Pattern Avoidance in Parking Functions

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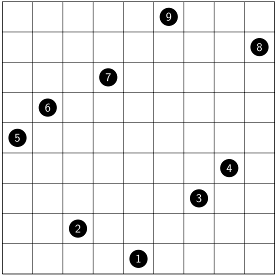
(joint work with Lara Pudwell)

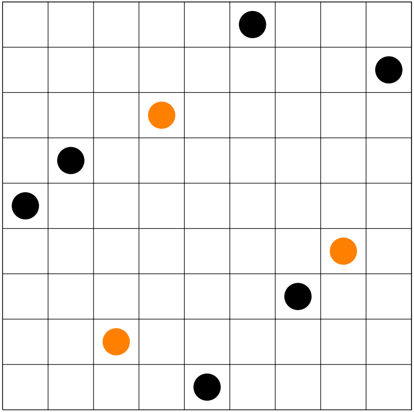
Permutation Patterns
June 20-24, 2022

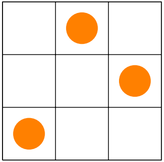
Definition

A permutation is a list where order matters.
$\mathcal{S}_n$ is the set of all permutations of $\{1, 2, \ldots, n\}$.

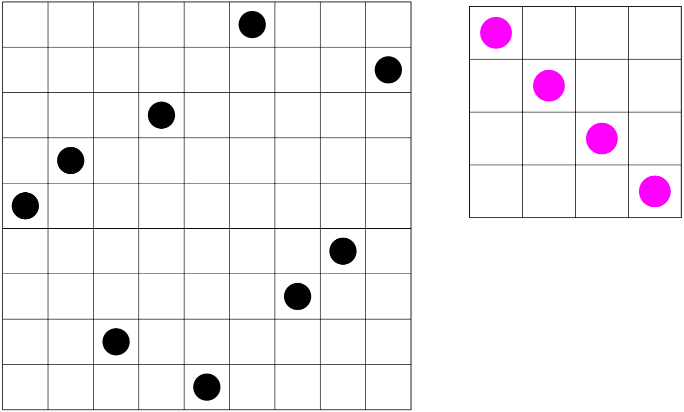


E.g. $\pi = 562719348$





562719348 contains the pattern 132



562719348 avoids the pattern 4321

Big question

How many permutations of length n contain the pattern ρ?

Or, alternatively...

Big question

How many permutations of length n avoid the pattern ρ?

(depends on what ρ is!)

Notation

$\mathcal{S}_n(\rho)$ is the set of permutations of length n *avoiding* ρ.

Definition

A parking function is a sequence $a_1 \cdots a_n \in [n]^n$ such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1 \cdots a_n$ then $b_i \leq i$ for all $1 \leq i \leq n$.

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11111, 12233, 12345

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Observations

- There are $(n+1)^{n-1}$ parking functions of size n.
- Every permutation of size n is a parking function of size n.

History/Motivation

Jelínek and Mansour (2009)

- Consider parking functions as words on $[n]^n$
- Determined all equivalence classes of patterns of length at most 5

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- Each Dyck path is associated with a permutation (many-to-one correspondence)
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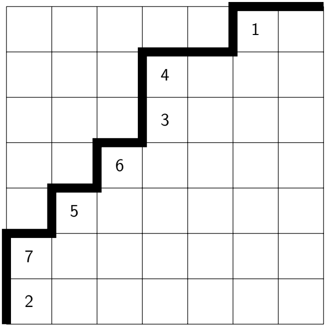
Current project:

- Follow Remmel and Qiu's definitions
- Count parking functions avoiding a subset of $\mathcal{S}_3$.

Parking function: 6144231

Blocks: $\{2,7\},\{5\},\{6\},\{3,4\},\emptyset,\{1\},\emptyset$

Dyck path:



Associated permutation: 2756341

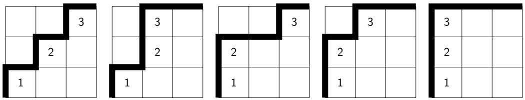
Warmup

Notation

Let $\mathrm{pf}_n(\rho)$ be the number of parking functions of size n whose associated permutations avoid ρ.

