}V_1 \to -\infty$ as $|x| \to \infty$ whenever $2\beta > \alpha$ since $p - 1 < 0$. □

Lemma 3.3. *$V_2 = |y|^p$ is a local Lyapunov function on the region $\mathcal{R}_2 = \{(x,y) : |x| \leq 2a$ and $|y| \geq 1\}$ where $0 < p < 1$ and $a > 0$.*

Proof. First, note that V_2 meets the first and second Lyapunov conditions on $\mathcal{R}_2$, depicted in Figure 4. For the third Lyapunov condition we apply the generator to get

$$
\begin{aligned}
\mathcal{L}V_2 &= -\alpha cmp|y|^{p-1}\text{sgn}(y)x^{m-1}y^n e^{\alpha x^m} + \frac{1}{2}\epsilon_2^2|y|^{2s}p(p-1)|y|^{p-2} \\
&= -\alpha cmp|y|^p x^{m-1}y^{n-1}e^{\alpha x^m} + \frac{1}{2y^2}\epsilon_2^2|y|^{2s}p(p-1)|y|^p \\
&= p|y|^p\big(-\alpha cmx^{m-1}y^{n-1}e^{\alpha x^m} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big) \\
&\le p|y|^p\big(-\alpha|c|mx^{m-1}y^{n-1}e^{\alpha x^m} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big) \\
&\le p|y|^p\big(-\alpha|c|m(2a)^{m-1}y^{n-1}e^{\alpha x^m} + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)
\end{aligned}
$$

Thus, $\mathcal{L}V_2 \to -\infty$ as $|y| \to \infty$ whenever $2s-2 > n-1 \implies s > \frac{n+1}{2}$ since $p-1<0$. ☐

Lemma 3.4. *$V_3 = |x|^p|y|^p$ is a local Lyapunov function on the region $\mathcal{R}_3 = \{(x,y) : |y| \ge a$ and $|x| \ge a\}$ where $0<p<1$ and $a>0$.*

Proof. First, note that V_3 meets the first and second Lyapunov conditions on $\mathcal{R}_3$, depicted in Figure 5. For the third Lyapunov condition we apply the generator and consider two cases
Case 1 ($|y| \ge 1$):

$$
\begin{aligned}
\mathcal{L}V_3 &= cny^{n-1}e^{\alpha x^m}p|x|^{p-1}\text{sgn}(x)|y|^p - \alpha cmx^{m-1}y^n e^{\alpha x^m}|x|^p p|y|^{p-1}\text{sgn}(y) \\
&+ \frac{1}{2}\epsilon_1^2 e^{2\beta x^m}p(p-1)|x|^{p-2}|y|^p + \frac{1}{2}\epsilon_2^2|y|^{2s}|x|^p p(p-1)|y|^{p-2} \\
&= cn\frac{y^{n-1}}{x}e^{\alpha x^m}p|x|^p|y|^p - \alpha cmx^{m-1}y^{n-1}e^{\alpha x^m}|x|^p p|y|^p \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}p(p-1)|x|^p|y|^p + \frac{1}{2}\epsilon_2^2|y|^{2s-2}|x|^p p(p-1)|y|^p \\
&= p|x|^p|y|^p\big(cn\frac{y^{n-1}}{x}e^{\alpha x^m} - \alpha cmx^{m-1}y^{n-1}e^{\alpha x^m} \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big) \\
&\le p|x|^p|y|^p\big(|c|n\frac{y^{n-1}}{x}e^{\alpha x^m} - \alpha|c|mx^{m-1}y^{n-1}e^{\alpha x^m} \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big) \\
&\le p|x|^p|y|^p\big(|c|n\frac{y^{n-1}}{x}e^{\alpha x^m} - \alpha|c|mx^{m-1}y^{n-1}e^{\alpha x^m} \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)
\end{aligned}
$$

When $|y|^{n-1}e^{\alpha x^m} << e^{2\beta x^m}$, the dominant term in $\mathcal{L}V_3$ is $\frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)$, which conver This inequality holds for $\mathcal{R}_3$ when $\beta > \frac{\alpha}{2}$ and $s > \frac{(n-1)\alpha}{2(2\beta-\alpha)} + \frac{n+1}{2}$.

