On the generating functions of pattern-avoiding Motzkin paths

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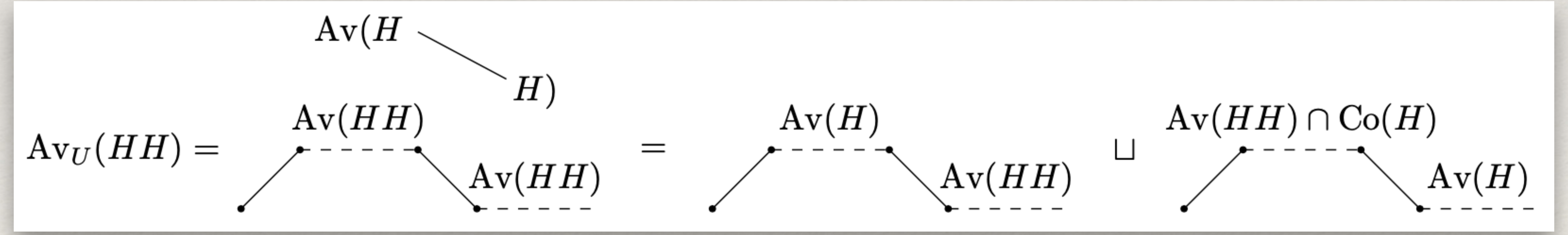
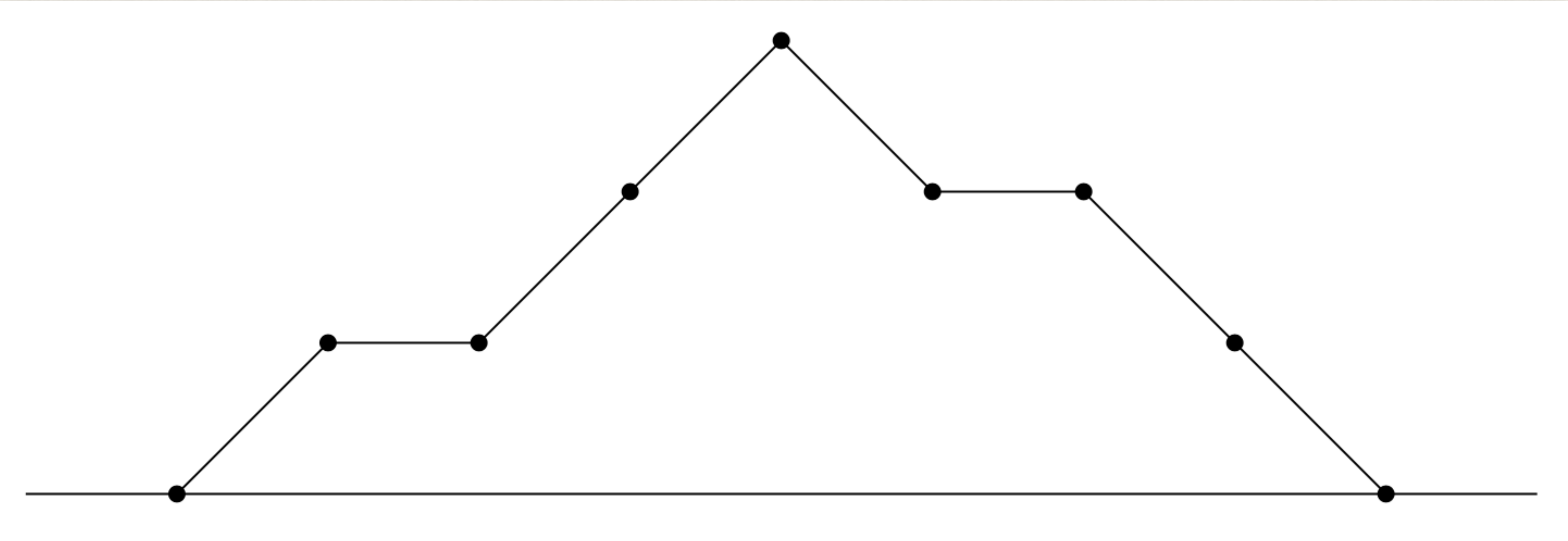


Motzkin paths

A *Motzkin path* of length n is a lattice path starting at $(0, 0)$ and ending at $(n, 0)$ consisting of *up steps* ($U = (1, 1)$), *down steps* ($D = (1, -1)$), and *horizontal steps* ($H = (1,0)$) that never goes below the x-axis.