Proposition

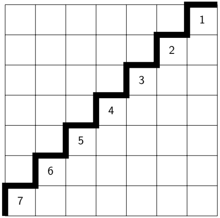
$\mathrm{pf}_n(21) = C_n$ (nth Catalan number)



Warmup

Proposition

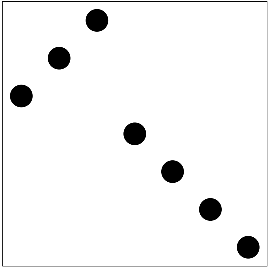
$\text{pf}_n(12) = 1$



Theorem

$$\mathrm{pf}_n(132, 213, 312) = \mathrm{pf}_n(213, 231, 312) = \frac{3(2n)!}{(n+2)!(n-1)!} = C_{n+1} - C_n$$

$\mathcal{S}_n(132, 213, 312)$



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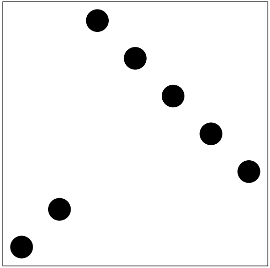
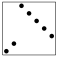
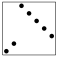


$$\mathcal{S}_n(213, 231, 312)$$ 

Let $a(n, k)$ be the number of size n parking functions whose associated permutation begins with $k - 1$ ascents.

- $a(n, 1) = 1$
- $a(n, n) = C_n$

$\mathcal{S}_n(213, 231, 312)$ 

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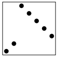
$\mathcal{S}_n(213, 231, 312)$ 

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Two cases:

1. Last block has one element ($a(n-1,k)$)
2. Last block is empty

Case 1? Deleting/reinserting last block (and standardizing) is bijection

$$\{1,2\}, \emptyset, \{7\}, \{6\}, \{5\}, \{4\}, \cancel{\{3\}} \leftrightarrow \{1,2\}, \emptyset, \{6\}, \{5\}, \{4\}, \{3\}$$

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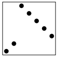

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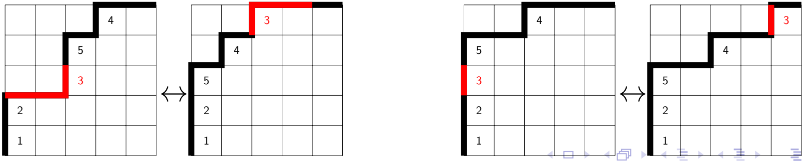
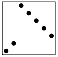
Case 2? Bijection via moving last element before decreasing run.



$\mathcal{S}_n(213, 231, 312)$ 

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In general:

$$a(n, k) = a(n-1, k) + a(n, k-1).$$

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Theorem

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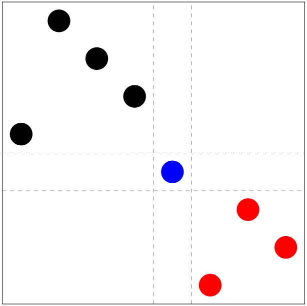

Encoding $\{123, 213\}$-avoiding parking functions:

- One dot per element
- Left paren at start of each interval.
- If corresponding right paren encloses one element, all numbers in blocks of size 1.
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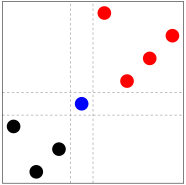
Can confirm via CAS that $f(n) = C_{n+1} - C_n$ matches initial conditions and satisfies recurrence.

Theorem

$$\mathrm{pf}_n(231, 321) = \frac{\binom{3n}{n}}{2n+1} \quad \text{(OEIS A001764)}$$

$\mathcal{S}_n(231, 321)$



$\frac{\binom{3n}{n}}{2n+1}$ counts

- ternary trees
- non-crossing trees

Strategy for $\mathrm{pf}_n(231, 321)$

1. bijection between Dyck paths and rooted ordered trees
2. bijection between parking functions and non-crossing trees via...
   - labeling Dyck paths
   - arranging tree vertices on circle

Labelling the Dyck path to avoid $\{231, 321\}$:

Characterization of $\{231, 321\}$-avoiding permutations

The digit d must be either first or second among the digits $\{d, d+1, \ldots, n\}$.

