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Recursive maps for derangements and nonderangements

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Definitions

A *derangement* is a permutation $\sigma \in S_n$ such that for all $i \in [n]$,

$$\sigma(i) \neq i.$$

We denote by D_n the set of derangements on n elements, and let $d_n = \#D_n$. We denote by E_n the set of permutations of n elements with exactly one fixed point, and let $e_n = \#E_n$.

Recurrence relations

Two well-known recurrences for derangements are

$$d_n = (n-1)d_{n-1} + (n-1)d_{n-2} \quad (1) \qquad \text{and} \qquad d_n = nd_{n-1} + (-1)^n \quad (2)$$

with $d_0 = 1$ and $d_1 = 0$. It can also be shown that

$$e_n = nd_{n-1}. \quad (3)$$

In view of equation (3), we can rewrite (1) as follows:

$$d_n = (n-1)d_{n-1} + e_{n-1}. \quad (4)$$

Also, substituting (3) into (2), we obtain

$$d_n = e_n + (-1)^n. \quad (5)$$

Bijections exhibiting recurrences

Notation

We make use of some notation provided in [2]: Given a permutation $\sigma \in S_n$ and $a \in [n]$, we denote by $\sigma \setminus a$ the permutation given by removing a from the cycle notation of σ.

Equation (3) can be proven via the map

$$f_n : [n] \times D_{n-1} \to E_n, \quad (m,\sigma) \mapsto (mn)\sigma(mn), \qquad \text{with} \qquad f_n^{-1} : E_n \to [n] \times D_{n-1}, \quad \tau \mapsto (a, (an)\tau(an) \setminus n),$$

where a is the unique fixed point of τ. The map f_n constructs a permutation with exactly one fixed point by replacing m with n in the cycle notation of σ and fixing m if $m < n$, and otherwise just appending the one-cycle (n). The map f_n^{-1} essentially swaps the fixed point of τ with n and then removes (n) to get a permutation in D_{n-1}.

For $n > 1$, equation (4) can be proven by exhibiting a bijection between D_n and $([n-1] \times D_{n-1}) \cup E_{n-1}$. Let

$$\varphi_n : D_n \to ([n-1] \times D_{n-1}) \cup E_{n-1}, \quad \sigma \mapsto \begin{cases} (\sigma(n), \sigma \setminus n) & \text{if } \sigma \setminus n \in D_{n-1} \\ \sigma \setminus n & \text{otherwise,} \end{cases} \qquad \text{with} \qquad \varphi_n^{-1} : ([n-1] \times D_{n-1}) \cup E_{n-1} \to D_n, \quad (m,\sigma) \mapsto (nm)\sigma, \quad \tau \mapsto (an)\tau.$$

Removing n from the cycle notation of a derangement yields a permutation in E_{n-1} exactly when n was in a transposition in σ, so the first case of φ_n occurs exactly when n is not in a transposition in σ. The inverse map φ_n^{-1} sends a pair (m, σ) to the permutation $(nm)\sigma$, which essentially adds n in the cycle notation of σ before m. In the second case, φ_n^{-1} sends elements $\tau \in E_{n-1}$ to the permutation obtained by inserting n into a cycle with the unique fixed point of τ.

The proof of relation (2) is by induction using equation (1).

Obtaining new maps

We will use the map φ_n to obtain a bijection showing of the relation (5). This yields a map which is conjugate to the one presented in [2]; here we show how such a map can be deduced using the bijective proof of the two-term identity (4).

Definition

Define $\pi_n \in S_n$ to be the product of disjoint simple transpositions $(1\ 2)(3\ 4)\cdots(n-1\ n)$ if n is even, and if n is odd, $\pi_n = (1\ 2)(3\ 4)\cdots(n-2\ n-1)(n)$. Let Π_n denote the singleton set $\{\pi_n\}$. Define $\pi_{n,j}$ to be a permutation on $[n] \setminus [j-1]$ given by

$$\pi_{n,j} = \begin{cases} (j\ j+1)\cdots(n-1\ n) & \text{if } n-j+1 \text{ is even} \\ (j\ j+1)\cdots(n-2\ n-1)(n) & \text{if } n-j+1 \text{ is odd.} \end{cases}$$

Recursive maps

Notation

Let A, B be disjoint sets. Given functions $f : A \to C$ and $g : B \to D$, we define

$$f \oplus g : A \cup B \to C \cup D, \quad x \mapsto \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Also, denote by $()$ the empty permutation from $\emptyset$ to $\emptyset$.