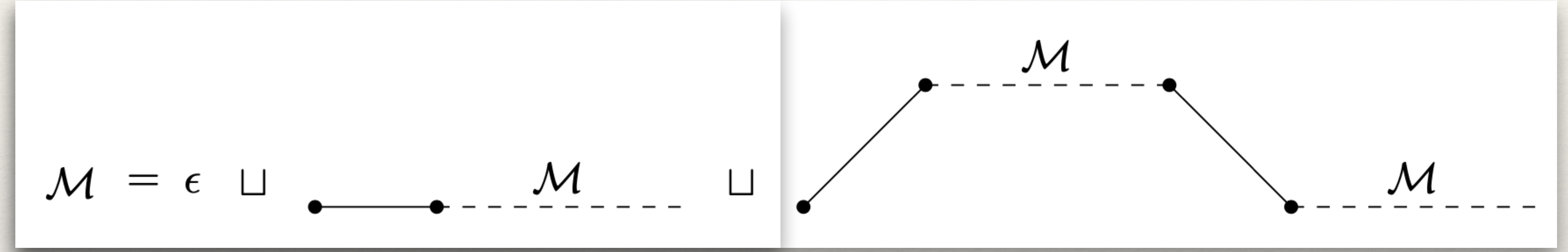


The Motzkin path *UHUUDHDD*.

Building Motzkin paths

Every Motzkin path beginning with H can be written as Hw for some Motzkin path w.

Every Motzkin path beginning with U can be written as $UxDy$ for some Motzkin paths x and y.



Let $M(x) = \sum_{n \geq 0} m_n x^n$ be the generating function for the number of length n Motzkin paths, then $M(x)$ satisfies $M(x) = 1 + xM(x) + x^2 M(x)^2$.

Patterns in Motzkin paths

A Motzkin path p contains a pattern q in $\{U, H, D\}^*$ if q occurs as a subword in p. If it does not contain q we say it avoids q.

For a set of patterns P we say a path avoids P if it avoids every pattern in P. If it does not avoid P we say it contains P.

Let $\mathscr{M}$ be the set of Motzkin paths, then define

$$Av(P) = \{p \in \mathscr{M} \mid p \text{ avoids } P\}$$

$$Co(P) = \{p \in \mathscr{M} \mid p \text{ contains } P\}$$

Dyck paths

The set $Av(H)$ is the set of Dyck paths which are counted by the Catalan numbers.

Every Dyck path starting with U can be written as $UxDy$ for some Dyck paths x and y.

