Descents on nonnesting multipermutations

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Descents and plateaus

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$A_1(t) = 1$	1
$A_2(t) = 1 + t$	12·, 2·1
$A_3(t) = 1 + 4t + t^2$	123, 13·2, 2·13, 23·1, 3·12, 3·2·1
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These polynomials appear in work of Euler from 1755, and they satisfy

$$\sum_{m \geq 0} m^n t^m = \frac{t\,A_n(t)}{(1-t)^{n+1}}.$$

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1, 1, 2, 2, \ldots, n, n\}$.

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We have $|\mathcal{Q}_n| = (2n-1) \cdot (2n-3) \cdot \dots \cdot 3 \cdot 1$, since every permutation in $\mathcal{Q}_n$ can be obtained by inserting nn into one of the $2n-1$ spaces of a permutation in $\mathcal{Q}_{n-1}$.

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Let $S(\ ,\ )$ denote the Stirling numbers of the second kind.

Theorem (Gessel–Stanley '78)

$$\sum_{m\geq 0} S(m+n,m)\,t^m = \frac{t\,\sum_{\pi\in\mathcal{Q}_n} t^{\mathrm{des}(\pi)}}{(1-t)^{2n+1}}.$$

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There is an extensive literature on the distribution of statistics on Stirling permutations and generalizations to other multisets [Brenti'89, Park'94, Bóna'08, Janson'08, Janson–Kuba–Panholzer'11, Haglund–Visontai'12].

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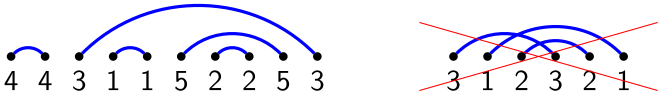

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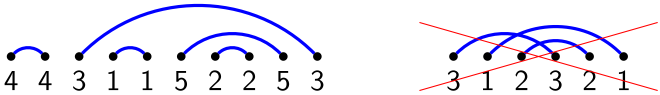

They are in bijection with labeled noncrossing matchings. It follows that

$$|\overline{\mathcal{Q}}_n| = n!\mathrm{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

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Theorem (E. '21)

The number of $\pi \in \overline{\mathcal{Q}}_n$ *with* $\operatorname{des}(\pi) = n-1$ *is equal to* $(n+1)^{n-1}$.

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More generally, consider the generating function

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There is a generalization that also keeps track of the number of plateaus and extends to the multiset with k copies of each number in $[n]$.

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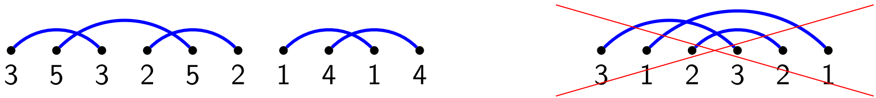

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Our goal is to count nonnesting permutations with respect to the number of descents and plateaus. Consider the polynomials

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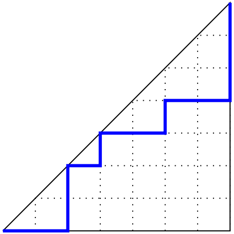
Even though $|\mathcal{C}_n| = |\overline{\mathcal{Q}}_n|$, we have $\displaystyle\sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} \neq \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\mathrm{des}(\pi)}$.

Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to (n,n) with steps $\mathtt{e} = (1,0)$ and $\mathtt{n} = (0,1)$ that do not go above the diagonal $y = x$.



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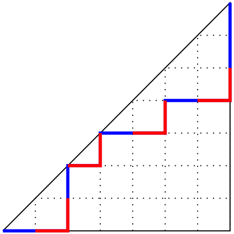

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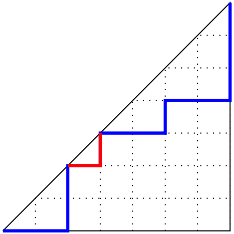

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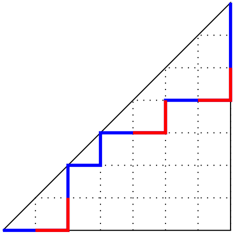

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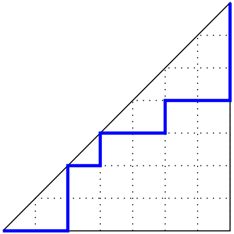

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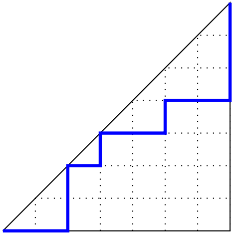

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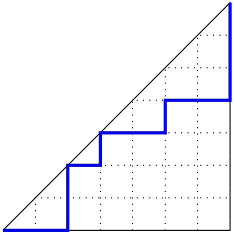

