Èdouard Lucas:

The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.

Enumerating Orderings on Matched Product Graphs

Amir Barghi[1] & Daryl DeFord[2]

[1]Department of Mathematics and Statistics
Saint Michael's College - Colchester, VT

[2]Department of Mathematics and Statistics
Washington State University - Pullman, WA

Permutation Patterns
Valparaiso University
June 21, 2022

Outline

1. Combinatorial Motivation
2. Graph Stirling Numbers
3. Planarity for Matched Path Products (Square Permutations)
4. Enumeration on Graph Products
5. Path Decompositions of Paths (???)

Warmup Problem (Honsberger)

Warmup Question

A classroom has 5 rows of 5 desks per row. The teacher **requires** *that each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). In how many ways can the students rearrange themselves?*

Warmup Problem (Honsberger)

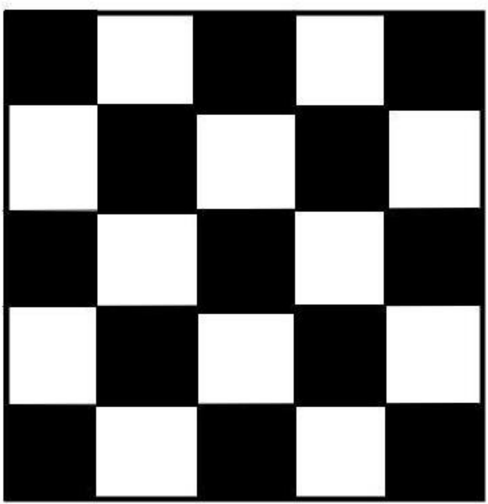
Warmup Question

A classroom has 5 rows of 5 desks per row. The teacher **requires** *that each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). In how many ways can the students rearrange themselves?*

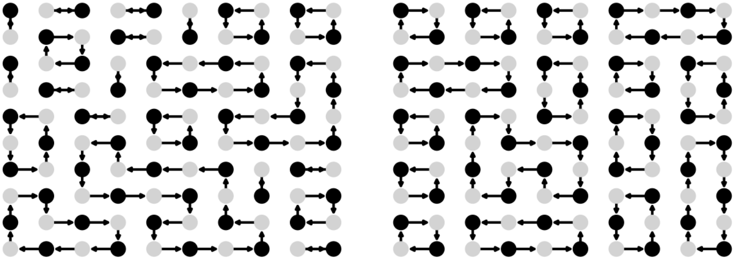
Answer

Zero.

Warmup Problem Solution



Generalizations



More interesting problem

Warmup Question II

A classroom has 5 rows of 5 desks per row. The teacher **allows** *each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right* **or to remain in place**. *In how many ways can the students rearrange themselves?*

More interesting problem

Warmup Question II

A classroom has 5 rows of 5 desks per row. The teacher **allows** *each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right* **or to remain in place**. *In how many ways can the students rearrange themselves?*

Answer

19,114,420

More interesting problem

Warmup Question II

A classroom has 5 rows of 5 desks per row. The teacher **allows** *each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right* **or to remain in place**. *In how many ways can the students rearrange themselves?*

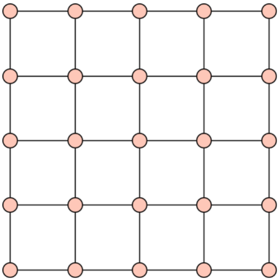
Answer

19,114,420

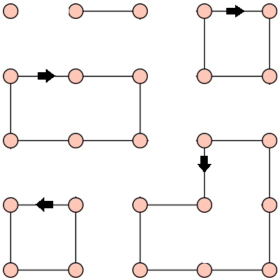
Definition (Graph Factorial)

The factorial of a graph G is the number of ways to decompose the vertices of G into a collection of disjoint cycles.