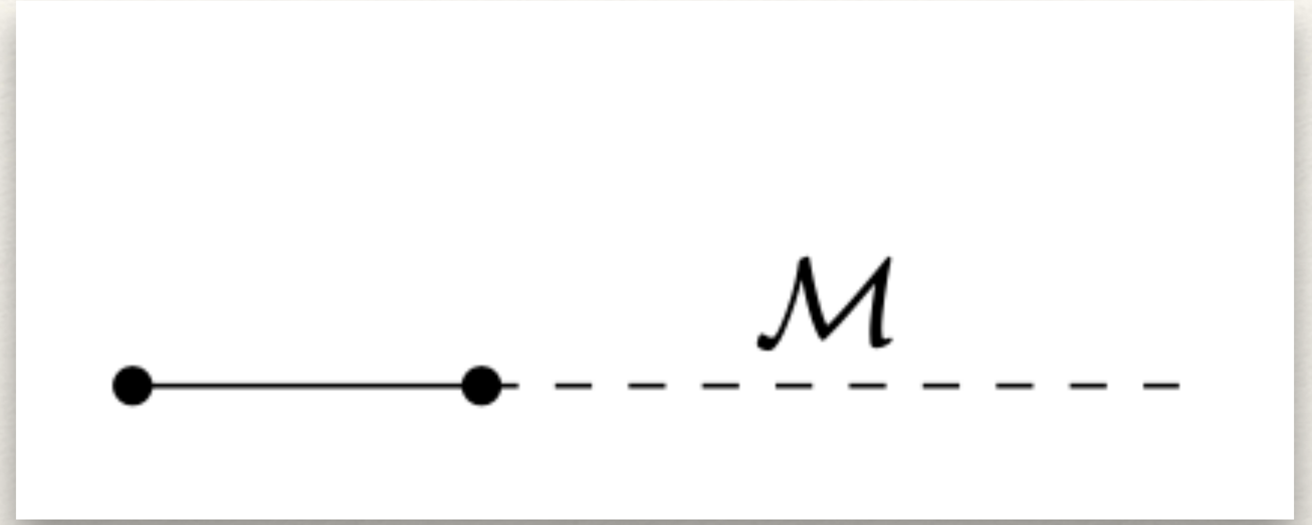
Let $C(x) = \sum_{n \geq 0} c_n x^n$ be the generating function for the number of length n Dyck paths, then $C(x)$ is

$$C(x) = 1 + x^2 C(x)^2 = \frac{1 - \sqrt{1 - 4x^2}}{2x^2} = 1 + x^2 + 2x^4 + 5x^6 + 14x^8 + \dots$$

This is the Catalan numbers, and we call C(x) the Catalan generating function.

Avoiding a set of patterns

For a set of patterns P, define $Av_H(P)$ to be the set of Motzkin paths avoiding P and starting with H.



A Motzkin path Hp avoids Ux if p avoids Ux, and avoids Dx if p avoids Dx.

A Motzkin path Hp avoids Hx if p avoids x.

Theorem 2.1. For a set of patterns P, let P_U, P_D, and P_H be the sets of patterns in P beginning with U, D and H, respectively, and

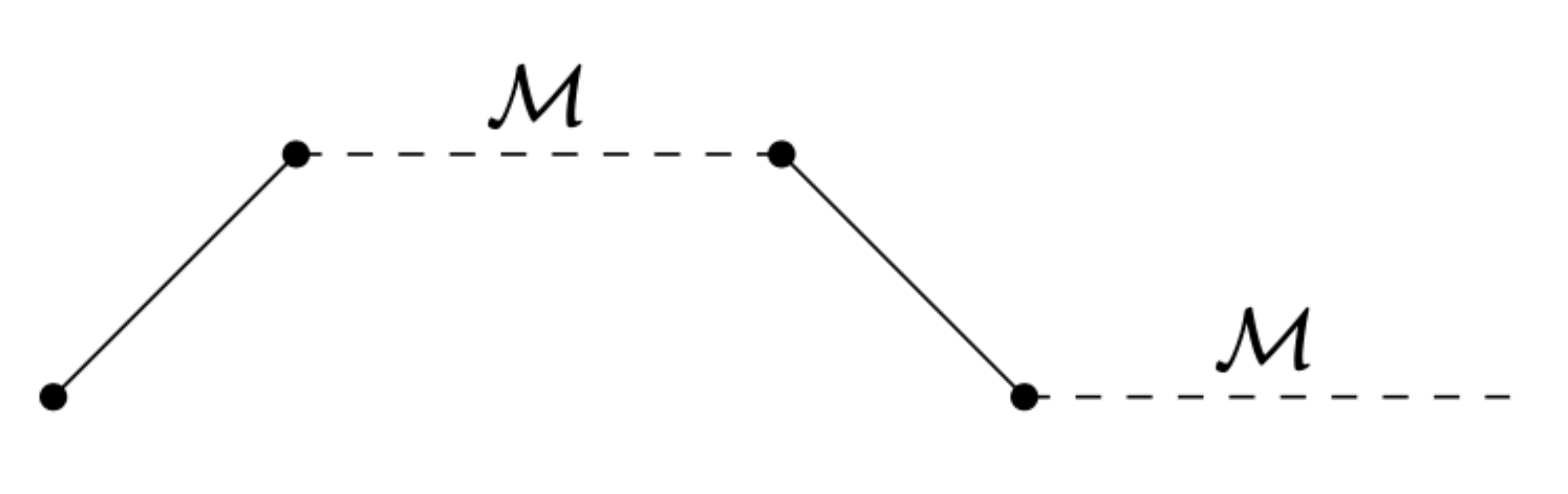
$$P' = P_U \cup P_D \cup \{p \mid Hp \in P_H\}$$

