An analogue of direct and skew sum of permutations, chain permutational posets to words

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Introduction

In this presentation, I will describe a way to construct direct sum and skew sum of words in a similar method they are defined for permutations. Here I discuss some familiar statistics for example four fundamental Statistics lb,rb,ls,rs by Wachs and White towards direct and skew sum of words along with along with lsg,rsg. An analogue to chain permutational poset is defined. Examples of such posets like divisor posets for any positive integer n, etc. are given.

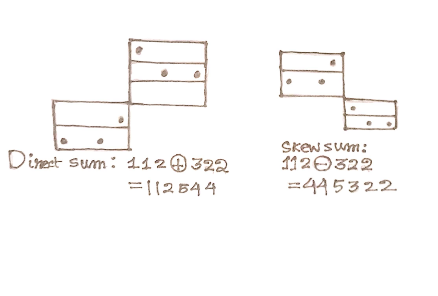
Definition

Direct and skew sum

Let σ, τ be two words of length k, l respectively. Then define

i $\sigma \oplus \tau$=$\sigma(i), i \leqslant k$,
$= |\sigma|+\tau(i-k), k+1 \leqslant i \leqslant k+l$.
where $|\sigma|$ is the largest digit of σ. For example:
$112 \oplus 322 = 112544$

ii $\sigma \ominus \tau$=$\sigma(i) + |\tau|, i \leqslant k$,
$= \tau(i-k), k+1 \leqslant i \leqslant k+l$,
For example:
$112 \ominus 322 = 445322$ as in the following diagram.



Remarks So, for any two words π, σ, we have,

i lb($\sigma \oplus \tau$)=lb(σ)+lb(τ)

ii ls($\sigma \oplus \tau$)=ls(σ)+lb(τ)+mn, where m is the length of τ and n is the number of distinct digits of σ.

iii rb($\sigma \oplus \tau$) = rb(σ)+rb(τ)+kr, where k is the length of σ and r is the number of distinct digits of τ.

iv rs($\sigma \oplus \tau$) = rs(σ)+rs(τ).

v lb($\sigma \ominus \tau$)=lb(σ)+lb(τ) +mn

vi ls($\sigma \ominus \tau$)= ls(σ)+ls(τ)

vii rb($\sigma \ominus \tau$)=rb(σ)+rb(τ)

viii rs($\sigma \oplus \tau$)=rs(σ)+rs(τ)+kr.

ix rev(rev(σ) $\ominus$ rev(τ)) =$\tau \oplus \sigma$, as in permutations of finite lengths.

Middle direct, skew sum

Definition

Let σ, τ be two words of length k, l respectively. Then define words of length $k+l+1$ as

i $\sigma \vee \tau$=$\sigma(i)$, if $\quad i \leqslant k$,
$= |\sigma| + |\tau| + 1$, if $\quad i = k+1$
$= |\sigma| + \tau(i-k-1), k+2 \leqslant i \leqslant k+l+1$,
where $|\sigma|, |\tau|$, are the largest digits of σ, τ respectively. For example:
$112 \vee 322 = 1126544$

ii $\sigma \wedge \tau$=$\sigma(i) + |\tau|$, if $\quad i \leqslant k$,
$= |\sigma| + |\tau| + 1$, if $\quad i = k+1$
$= \tau(i-k-1), k+2 \leqslant i \leqslant k+l+1$. For example: $112 \wedge 322 = 4456322$

Lemma

It follows the same way as in [1], if σ, τ are two words that avoid 132, then $\sigma \wedge \tau$ avoid 132.

Lemma

Any non empty word π that avoids 132, if it's largest digit is one more than the sum of largest digits of the subwords to it's left and right, then there are unique words σ and τ so that $\pi = \sigma \wedge \tau$. Proof follows as in [1].