Case 2 ($|y| < 1$):

$$
\begin{aligned}
\mathcal{L}V_3 &= cny^{n-1}e^{\alpha x^m}p|x|^{p-1}\text{sgn}(x)|y|^p - \alpha cmx^{m-1}y^n e^{\alpha x^m}|x|^p p|y|^{p-1}\text{sgn}(y) \\
&+ \frac{1}{2}\epsilon_1^2 e^{2\beta x^m}p(p-1)|x|^{p-2}|y|^p + \frac{1}{2}\epsilon_2^2|x|^p p(p-1)|y|^{p-2} \\
&= cn\frac{y^{n-1}}{x}e^{\alpha x^m}p|x|^p|y|^p - \alpha cmx^{m-1}y^{n-1}e^{\alpha x^m}|x|^p p|y|^p \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}p(p-1)|x|^p|y|^p + \frac{1}{2y^2}\epsilon_2^2|x|^p p(p-1)|y|^p \\
&= p|x|^p|y|^p\big(cn\frac{y^{n-1}}{x}e^{\alpha x^m} - \alpha cmx^{m-1}y^{n-1}e^{\alpha x^m} \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2y^2}\epsilon_2^2(p-1)\big) \\
&\leq p|x|^p|y|^p\big(|c|n\frac{y^{n-1}}{x}e^{\alpha x^m} - \alpha|c|mx^{m-1}y^{n-1}e^{\alpha x^m} \\
&+ \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2y^2}\epsilon_2^2(p-1)\big) \\
&\simeq \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)
\end{aligned}
$$

Thus $\mathcal{L}V_3 \to -\infty$ as $|(x,y)| \to \infty$ in $\mathcal{R}_3$ since $\beta > \frac{\alpha}{2}$ and $p - 1 < 0$. □

We utilize a local Lyapunov function that works over the overlap regions (shown in Figure 6) between the regions shown in Lemma 3.2 and Lemma 3.4. This local Lyapunov function is a convex combination of the local Lyapunov functions of Lemma 3.2 and Lemma 3.4. This convex combination allows us to smoothly transition from one local region to another.

Lemma 3.5. *$V_{13} = \phi V_1 + (1-\phi)V_3$ is a local Lyapunov function on $\mathcal{R}_1 \cap \mathcal{R}_3 = \{(x,y) : |x| \geq 1$ and $a \leq |y| \leq 2a\}$ where $a > 0$, $\phi = \phi(\omega_{13}(x,y))$ is a smooth function from $[0,1]$, and*

$$\omega_{13}(x,y) = \frac{2a - |y|}{a}.$$

Proof. Since the first two Lyapunov conditions hold for V_1 and V_3, the first two conditions hold for V_{13}. Then, applying the generator we get

$$
\begin{aligned}
\mathcal{L}V_{13} &= \mathcal{L}[\phi V_1 + (1-\phi)V_3] \\
&= \phi\mathcal{L}V_1 + (1-\phi)\mathcal{L}V_3 + (V_1 - V_3)\mathcal{L}\phi \\
&+ \sigma_1^2\frac{\partial\phi}{\partial x}\frac{\partial(V_1 - V_3)}{\partial x} + \sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_1 - V_3)}{\partial y}
\end{aligned}
$$

For the first two terms we have

$$
\begin{aligned}
\phi\mathcal{L}V_1 + (1-\phi)\mathcal{L}V_3 = {} & \phi p|x|^p\big(cn\frac{y^{n-1}}{x}e^{\alpha x^m} + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)\big) \\
& + (1-\phi)p|x|^p|y|^p\big(cn\frac{y^{n-1}}{x}e^{\alpha x^m} - \alpha cmx^{m-1}y^{n-1}e^{\alpha x^m} \\
& + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)
\end{aligned}
$$