then

$$\mathrm{Av}_H(P) = \{Hp \mid p \in \mathrm{Av}(P')\}.$$

Starting with U

For a set of patterns P, define $Av_U(P)$ to be the set of Motzkin paths avoiding P and starting with U.



That is we can write every Motzkin path as $UxDy$ for some Motzkin paths x and y.

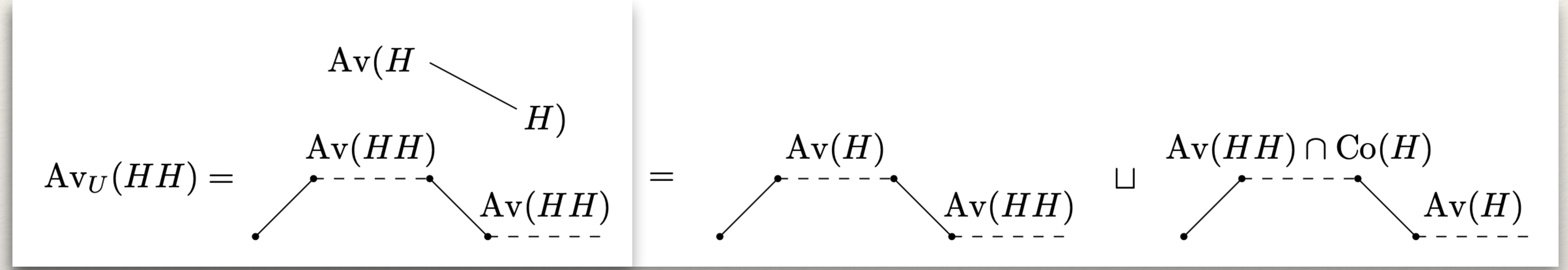
If $P = \{HH\}$, then not every choice of x and y in $Av(P)$ results in a path avoiding P.

For example, if $x = H$ and $y = H$ then $UxDy = UHDH$ contains HH.

Crossing patterns

A Motzkin path $UxDy$ *contains* the *crossing pattern* $\ell - r$, where $\ell, r \in \{U, H, D\}^*$, if UxD contains ℓ and y contains r. Otherwise we say it *avoids* $\ell - r$.

For $Av_U(HH) = Av_U(HH - , - HH, H - H)$, that is



Let $\Delta_{HH}(x)$ be the generating function for the set of Motzkin paths avoiding HH. Then,

$$\Delta_{HH}(x) = 1 + xC(x) + x^2C(x)\Delta_{HH}(x) + x^2(\Delta_{HH}(x) - C(x))C(x)\,.$$

Case analysis

We generalise this idea.

Theorem 2.2. For any finite sets P and Q of patterns there exist sets of local crossing patterns $P_1, P_2, \ldots, P_k$ and $Q_1, Q_2, \ldots, Q_k$ such that

$$\mathrm{Av}_U(P) \cap \bigcap_{q \in Q} \mathrm{Co}(q) = \bigsqcup_{i=1}^{k} \left(\mathrm{Av}_U(P_i) \cap \bigcap_{q \in Q_i} \mathrm{Co}_U(q) \right).$$

Once all of the patterns are local, it is just a Cartesian product.