Example



Original



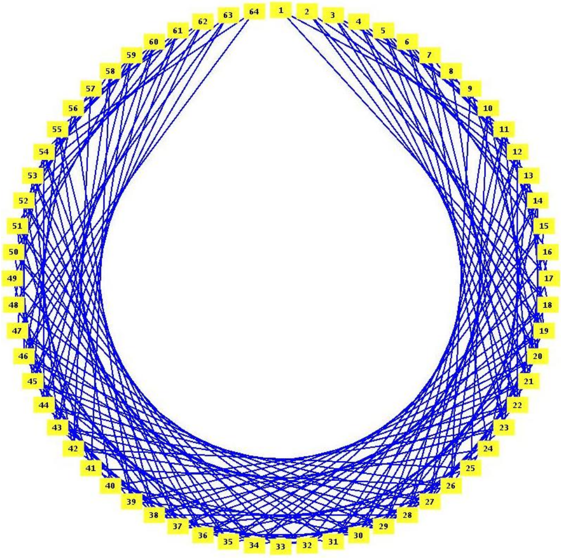
Cycle Decomposition

Chess Moves

Warmup Question III

A classroom has m rows of n desks per row. The teacher allows each pupil to change his seat by going either moving like **a given set of chess pieces** *or to remain in place. In how many ways can the students rearrange themselves?*

Knight Rearrangements



Knight's Tour

- 8×8 Knight's Tour (Hamiltonian Cycles)
- 26,534,728,821,064 [1,2]

[1] M. Löbbing and I. Wegener, The Number of Knight's Tours Equals 33,439,123,484,294 Counting with Binary Decision Diagrams, Electronic Journal of Combinatorics, (1996).

[2] B. McKay, Knight's Tours on an 8×8 Chessboard, Technical Report TR-CS-97-03, Australian National University, (1997).

Knight's Tour

- 8×8 Knight's Tour (Hamiltonian Cycles)
- 26,534,728,821,064 [1,2]
- 8×8 Knight's Graph Factorial
- 8,121,130,233,753,702,400

[1] M. Löbbing and I. Wegener, The Number of Knight's Tours Equals 33,439,123,484,294 Counting with Binary Decision Diagrams, Electronic Journal of Combinatorics, (1996).

[2] B. McKay, Knight's Tours on an 8×8 Chessboard, Technical Report TR-CS-97-03, Australian National University, (1997).

Stirling Numbers for Arbitrary Graphs

Definition

Let G be a graph. The kth Stirling number of the first kind for G, denoted by $\begin{bmatrix} G \\ k \end{bmatrix}$, is the number of vertex-disjoint partitions of G into k cycles, where 1-cycles and 2-cycles are allowed and cycles of order three or higher have two orientations. The graphical factorial $G!$ is then $\sum_k \begin{bmatrix} G \\ k \end{bmatrix}$.

Simple Graph Families

Theorem (D. 2014[1])

Let $n, m \in \mathbb{N}$. *Then*

1. $K_n! = n!$;
2. $P_n! = f_{n+1}$;
3. $C_n! = f_{n+1} + f_{n-1} + 2$, *for* $n \geq 3$;

Theorem (B. 2018[2])

Let $n, m, k \in \mathbb{N}$. *Then*

1. $\begin{bmatrix} K_n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}$;
2. $\begin{bmatrix} P_n \\ k \end{bmatrix} = \binom{k}{n-k}$;
3. $\begin{bmatrix} C_n \\ k \end{bmatrix} = \binom{k-1}{n-k} + 2\binom{k-1}{n-k-1}$ *and* $\begin{bmatrix} C_n \\ 1 \end{bmatrix} = 2$, *for* $n \geq 3$ *and* $k \geq 2$;

[1] D. DeFord, Seating rearrangements of arbitrary graphs, Involve, (2014).

[2] A. Barghi, Stirling numbers of the first kind for graphs, Australasian Journal of Combinatorics, (2018).

Obstructions and Observations

- For arbitrary graphs enumeration is $\#P$ complete by reduction to matrix permanent
- Problem of convertible matrices $\det(A') = \mathrm{per}(A)$ raised by Pòlya (1913). Progress and combinatorial characterizations Beineke and Harary (1966), Little (1975), Vazarani and Yannakakis (1988). Excellent survey Kuperberg (1998). Resolved by Robertson, Seymour, and Thomas (1999).
- Lots of interesting (combinatorially tractable) sequences from families of graphs e.g. W_n as $n \to \infty$
- Distributions over larger families e.g. Trees/Forests on n nodes