We know $a \le |y| \le 2a$

$$
\begin{aligned}
& \le \phi p|x|^p\big(|c|n\frac{(2a)^{n-1}}{x}e^{\alpha x^m} + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)\big) \\
& + (1-\phi)p|x|^p a^p\big(|c|n\frac{(2a)^{n-1}}{x}e^{\alpha x^m} - \alpha|c|mx^{m-1}a^{n-1}e^{\alpha x^m} \\
& + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1) + \frac{1}{2}\epsilon_2^2 a^{2s-2}(p-1)\big) \\
& = p|x|^p\big(|c|n\frac{(2a)^{n-1}}{x}e^{\alpha x^m}\phi + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)\phi \\
& + |c|n\frac{(2a)^{n-1}}{x}a^p e^{\alpha x^m}(1-\phi) - \alpha|c|mx^{m-1}a^{p+n-1}e^{\alpha x^m}(1-\phi) \\
& + \frac{1}{2x^2}a^p\epsilon_1^2 e^{2\beta x^m}(p-1)(1-\phi) + \frac{1}{2}\epsilon_2^2 a^{p+2s-4}(p-1)(1-\phi)\big) \\
& \le p|x|^p\Bigg(|c|n\frac{(2a)^{n-1}}{x}e^{\alpha x^m}\big(\phi + a^p(1-\phi)\big) \\
& - \alpha|c|mx^{m-1}a^{p+n-1}e^{\alpha x^m}(1-\phi) \\
& + \frac{1}{2x^2}\epsilon_1^2 e^{2\beta x^m}(p-1)\big(\phi + a^p(1-\phi)\big)\Bigg) \\
& \simeq \frac{1}{2x^2}\epsilon_1^2 p(p-1)|x|^p e^{2\beta x^m}\big(\phi + a^p(1-\phi)\big)
\end{aligned}
$$

Then for the third term we have

$$
\begin{aligned}
\mathcal{L}\phi_{13} & = (|x|^p - |x|^p|y|^p)\big(-\alpha cmx^{m-1}y^n e^{\alpha x^m}\phi'(\frac{-1}{a}) + \frac{1}{2}\epsilon_2^2\phi''|y|^{2s}(\frac{-1}{a})^2\big) \\
& = (|x|^p - |x|^p|y|^p)(\frac{\alpha cmx^{m-1}y^n e^{\alpha x^m}\phi'}{a} + \frac{\epsilon_2^2\phi''|y|^{2s}}{2a^2})
\end{aligned}
$$

We can say that ϕ', $\phi'' \le M$ for some positive real number M.

$$\begin{aligned}
&\le (|x|^p - |x|^p|y|^p)(\frac{\alpha cm|x|^{m-1}|y|^n e^{\alpha x^m} M}{a} + \frac{\epsilon_2^2 M|y|^{2s}}{2a^2}) \\
&\le (|x|^p - |x|^p a^p)(\frac{\alpha|c|m|x|^{m-1}(2a)^n e^{\alpha x^m} M}{a} + \frac{\epsilon_2^2 M|2a|^{2s}}{2a^2}) \\
&= (1 - a^p)(\frac{\alpha|c|m|x|^{p+m-1}(2a)^n e^{\alpha x^m} M}{a} + \frac{\epsilon_2^2 M|x|^p|2a|^{2s}}{2a^2})
\end{aligned}$$

Finally for the last two terms we have

$$\begin{aligned}
\epsilon_1^2\sigma_1^2\frac{\partial\phi}{\partial x}\frac{\partial(V_1 - V_3)}{\partial x} + \epsilon_2^2\sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_1 - V_3)}{\partial y} &= -p\epsilon_2^2|y|^{2s}\phi'\frac{dr}{dy}|x|^p|y|^{p-1}\text{sgn}(y) \\
&= -p\epsilon_2^2|y|^{2s}\phi'\frac{-1}{a}|x|^p|y|^{p-1}\text{sgn}(y) \\
&= \epsilon_2^2 y|y|^{2s}\phi'\frac{p}{a}|x|^p|y|^{p-2} \\
&\le \epsilon_2^2(2a)^{2s+1}\phi'\frac{p}{a}|x|^p|2a|^{p-2}
\end{aligned}$$

The first two terms in the convex combination are larger in magnitude when the generator is applied. Hence, these these terms determine the behavior at infinity. Because the first two terms go to negative infinity, it follows that V_{13} goes to negative infinity when the generator is applied. Thus, V_{13} meets the third condition for a Lyapunov function. □

We utilize another local Lyapunov function that works over the overlap regions (shown in Figure 7) between the regions shown in Lemma 3.3 and Lemma 3.4. This local Lyapunov function is a convex combination of the local Lyapunov functions of Lemma 3.3 and Lemma 3.4. This convex combination allows us to smoothly transition from one local region to another.