Theorem 2.3. Let P and Q be sets of local crossing patterns. Let P_r (Q_r) be the right local patterns in P (Q). Let P_ℓ (Q_ℓ) be the patterns obtained by taking the left local patterns in P (Q) and removing a single U from the left and single D from the right if such exists. Then,

$$\mathrm{Av}_U(P) \cap \bigcap_{q \in Q} \mathrm{Co}_U(q) = \{UxDy \mid x \in \mathrm{Av}_U(P_\ell) \cap \bigcap_{q \in Q_\ell} \mathrm{Co}_U(q), y \in \mathrm{Av}_U(P_r) \cap \bigcap_{q \in Q_r} \mathrm{Co}_U(q)\}.$$

Combinatorial exploration

Combinatorial exploration is a domain-agnostic algorithmic framework for discovering combinatorial specifications. It systematically applies strategies to create rules that describe how to build a class from other classes. Here, we have only used disjoint union and Cartesian product rules.

The recursive application these theorems results in either a shortening of the patterns being avoided and contained or reducing the size of the sets being avoided and contained. This gives a finite process to find a specification.

Theorem 2.1. For a set of patterns P, let P_U, P_D, and P_H be the sets of patterns in P beginning with U, D and H, respectively, and

$$P' = P_U \cup P_D \cup \{p \mid Hp \in P_H\}$$

then

$$\mathrm{Av}_H(P) = \{Hp \mid p \in \mathrm{Av}(P')\}.$$

Theorem 2.2. For any finite sets P and Q of patterns there exist sets of local crossing patterns P_1, P_2, ..., P_k and Q_1, Q_2, ..., Q_k such that

$$\mathrm{Av}_U(P) \cap \bigcap_{q \in Q} \mathrm{Co}(q) = \bigsqcup_{i=1}^{k} \left(\mathrm{Av}_U(P_i) \cap \bigcap_{q \in Q_i} \mathrm{Co}_U(q) \right).$$

Theorem 2.3. Let P and Q be sets of local crossing patterns. Let P_r (Q_r) be the right local patterns in P (Q). Let P_ℓ (Q_ℓ) be the patterns obtained by taking the left local patterns in P (Q) and removing a single U from the left and single D from the right if such exists. Then,

$$\mathrm{Av}_U(P) \cap \bigcap_{q \in Q} \mathrm{Co}_U(q) = \{UxDy \mid x \in \mathrm{Av}_U(P_\ell) \cap \bigcap_{q \in Q_\ell} \mathrm{Co}_U(q)\,, y \in \mathrm{Av}_U(P_r) \cap \bigcap_{q \in Q_r} \mathrm{Co}_U(q)\}.$$

The specification found can then be used to count the number of Motzkin paths of each length, give equations for the generating function, and sample uniformly at random.

Our Python implementation, which uses the comb_spec_searcher package, can be found on GitHub.

Dyck paths are rational

Theorem (Bacher, Bernini, Ferrari, Gunby, Pinzani, and West, 2014)

Every set of Dyck paths avoiding a fixed pattern has a rational generating function.

Theorem (B., Bernini, Cervetti, and Ferrari, 2022)

Every set of Motzkin paths avoiding a fixed pattern has a generating function that is rational over x and $C(x)$.

A *Motzkin prefix* is a lattice path using steps U, D, and H that never goes below the x-axis.

For a Motzkin prefix p, let p^- be the Motzkin prefix obtained by removing the last step.

Let $\mathscr{MP}$ be the set of Motzkin prefixes, then define

$$\mathrm{MinCo}(P) = \{p \in \mathscr{MP} \mid p \text{ contains } P \text{ and } p^- \text{ avoids } P\}$$

Counting Motzin prefixes

Let q be any Motzkin prefix, recall

$$\mathrm{MinCo}(q) = \{p \in \mathscr{MP} \,|\, p \text{ contains } q \text{ and } p^{-} \text{ avoids } q\}$$

and let $\Gamma_q(x, y)$ be the bivariate generating function of MinCo(q)