Two other results

Theorem

$\mathrm{pf}_n(231, 312, 321)$ *is equal to the number of dissections of a convex (n+2)-gon into triangles and quadrilaterals by non-intersecting diagonals.*

Corollary

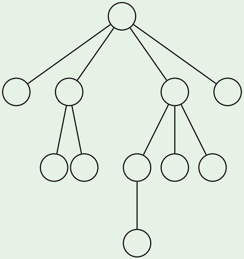
$$\mathrm{pf}_n(231, 312, 321) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n+1} \binom{2n-k}{n+k} \binom{n+k}{k}$$

Two other results

Theorem

$\mathrm{pf}_n(123, 132, 213)$ *is equal to the number of rooted ordered trees with* $n+1$ *edges such that every vertex is either a leaf or adjacent to a leaf (OEIS A143353).*

Example



corresponds to $\{9\}|\{7,8\}|\{5\}|\{6\}|\{4\}|\{2,3\}|\emptyset|\emptyset|\{1\}$.

Summary

Three cases:

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- $\mathrm{pf}_n(123, 213) = C_{n+1} - C_n$
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A few others:

- $\mathrm{pf}_n(123, 231) = \binom{n+1}{3} + \binom{n}{2} + 1$
- $\mathrm{pf}_n(132, 213, 321) = \binom{2n-1}{n}$
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Forthcoming:

results for avoiding any set of 2 or more patterns in $\mathcal{S}_3$

Patterns P	$\mathrm{pf}_n(P)$, $1 \leq n \leq 6$	OEIS
123, 132, 231	1, 3, 5, 7, 9, 11	A005408
123, 132, 312 123, 213, 231 123, 231, 312	1, 3, 6, 10, 15, 21	A000217
123, 213, 312	1, 3, 7, 13, 21, 31	A002061
123, 132, 213	1, 3, 6, 17, 43, 123	A143363
132, 213, 231 132, 231, 312	1, 3, 8, 22, 64, 196	A014138
132, 213, 312 213, 231, 312	1, 3, 9, 28, 90, 297	A000245
132, 231, 321	1, 3, 9, 29, 98, 342	A077587
132, 213, 321 132, 312, 321 213, 231, 321	1, 3, 10, 35, 126, 462	A001700
213, 312, 321	1, 3, 11, 41, 154, 582	A076540
231, 312, 321	1, 3, 10, 38, 154, 654	A001002

Patterns P	$\mathrm{pf}_n(P)$, $1 \leq n \leq 6$	OEIS
123, 231	1, 3, 8, 17, 31, 51	A105163
123, 312	1, 3, 9, 21, 41, 71	A064999
123, 132	1, 3, 8, 24, 75, 243	A000958
123, 213	1, 3, 9, 28, 90, 297	A000245
132, 231	1, 3, 10, 36, 137, 543	A002212
132, 213 132, 312 213, 231 231, 312	1, 3, 11, 45, 197, 903	A001003
132, 321	1, 3, 12, 52, 229, 1006	new
213, 321	1, 3, 13, 60, 275, 1238	new
213, 312	1, 3, 12, 54, 259, 1293	new
231, 321	1, 3, 12, 55, 273, 1428	A001764
312, 321	1, 3, 13, 63, 324, 1736	new

Pattern P	$\mathrm{pf}_n(P)$, $1 \leq n \leq 6$	OEIS
123	1, 3, 11, 48, 232, 1207	new (Remmel & Qiu)
132 231	1, 3, 13, 69, 417, 2759	A243688*
213 312	1, 3, 14, 81, 533, 3822	new
321	1, 3, 15, 97, 728, 6024	new

* "Number of Sylvester classes of 1-multiparking functions of length n."

References

- V. Jelínek and T. Mansour, Wilf-equivalence on k-ary words, compositions, and parking functions, *Electron. J. Combin.* **16** (2009), #R58, 9pp.
- J. Remmel and D. Qiu, Patterns in ordered set partitions and parking functions, Permutation Patterns 2016 (slides), available electronically at `https://www.math.ucsd.edu/~duqiu/files/PP16.pdf`.
- Richard Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 2001.
- The On-Line Encyclopedia of Integer Sequences at `oeis.org`.

References

- V. Jelínek and T. Mansour, Wilf-equivalence on k-ary words, compositions, and parking functions, *Electron. J. Combin.* **16** (2009), #R58, 9pp.
- J. Remmel and D. Qiu, Patterns in ordered set partitions and parking functions, Permutation Patterns 2016 (slides), available electronically at `https://www.math.ucsd.edu/~duqiu/files/PP16.pdf`.
- Richard Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 2001.
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Thanks for listening!

Theorem (Remmel, Qiu)

$$\mathrm{pf}_n(123) = \sum_{k=\frac{n}{2}}^{n} \frac{C_k}{n-k+1}\binom{n}{k}\binom{k}{n-k}$$