Lemma 3.6. *$V_{23} = \phi V_2 + (1-\phi)V_3$ is a local Lyapunov function on $\mathcal{R}_2 \cap \mathcal{R}_3 = \{(x,y) : |x| \ge 1$ and $a \le |y| \le 2a\}$ where $a > 0$, $\phi = \phi(\omega_{23}(x,y))$ is a smooth function from $[0,1]$, and*

$$\omega_{23}(x,y) = \frac{2a - |x|}{a}.$$

Proof. Since the first two Lyapunov conditions hold for V_2 and V_3, the first two conditions hold for V_{23}. Then, applying the generator we get

$$\begin{aligned}
\mathcal{L}V_{23} &= \mathcal{L}[\phi V_2 + (1-\phi)V_3] \\
&= \phi\mathcal{L}V_2 + (1-\phi)\mathcal{L}V_3 + (V_2 - V_3)\mathcal{L}\phi \\
&+ \sigma_1^2\frac{\partial\phi}{\partial x}\frac{\partial(V_2 - V_3)}{\partial x} + \sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_2 - V_3)}{\partial y}
\end{aligned}$$

For the first two terms we have

$$\begin{aligned}\phi\mathcal{L}V_2+(1-\phi)\mathcal{L}V_3 &= \phi p|y|^p\big(-\alpha cmx^{m-1}y^{n-1}e^{\alpha x^m}+\frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)\\ &+(1-\phi)p|x|^p|y|^p\big(cn\frac{y^{n-1}}{x}e^{\alpha x^m}-\alpha cmx^{m-1}y^{n-1}e^{\alpha x^m}\\ &+\frac{1}{2x^2}\epsilon_1^2e^{2\beta x^m}(p-1)+\frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)\end{aligned}$$

We know $a\leq|x|\leq 2a$

$$\begin{aligned}&\leq \phi p|y|^p\big(-\alpha|c|ma^{m-1}y^{n-1}e^{\alpha a^m}+\frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)\\ &+(1-\phi)pa^p|y|^p\big(|c|n\frac{y^{n-1}}{2a}e^{\alpha(2a)^m}-\alpha|c|ma^{m-1}y^{n-1}e^{\alpha a^m}\\ &+\frac{1}{2a^2}\epsilon_1^2e^{2\beta a^m}(p-1)+\frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big)\\ &=p|y|^p\big(-\alpha|c|ma^{m-1}y^{n-1}e^{\alpha a^m}\phi+\frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\phi\\ &+|c|n\frac{y^{n-1}}{2a}a^pe^{\alpha(2a)^m}(1-\phi)-\alpha|c|ma^{p+m-1}y^{n-1}e^{\alpha a^m}(1-\phi)\\ &+\frac{1}{2}a^{p-2}\epsilon_1^2e^{2\beta a^m}(p-1)(1-\phi)+\frac{1}{2}a^p\epsilon_2^2|y|^{2s-2}(p-1)(1-\phi)\big)\\ &\leq p|y|^p\Bigg(-\alpha|c|ma^{m-1}y^{n-1}e^{\alpha a^m}\big(\phi+a^p(1-\phi)\big)\\ &+|c|n\frac{y^{n-1}}{2}a^{p-1}e^{\alpha(2a)^2}(1-\phi)\\ &+\frac{1}{2}\epsilon_2^2|y|^{2s-2}(p-1)\big(\phi+a^p(1-\phi)\big)\Bigg)\\ &\simeq\frac{1}{2}\epsilon_2^2p(p-1)|y|^{p+2s-2}\big(\phi+a^p(1-\phi)\big)\end{aligned}$$

Then for the third term we have

$$\begin{aligned}\mathcal{L}\phi_{23} &= (|y|^p-|x|^p|y|^p)\big(cny^{n-1}e^{\alpha x^m}\phi'(\frac{-1}{a})+\frac{1}{2}\epsilon_1^2\phi''e^{2\beta x^m}(\frac{-1}{a})^2\big)\\ &=(|y|^p-|x|^p|y|^p)(\frac{-cny^{n-1}e^{\alpha x^m}\phi'}{a}+\frac{\epsilon_1^2\phi''e^{2\beta x^m}}{2a^2})\end{aligned}$$