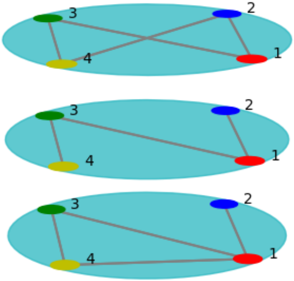
Obstructions and Observations

- For arbitrary graphs enumeration is $\#P$ complete by reduction to matrix permanent
- Problem of convertible matrices $\det(A') = \mathrm{per}(A)$ raised by Pòlya (1913). Progress and combinatorial characterizations Beineke and Harary (1966), Little (1975), Vazarani and Yannakakis (1988). Excellent survey Kuperberg (1998). Resolved by Robertson, Seymour, and Thomas (1999).
- Lots of interesting (combinatorially tractable) sequences from families of graphs e.g. W_n as $n \to \infty$
- Distributions over larger families e.g. Trees/Forests on n nodes
- .. but then what?!?

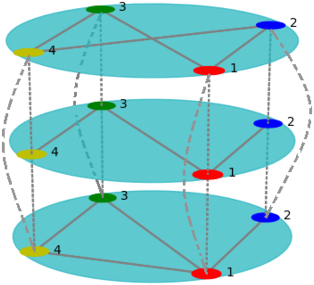
Obstructions and Observations

- For arbitrary graphs enumeration is $\#P$ complete by reduction to matrix permanent
- Problem of convertible matrices $\det(A') = \operatorname{per}(A)$ raised by Pòlya (1913). Progress and combinatorial characterizations Beineke and Harary (1966), Little (1975), Vazarani and Yannakakis (1988). Excellent survey Kuperberg (1998). Resolved by Robertson, Seymour, and Thomas (1999).
- Lots of interesting (combinatorially tractable) sequences from families of graphs e.g. W_n as $n \to \infty$
- Distributions over larger families e.g. Trees/Forests on n nodes
- .. but then what?!?
- Graph products

Supra–Adjacency



Disjoint Layers



Supra–Adjacency

M. Kivelä, A. Arenas, M. Barthelemy, James P. Gleeson, Y. Moreno, M. A. Porter, Multilayer networks, Journal of Complex Networks, (2014).

Matched Product

Definition (Matched Product)

Let $G_1, G_2, \ldots, G_k$ be an ordered list of graphs, each with n nodes and a common labeling of the nodes and let C be a graph with k ordered nodes. The matched product $\boxed{C}\,(G_1, G_2, \ldots, G_k)$ is the graph with node set $\bigcup V_i$ and two nodes v_i^α and v_j^β in $\boxed{C}\,(G_1, G_2, \ldots, G_k)$ are connected if and only if either

1. $c_\alpha \sim c_\beta$ and $i = j$
2. $\alpha = \beta$ and $v_i^\alpha \sim v_j^\alpha$

where c_α and c_β are nodes in C and v_i^α represents the copy of node i in G_α.

Example: Petersen Graph

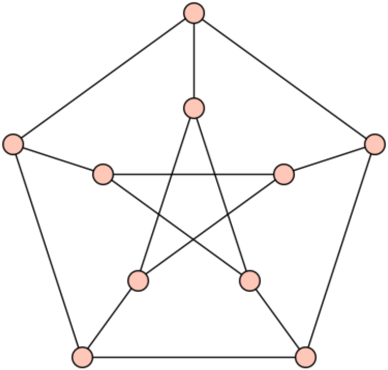


Figure: $\boxed{P_2}$ (C_5, C_5)

Relationship to Other Graph Products

Theorem

There are labelings of the graphs below such that the following hold:

1. *The cartesian product of G and H can be represented by* $\boxed{H}\,(G, G, \ldots, G)$
2. *The rooted product of G and H can be represented by* $\boxed{H}\,(G, E_n, E_n, \ldots, E_n)$
3. *The hierarchical product*[1,2] *of G and H with subset $\{a_i\} \subset H$ can be represented by* $\boxed{H}\,(G_1, G_2, \ldots, G_k)$ *where*
$$G_i = \begin{cases} G & \text{if } i \in \{a_i\} \\ E_n & \text{otherwise} \end{cases}.$$

[1] L. Barrièrea, C. Dalfóa, M. A. Fiola, M. Mitjanab, The generalized hierarchical product of graphs, Discrete Mathematics, (2009).

[2] P. S. Skardal and K. Wash, Spectral properties of the hierarchical product of graphs, Physical Review E, (2016).

