2-AVOIDANCE

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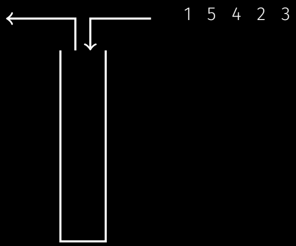
Definition (Pop stack)

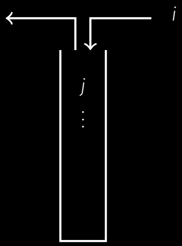
A pop stack operates as follows: either

- push the next input token onto the stack
- pop the *entire contents* of the stack

Eg:

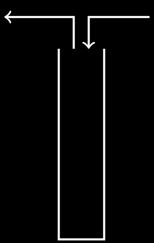


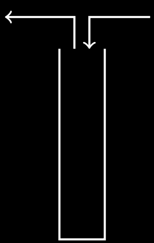
A *deterministic* pop stack pushes if $i < j$, else pops entire contents of the stack:



MULTI-PASS POP STACKS: AKA K-POP STACKS

A permutation is *k-pass pop stack sortable* if it can be sorted using *k* passes through a deterministic pop stack.



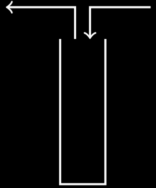


Eg: 2-pass?

- 41352
- 3241

Eg: 1-pass?

- 123
- 132
- 213
- 231
- 312
- 321



Theorem (Avis, Newborn 1981)

A permutation is 1-pass pop stack sortable if and only if it avoids 231, 312.

2-pass?

Recall: we showed $p = 41352$ is 2-pass pop stack sortable, and $q = 3241$ is not.

But p contains q ?!

A *barred permutation* is a permutation where some entries have bars on them.

Eg: $4\bar{1}32$. Let $\mathrm{unbar}(4\bar{1}32) = 4132$ and $\mathrm{removebar}(4\bar{1}32) = 432 \sim 321$.

A permutation p is said to *avoid* a barred pattern β if

- any subpermutation q of p with $q \sim \mathrm{removebar}(\beta)$

  is contained in a subpermutation r of p with $r \sim \mathrm{unbar}(\beta)$.

Eg: 15243 avoids $4\bar{1}32$

Eg: 52413 does not avoid $4\bar{1}32$

RESULT: 2-POP

Theorem (Pudwell, Smith[1])

A permutation is 2-pass pop stack sortable if and only if it avoids each of the following:

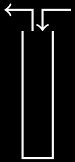
$$2341 \quad 3412 \quad 3421 \quad 4123 \quad 4231 \quad 4312 \quad 4\bar{1}352 \quad 413\bar{5}2$$

Claesson and Guðmundsson[2] : is there a

“useful permutation pattern characterization of the k-pop stack-sortable permutations” [for $k \geqslant 3$]?

[1] Pudwell and Smith. Two-stack-sorting with pop stacks. Australasian Journal of Combinatorics, 2019.

[2] Claesson and Guðmundsson, Enumerating permutationssortable by k passes through a pop-stack. Sém. Lothar. Combin., 2018.



32451

4631572

4731562

If we were to characterise 3-pop stack sortable permutations as those avoiding some list containing $4\bar{6}3\bar{1}572$ and $4\bar{7}3\bar{1}562$, then we would be mistaken:

4731562 does not avoid this list since it fails to avoid $4\bar{6}3\bar{1}572$.

So we made a new definition.

Let $F, G \subseteq S^\infty$. Let $\sim$ denote "order-isomorphic".

Definition

A permutation p is said to *2-avoid* (F, G) if

- if p_1 is a subpermutation of p with $p_1 \sim f \in F$,

  then p_1 is a subpermutation of p_2 of p and $p_2 \sim g \in G$.

We say that p_1 can be *"saved"* by some element of G.

Eg: $p = 4731562$ 2-avoids $(\{32451\}, \{4631572, 4731562\})$.

Theorem (Pudwell and Smith)

The set of 2-pass pop stack sortable permutations is equal to

$$Av_B\left(\{2341, 3412, 3421, 4123, 4231, 4312, 3241, 4\bar{1}352, 413\bar{5}2\}\right)$$

becomes

Theorem (Pudwell and Smith)

The set of 2-pass pop stack sortable permutations is equal to

$$Av_2(\{2341, 3412, 3421, 4123, 4231, 4312, 3241, 4132\}, \{41352\})$$

Theorem (E, Goh 2021)

For each $k \in \mathbb{N}$, there exist finite sets F_k, G_k such that a permutation is k-pass pop stack sortable if and only if it 2-avoids (F_k, G_k).

$F_3 = \{32451, \ldots\}$, $G_3 = \{4631572, 4731562, \ldots\}$. (Actually the sets we construct are quite large)

We also give an algorithm to construct the sets F_k, G_k from F_{k-1}, G_{k-1}. See `https://www.combinatorics.org/ojs/index.php/eljc/article/view/v28i1p54/pdf`

$F = \{1\}, G = \{12, 21\}$. Then $\mathrm{Av}_2(F, G) = S^\infty \setminus \{1\}$.

This shows that the growth of (proper) 2-avoidance sets can be factorial (in contrast to [3]),

Question: what growth rates are possible for 2-avoidance?

- super-exponential but sub-factorial?
- super-polynomial but sub-exponential? (in contrast to [4])

[3] Marcus and Tardos, Excluded permutation matrices and the Stanley-Wilf conjecture, J. Combin. Theory Ser. A 2004

[4] Kaiser and Klazar, On growth rates of closed permutation classes, EJC 2003

THANKS