We can say that ϕ', $\phi''\leq M$ for some positive real number M.

$$
\begin{aligned}
&\leq (|y|^p - |x|^p|y|^p)(\frac{cn|y|^{n-1}e^{\alpha x^m}M}{a} + \frac{\epsilon_1^2 M e^{2\beta x^m}}{2a^2}) \\
&\leq (|y|^p - a^p|y|^p)(\frac{|c|n|y|^{n-1}e^{\alpha(2a)^m}M}{a} + \frac{\epsilon_1^2 M e^{2\beta(2a)^m}}{2a^2}) \\
&= (1-a^p)(\frac{|c|n|y|^{p+n-1}e^{\alpha(2a)^m}M}{a} + \frac{\epsilon_1^2 M|y|^p e^{2\beta(2a)^m}}{2a^2})
\end{aligned}
$$

Finally for the last two terms we have

$$
\begin{aligned}
\epsilon_1^2\sigma_1^2\frac{\partial\phi}{\partial x}\frac{\partial(V_2-V_3)}{\partial x} + \epsilon_2^2\sigma_2^2\frac{\partial\phi}{\partial y}\frac{\partial(V_2-V_3)}{\partial y} &= -p\epsilon_2^2 e^{2\beta x^m}\phi'\frac{dr}{dy}|x|^{p-1}|y|^p\mathrm{sgn}(x) \\
&= -p\epsilon_1^2 e^{2\beta x^m}\phi'\frac{-1}{a}|x|^{p-1}|y|^p\mathrm{sgn}(x) \\
&= \epsilon_1^2 x e^{2\beta x^m}\phi'\frac{p}{a}|x|^{p-2}|y|^p \\
&\leq \epsilon_1^2 2e^{2\beta(2a)^m}\phi' p|2a|^{p-2}|y|^p
\end{aligned}
$$

The first two terms in the convex combination are larger in magnitude when the generator is applied. Hence, these these terms determine the behavior at infinity. Because the first two terms go to negative infinity, it follows that V_{23} goes to negative infinity when the generator is applied. Thus, V_{23} meets the third condition for a Lyapunov function. □

From the above lemmas, we construct a global Lyapunov function $V(x,y)$ that works over the entire plane, it satisfies all criteria for being a Lyapunov function.

$$
V(x,y) = \begin{cases} \overline{V}(x,y) & \text{for } x^2+y^2 > 4 \\ \text{arbitrary positive and smooth} & \text{for } x^2+y^2 \leq 4 \end{cases}
$$

where $\overline{V}(x,y)$ is a piecewise Lyapunov function that works over the parts of the plane that go to infinity, given by

$$
\overline{V}(x,y) = \begin{cases} V_1(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3^c \\ V_2(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \cap \mathcal{R}_3^c \\ V_3(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2^c \cap \mathcal{R}_3 \\ V_{13}(x,y) & \text{for } (x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2^c \cap \mathcal{R}_3 \\ V_{23}(x,y) & \text{for } (x,y) \in \mathcal{R}_1^c \cap \mathcal{R}_2 \cap \mathcal{R}_3 \end{cases}
$$

The region where $\overline{V}(x,y)$ is defined is shown in figure 8.

4 Conclusion

In this paper we show that two different classes of Hamiltonian systems exhibit noise-induced stabilization with the addition of multiplicative noise. We utilize Lyapunov functions to rigorously prove the stability of the stochastic differential equations that correspond to our chosen unstable Hamiltonian systems. The approach we use use in this paper demonstrates a systematic process that might be applied to stabilize other unstable classes of Hamiltonian systems.

The minimum amount of multiplicative noise required for noise-induced stabilization for one dimensional systems of SDEs was explored in [AGKK17]. However, the minimum amount of multiplicative noise needed to stabilize two dimensional systems such as the Hamiltonian systems discussed in this paper, remains unanswered. This would be an interesting area for further investigation.

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