Property Preservation

Proposition

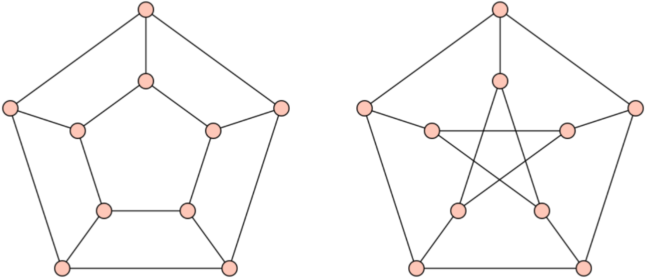
1. *Let $G_1, G_2, \ldots, G_k$ and C be Eulerian graphs then any labeling of* $\boxed{C}$ $(G_1, G_2, \ldots, G_k)$ *is Eulerian.*
2. *Let $(G_1, G_2, \ldots, G_k)$ and C have Hamiltonian cycles. Then* $\boxed{C}$ $(G_1, G_2, \ldots, G_k)$ *is Hamiltonian.*

Proof.

1. A graph is Eulerian if each vertex has even degree. Since the G_i and C are Eulerian each vertex in the product has even degree.
2. Label the nodes in $(G_1, G_2, \ldots, G_k)$ arbitrarily and let $(a_1, a_2, \ldots, a_k)$ be a Hamiltonian cycle in C. Starting at an arbitrary vertex in G_{a_1} traverse the cycle in G_{a_1} and then travel along the edge to G_{a_2}. The hypotheses guarantee that we can continue to traverse each layer, ending at a copy of the original node on G_{a_k}.

□

Labeling Matters!



Cylinder Graph　　　Petersen Graph

Figure: Two labelings of $\boxed{P_2}\,(C_5, C_5)$

Planarity Result

Proposition (Planarity)

Let G and H be connected graphs on n nodes. There exists a labeling so that of $\boxed{P_2}\,(G,H)$ is planar if and only if G and H are outerplanar[1].

[1] G. Chartrand, and F. Harary, Planar permutation graphs , Annales de l'Institut Henri Poincar B, (1967).

Planarity Result

Proposition (Planarity)

Let G and H be connected graphs on n nodes. There exists a labeling so that of $\boxed{P_2}$ (G, H) is planar if and only if G and H are outerplanar[1].

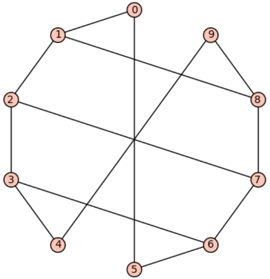


Figure: A labeling of $\boxed{P_2}$ (P_5, P_5) that is not planar.

[1] G. Chartrand, and F. Harary, Planar permutation graphs , Annales de l'Institut Henri Poincar B, (1967).

Permutations of P_n

Theorem

Let $\pi \in S_n$. Then $\boxed{P_2}$ (P_n, P_n) with labelings $(1, 2, 3, \ldots, n)$ and $(\pi(1), \pi(2), \pi(3), \ldots, \pi(n))$ is planar if and only if π is a square permutation. There are $2(n+2)4^{n-2} - 4(2n-5)\binom{2n-6}{n-3}$ such permutations.

Proof.

A permutation is square if every point is a record (its consecutive–minima polygon has at most 4 'sides'). If π is square construct directly from diagram. If π is not square, there exists a vertex $2 < k < n-1$ such that contracting the edges between $1, \ldots, k-1$ and $k+1, \ldots, n$ is isomorphic to $K_{3,3}$. $\square$

Permutation Examples

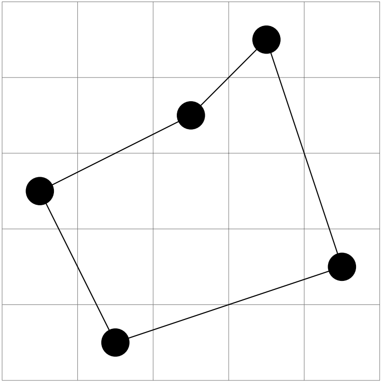


Figure: (3,1,4,5,2)