We give a recursive formula to compute $\Gamma_q(x, y)$

Proposition 3.1. For any given Motzkin prefix q, we have:

$$\Gamma_\epsilon(x,y) = 1, \tag{15}$$

$$\Gamma_{qU}(x,y) = \frac{xy}{(1-x)(x-y(1-x))}\left(x\Gamma_q\left(x, \frac{x}{1-x}\right) - y(1-x)\Gamma_q(x,y)\right), \tag{16}$$

$$\Gamma_{qH}(x,y) = \frac{2x}{\left(1-2xy+\sqrt{1-4x^2}\right)(y-xC(x))}\left(y\Gamma_q(x,y) - xC(x)\Gamma_q(x,xC(x))\right), \tag{17}$$

$$\Gamma_{qD}(x,y) = \frac{x}{y}\left(\frac{1}{1-xy-x}\Gamma_q(x,y) - \frac{1}{1-x}\Gamma_q(x,0)\right). \tag{18}$$

Computing $\Gamma_{qH}(x, y)$

Take $\pi \in \mathrm{MinCo}(qH)$ and π' be the smallest prefix of π containing q, $\delta^{(h)}$ be a Dyck factor starting at height h, then we write

$$\pi = \pi' \delta^{(h)} H$$

Let $D^{(h)}(x, y)$ be the generating function of the $\delta^{(h)}$ paths

Therefore,

$$\Gamma_{qH}(x, y) = \left(\sum_{h \geq 0} \left(\left[y^h \right] \Gamma_q(x, y) \right) D^{(h)}(x, y) \right) x \ . \qquad (22)$$

Computing $D^{(h)}(x,y)$

A Dyck factor can be written as

$$\delta^{(h)} = (\gamma_1 D)(\gamma_2 D)\cdots(\gamma_r D)\gamma$$

Where the D steps are the first time the path is at heights $(h-1)$, $(h-2)$, etc.

$$\begin{aligned} D^{(h)}(x,y) &= \mathcal{DP}(x,y)y^h + C(x)x\mathcal{DP}(x,y)y^{h-1} + \\ & C(x)^2x^2\mathcal{DP}(x,y)y^{h-2} + \cdots + C(x)^h x^h \mathcal{DP}(x,y)y^{h-h} \\ &= \mathcal{DP}(x,y)\sum_{i=0}^{h} x^i y^{h-i} C(x)^i \ , \end{aligned}$$

leading to

$$D^{(h)}(x,y) = \frac{2}{1-2xy+\sqrt{1-4x^2}} \cdot \frac{y^{h+1}-x^{h+1}C(x)^{h+1}}{y-xC(x)} \ . \tag{23}$$

Rewriting $\Gamma_{qH}(x, y)$

$$\Gamma_{qH}(x,y) = \left(\sum_{h \geq 0} \left(\left[y^h \right] \Gamma_q(x,y) \right) D^{(h)}(x,y) \right) x$$

$$D^{(h)}(x,y) = \frac{2}{1 - 2xy + \sqrt{1-4x^2}} \cdot \frac{y^{h+1} - x^{h+1} C(x)^{h+1}}{y - xC(x)}$$

Plugging in our formula for $D^{(h)}(x, y)$ into our equation for $\Gamma_{qH}(x, y)$ gives

$$\Gamma_{qH}(x,y) = \frac{2x}{\left(1 - 2xy + \sqrt{1-4x^2}\right)\left(y - xC(x)\right)} \sum_{h \geq 0} \left(\left[y^h \right] \Gamma_q(x,y) \right) \left(y^{h+1} - x^{h+1} C(x)^{h+1}(x) \right)$$