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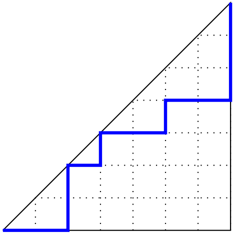

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$$\sum_{n \geq 0} N_n(t,u) z^n = \frac{1}{1 + (1+t-2u)z + \sqrt{1 - 2(1+t)z + (1-t)^2 z^2}}.$$

Main result

Recall:

$$C_n(t,u) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

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Example

$$\begin{aligned} C_3(t,u) &= u^3 + (1+2u+4u^3)t + (5+8u+u^3)t^2 + (5+2u)t^3 + t^4 \\ &= \left(1+4t+t^2\right)\left(u^3+(1+2u)t+t^2\right). \end{aligned}$$

Consequences

Since both $A_n(t)$ and $N_n(t,t)$ are palindromic, so is their product $C_n(t,t)$.

Example

$$C_3(t,t) = t + 7t^2 + 14t^3 + 7t^4 + t^5 = (1 + 4t + t^2)(t + 3t^2 + t^3).$$

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The distribution of weak descents on $\mathcal{C}_n$ is symmetric: for all r,

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We have bijective proofs but they are surprisingly complicated!

A refinement

Partition the set $\mathcal{C}_n$ according to the permutation $\sigma \in \mathcal{S}_n$ given by the first copy of each entry:

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Theorem

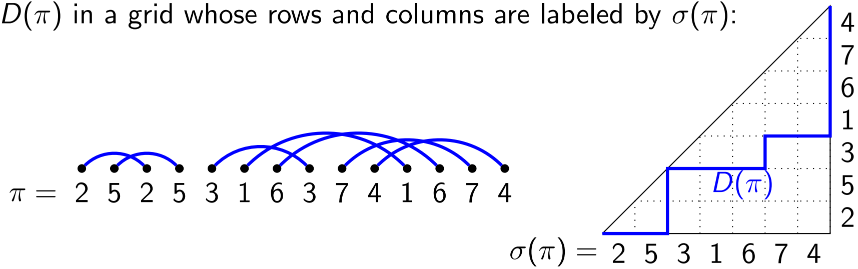
For all $\sigma \in \mathcal{S}_n$,

$$C_n^\sigma(t,u) = t^{\operatorname{des}(\sigma)} N_n(t,u).$$

Summing over $\sigma \in \mathcal{S}_n$, we obtain our main theorem.

About the proofs

Using the standard bijection between nonnesting matchings and Dyck paths, we can represent a nonnesting permutation $\pi \in \mathcal{C}_n$ as a Dyck path $D(\pi)$ in a grid whose rows and columns are labeled by $\sigma(\pi)$:



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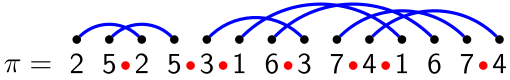

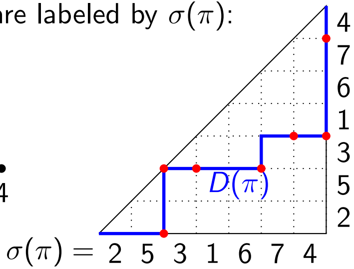

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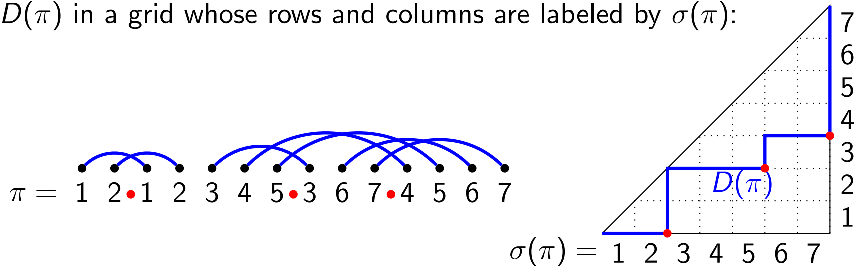

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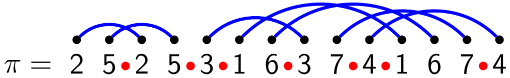

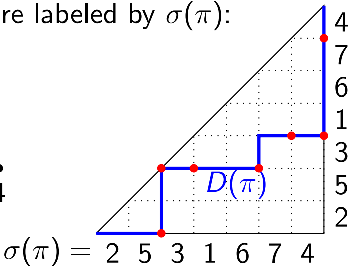

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In general, for each fixed $\sigma \in \mathcal{S}_n$, we get a different Dyck path statistic. We prove that they all have a (shifted) Narayana distribution.

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Our main result generalizes to permutations that have k copies of each number in $[n]$, for any given k.

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In the proof for the general case, the role of Dyck paths is played by standard Young tableaux of rectangular shape.

Thank you

- S.E., Descents on nonnesting multipermutations, arXiv:2204.00165.