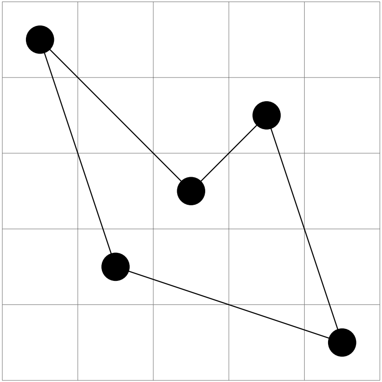


Figure: (5,2,3,4,1)

More Questions:

- Enumeration over relabelings:
  - Planarity
  - Factorial
  - Chromatic Number
  - etc.
- Products that are planar for all labelings

$$\boxed{P_2}\,(S_n, P_n) \text{ and } \boxed{P_2}\,(S_n, S_n)$$

- Products that are isomorphic for all labelings

$$\boxed{P_2}\,(K_n, G) \text{ and } \boxed{P_2}\,(S_n, C_n)$$

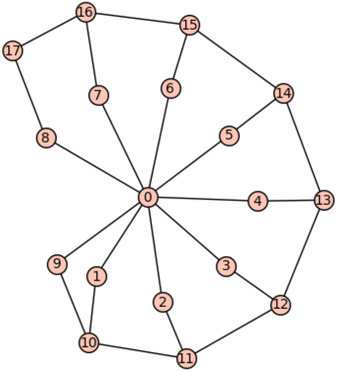
- Products that are never isomorphic for any pair of labelings

Comb Graph Factorials

G_n	$G_n!$
$\boxed{P_2}\ (E_n, E_n)$	2^n
$\boxed{P_2}\ (P_n, E_n)$	L_n
$\boxed{P_2}\ (S_n, E_n)$	$2^{n+1} + n2^n$
$\boxed{P_2}\ (C_n, E_n)$	$2L_{n-1} + 2L_{n-2} + 4$
$\boxed{P_2}\ (K_n, E_n)$	$\sum_\ell \binom{n}{\ell}(n-\ell)!$
$\boxed{P_2}\ (C_n, C_n)$	$6 + 4(-1)^n + \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$

The Pell numbers, L_n, are defined by $L_0 = 1$, $L_1 = 2$, and $L_n = 2L_{n-1} + L_{n-2}$.

$\boxed{P_2}$ (S_9, P_9)



$\boxed{P_2}\ (S_n, P_n)$

Examples

$$\begin{aligned}\boxed{P_2}\ (S_n, P_n)! =& 2L_{n+1} + (L_{j-1} + L_{j-2})L_{n-j-1} \\ &+ \sum_{j=1}^{n}[L_{j-1} + 2L_{j-2} + L_{n-3}]L_{n-j} \\ &+ 2\left(\sum_{j=1}^{n} L_{n-j} + \sum_{j=1}^{n-1}[L_{j-1} + L_{j-2}] \sum_{m=j+1}^{n} L_{n-m}\right)\end{aligned}$$

$\boxed{P_2}\,(G, H)$ Enumeration

n	$\boxed{P_2}\,\widehat{(C_n, S_n)}!$	$\boxed{P_2}\,\widehat{(K_n, S_n)}!$	$\boxed{P_2}\,(P_n, K_n)!$	$\boxed{P_2}\,\widehat{(P_n, K_n)}!$	$\boxed{P_2}\,(C_n, K_n)!$
2	9	48	4	9	4
3	49	293	9	48	20
4	140	2022	49	345	121
5	394	15657	216	2994	589
6	1093	135044	1773	30957	4820
7	2986	1287813	12113	369132	35293
8	8056	13480938	128036	4996761	365633
9	21504	153879977	1172341	75625710	3525212
10	56889	1903771512	14885241	1265833149	43894725

Star Enumeration

Definition

Let G be a graph. Define the m-star of G as the join of G and E_m, denoted by $S_m(G)$, i.e., $S_m(G) = G \bowtie E_m$.

Definition

Let G be a graph with n vertices. We denote the cardinality of the set of all partitions of G into j vertex-disjoint directed and ordered paths by $\left\langle {G \atop j} \right\rangle$, where we call each part in of one these partitions a partitioning directed path in G.

Theorem

Let G be a graph with $n \geq 2$ vertices. If $m \leq n$, then

$$\begin{bmatrix} S_m(G) \\ 2 \end{bmatrix} = m \begin{bmatrix} S_{m-1}(G) \\ 1 \end{bmatrix} + \begin{bmatrix} m \\ 2 \end{bmatrix} \left\langle \begin{matrix} G \\ m \end{matrix} \right\rangle;$$

otherwise, $\left[{S_m(G) \atop 2} \right] = 0$.