With some manipulation this is the same as (17) in our proposition

Proposition 3.1. For any given Motzkin prefix q, we have:

$$\Gamma_\epsilon(x,y) = 1, \tag{15}$$

$$\Gamma_{qU}(x,y) = \frac{xy}{(1-x)(x-y(1-x))} \left(x\Gamma_q\left(x, \frac{x}{1-x}\right) - y(1-x)\Gamma_q(x,y) \right), \tag{16}$$

$$\Gamma_{qH}(x,y) = \frac{2x}{\left(1-2xy+\sqrt{1-4x^2}\right)(y-xC(x))} \left(y\Gamma_q(x,y) - xC(x)\Gamma_q(x, xC(x)) \right), \tag{17}$$

$$\Gamma_{qD}(x,y) = \frac{x}{y} \left(\frac{1}{1-xy-x}\Gamma_q(x,y) - \frac{1}{1-x}\Gamma_q(x,0) \right). \tag{18}$$

Recursive formula for $\Delta_q(x)$

Let $\Delta_q(x)$ be the generating function for $Av(q)$

Proposition 3.2. For any Motzkin prefix q, the generating function $\Delta_q(x)$ is given by:

$$\Delta_\epsilon(x) = 0 \tag{24}$$

$$\Delta_{qD}(x) = \Delta_q(x) + \Gamma_q(x,0)\frac{1}{1-x} \tag{25}$$

$$\Delta_{qH}(x) = \Delta_q(x) + C(x)\cdot\Gamma_q\left(x, xC(x)\right) \tag{26}$$

$$\Delta_{qU}(x) = \Delta_q(x) + \frac{1}{1-x}\Gamma_q\left(x, \frac{x}{1-x}\right) \quad . \tag{27}$$

Computing $\Delta_{qH}(x)$

We can write π containing q but avoiding qH as the smallest prefix containing q, followed by a path using only U and D steps, i.e., a reversed Dyck prefix

$$\Delta_{qH}(x) = \Delta_q(x) + \sum_{h \geq 0} \left(\left[y^h\right] \Gamma_q(x,y) \right) \left(\left[y^h\right] \mathcal{DP}(x,y) \right)$$

By plugging in the following identity

$$\left[y^h\right] \mathcal{DP}(x,y) = \frac{2}{1+\sqrt{1-4x^2}} \left(\frac{2x}{1+\sqrt{1-4x^2}} \right)^h$$

We get the equation which is equivalent to our proposition.

$$\Delta_{qH}(x) = \Delta_q(x) + C(x) \sum_{h \geq 0} \left(\left[y^h\right] \Gamma_q(x,y) \right) \left(xC(x) \right)^h$$

The theorem again!

Proposition 3.1. For any given Motzkin prefix q, we have:

$$\Gamma_\epsilon(x,y) = 1, \tag{15}$$

$$\Gamma_{qU}(x,y) = \frac{xy}{(1-x)(x-y(1-x))}\left(x\Gamma_q\left(x, \frac{x}{1-x}\right) - y(1-x)\Gamma_q(x,y)\right), \tag{16}$$

$$\Gamma_{qH}(x,y) = \frac{2x}{\left(1-2xy+\sqrt{1-4x^2}\right)(y-xC(x))}\left(y\Gamma_q(x,y) - xC(x)\Gamma_q(x,xC(x))\right), \tag{17}$$

$$\Gamma_{qD}(x,y) = \frac{x}{y}\left(\frac{1}{1-xy-x}\Gamma_q(x,y) - \frac{1}{1-x}\Gamma_q(x,0)\right). \tag{18}$$

Proposition 3.2. For any Motzkin prefix q, the generating function $\Delta_q(x)$ is given by:

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$$\Delta_{qD}(x) = \Delta_q(x) + \Gamma_q(x,0)\frac{1}{1-x} \tag{25}$$

$$\Delta_{qH}(x) = \Delta_q(x) + C(x)\cdot\Gamma_q(x, xC(x)) \tag{26}$$

$$\Delta_{qU}(x) = \Delta_q(x) + \frac{1}{1-x}\Gamma_q\left(x, \frac{x}{1-x}\right). \tag{27}$$

These two propositions prove our theorem.

Theorem (B., Bernini, Cervetti, and Ferrari, 2022)

Every set of Motzkin paths avoiding a fixed pattern has a generating function that is rational over x and $C(x)$.

Thanks for listening!