Note: if it's largest digit is not one more than the sum of largest digits of the subwords to it's left and right, then the result may not hold. For example: 443 8 2122 avoids 132. Though, it can't be expressed as $\sigma \wedge \tau$, on the other hand, 443 5 2122= 221 $\wedge$ 2212.

Definition

Let π be a word, and that we decompose π into a sequence of increasing runs, separated by the descents in the word. The statistic rsg(π) is the sum of the number of runs of π strictly to the right of each entry i of π which contain elements both larger and smaller than i.

Lemma

For any non-empty word π, let last(π) denote the last entry of π. If π is the empty permutation, then we set last(π) = 0. The statistics rsg and last satisfy the following relations: $rsg(\sigma \wedge \tau) = rsg(\sigma) + rsg(\tau) +$ $|\tau|$-last $|\tau|$ The proof follows as in [1].

Similar results as above with middle skew sum, lsg statistics, 231 avoiding words follow as well.

P-chain words

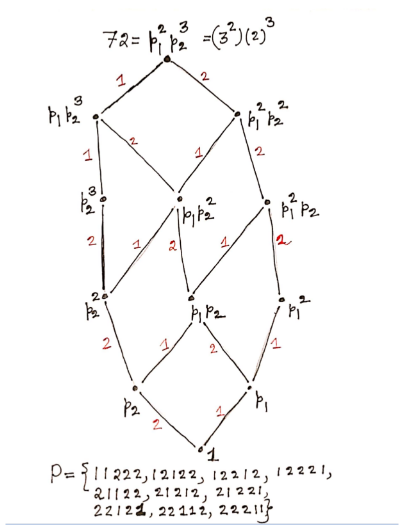
Motivated by the Chain permutational posets in [2] A P-chain word poset is defined.

Definition

A poset is called P-chain-word, where P is a set of words of length n. If it is possible to label the covering relations of P (i.e. the edges of the Hasse diagram of P) with numbers from I, 2, ... , n in such a way that along different maximal chains of P (whose length is necessarily n) the labels form different words from P, and every word in P arises in this manner.

Examples

i Let, n be a positive integer, with it's prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \cdots p_r$ are distinct primes. Let, $(D_n, \leqslant)$ be the poset, with D_n, the set of all positive integer divisor of $n, \forall a, b \in D_n, a \leqslant b$ iff a divides b,
In this case, D_n is P chain word, where P is the set of all words of length $\alpha = \sum_{i=1}^{r} \alpha_i$, with α_1 many 1's, α_2 many 2's $\cdots$ α_r many r's. (Because, when we label an edge in the covering relation, we label it by i if we take away the prime p_i. For example, the hass diagram of D_{72} is as follows)



ii **Definition**
[3]A restricted growth function (RGF) is a sequence $w = a_1 ... a_n$ of positive integers subject to the restrictions : $a_1 = 1$, for $i \geqslant 2$, $a_i \leqslant 1 + max\{a_1, ..., a_{i-1}\}$. There is a bijection between set partitions of $[n]$ and R.G.F.'s are described. For example the R.G.F of 1/23 is 122.
Consider the poset $(\Pi_n, \leqslant)$, where Π_n is the set of all partitions of the set $[n]$ and for any two partitions $\sigma, \tau, \sigma \leqslant \tau$, iff each block of σ is included in some block of τ.
Now lets adjoin another element say "." in the set Π_3 and extend the "$\leqslant$" as $x \leqslant ., \forall x \in \Pi_3$. This is a poset containing Π_3. Let's call it as $\Pi_{.,3}$.
Label the maximal chains by the R.G.F's of the corresponding set partition of [3] with 2 blocks present in that maximal chain. Then, $\Pi_{.,3}$ turns out to be a P-chain word poset, where P is the set of all R.G.F.'s of length 3 with 1 and 2's only as "." stays in the top of the Hass diagram of that of Π_3 and the top edge is always labeled by 1 as $a_1 = 1$ in an R.G.F.

Reference

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Restricted growth function patterns and statistics

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