Enumerative Schematic (Generic)

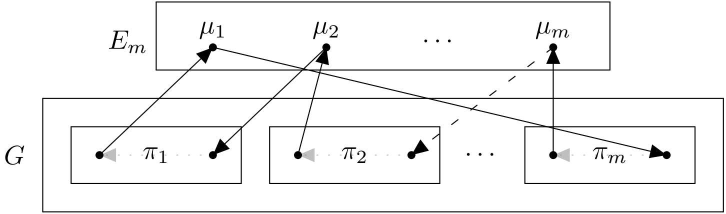


Figure: A cyclic partition of $S_m(G)$ into a single cycle

Enumeration Schematic (Complete)

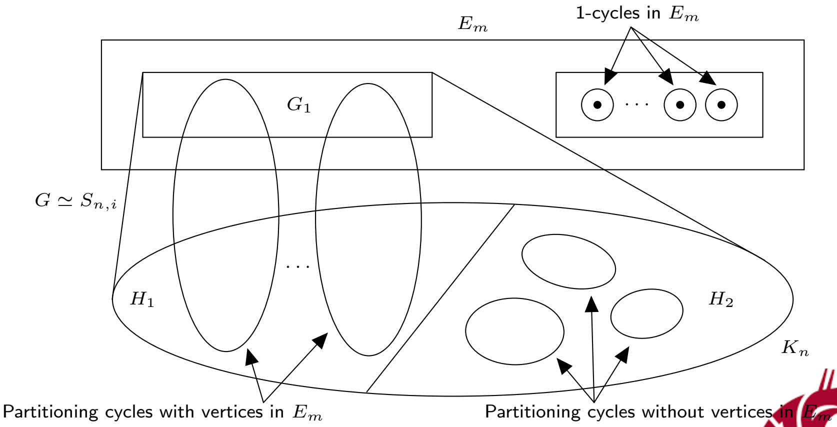
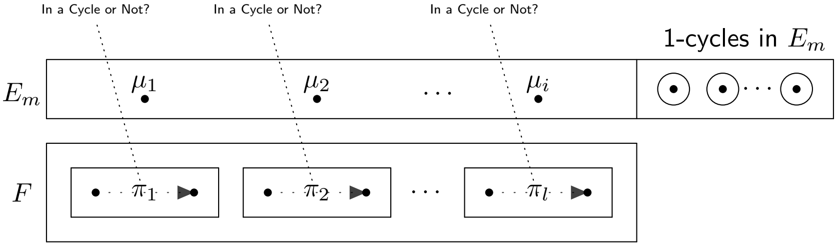


Figure: A cyclic partition of $S_{n,m}$

Enumerative Schematic (Forests)



Enumeration Examples

Theorem

If $n, m \in \mathbb{N}$, then $\begin{bmatrix} S_m(K_n) \\ k \end{bmatrix}$ is equal to

$$\begin{bmatrix} n \\ k-m \end{bmatrix} + \sum_{i \geq 1} \sum_{j=1}^{n} \sum_{l=0}^{n-1} \binom{m}{i} \begin{bmatrix} n-j \\ l \end{bmatrix} j! \binom{j-1}{i-1} \begin{bmatrix} i \\ k-(m-i+l) \end{bmatrix}.$$

Theorem

Suppose $m, n \in \mathbb{N}$. Let F be a forest of order n. Then, for $k \geq 2$,

$$\begin{bmatrix} S_m(F) \\ k \end{bmatrix} \leq \begin{bmatrix} F \\ k-m \end{bmatrix} + \sum_{i \geq 1} m^{\underline{i}} 2^{k-(m-i)} \left\langle \begin{matrix} F \\ k-(m-i) \end{matrix} \right\rangle.$$

One last Extension...

Cooldown Question (Path Decompositions of Paths)

For fixed n, consider the permutations π determined by ordered path decompositions of $\mathbb{P}_n$. Can this set be characterized by pattern avoidance?
Example:

$$1 \leftarrow 2 \leftarrow 3 \qquad 4 \qquad 5 \rightarrow 6$$

The End!

Thanks!

T-shaped Tetrominoes