We will define a map α_n that takes $D_n \to E_n \cup \Pi_n$ if n is even, and if n is odd, $\alpha_n : D_n \cup \Pi_n \to E_n$. Here are the bases cases:

$$\alpha_0 : D_0 \to E_0 \cup \Pi_0, \quad () \mapsto \pi_0 \qquad \text{and} \qquad \alpha_1 : D_1 \cup \Pi_1 \to E_1, \quad \pi_1 \mapsto (1).$$

For $n > 1$, we define the maps α_n and α_n^{-1} recursively. If n is even, $\alpha_n : D_n \to E_n \cup \Pi_n$ is as follows:

$$\alpha_n : D_n \xrightarrow{\varphi_n} ([n-1] \times D_{n-1}) \cup E_{n-1} \xrightarrow{\mathbb{1}_{[n-1]\times D_{n-1}} \oplus \alpha_{n-1}^{-1}} ([n-1] \times D_{n-1}) \cup D_{n-1} \cup \Pi_{n-1} \xrightarrow{g_n^{-1} \oplus \ell_n^{-1}} [n] \times D_{n-1} \cup \Pi_n \xrightarrow{f_n \oplus \mathbb{1}_{\Pi_n}} E_n \cup \Pi_n$$

where $\ell_n : \Pi_n \to \Pi_{n-1}$ sends π_n to $\pi_n \setminus n = \pi_{n-1}$ and g_n is the map which takes an ordered pair (m, σ) and removes the first coordinate if $m = n$. If n is odd, we similarly define $\alpha_n : D_n \cup \Pi_n \to E_n$ as follows:

$$\alpha_n : D_n \cup \Pi_n \xrightarrow{\varphi_n \oplus \ell_n} ([n-1] \times D_{n-1}) \cup E_{n-1} \cup \Pi_{n-1} \xrightarrow{\mathbb{1}_{[n-1]\times D_{n-1}} \oplus \alpha_{n-1}^{-1}} [n-1] \times D_{n-1} \cup D_{n-1} \xrightarrow{g_n^{-1}} [n] \times D_{n-1} \xrightarrow{f_n} E_n.$$

The inverse maps are given by reversing the arrows.

Following through the recursive definitions of the maps, we can show how to directly obtain the image of α_n depending on two possible cases for the input derangement. Using this combinatorial description, we see that α_n is conjugate to the map ψ_n in [2] by an involution on S_n given by "cycle-reversing" permutations. It follows that the combinatorial proof in [2] can be derived from the combinatorial proof for the identity (4).

If we modify the map α_n and extend its domain to $D_n \cup E_n$, we obtain an involution on all of S_n which exchanges the elements of each subset except for π_n.

Definition of $\lambda_n : D_n \cup E_n \to D_n \cup E_n$

If $\sigma = \pi_n$, then λ_n sends σ to itself. If σ has any fixed point m, we add the 2-cycle $(n+1\ m)$. The $n+1$ is a placeholder and will be discarded after applying the map to σ. Let $N = n$ if $n+1$ was not added as a placeholder, and otherwise let $N = n+1$. Then look at the cycle notation of σ, and find the smallest j such that

$$\sigma = \delta \circ \pi_{N,j}$$

for $\delta \in D_{j-1}$, having no copy of $\pi_{j-1,i}$ at the end. If there is no pattern of simple transpositions like this, we let $j = N+1$. Then the image of σ is as follows.

Case 1. If δ has $j-1$ in a 2-cycle, we have

$$\sigma = (\cdots j-2 \cdots)(j-1\ a) \circ \pi_{N,j}$$

and its image is

$$\lambda_n(\sigma) = (\cdots j-2\ a \cdots)(j-1\ j) \circ \pi_{N,j+1}$$

where any values above n are excluded from the cycle notation.

Case 2. If δ does not have $j-1$ in a 2-cycle, we have

$$\sigma = (\cdots j-1\ a \cdots) \circ \pi_{N,j}$$

and its image is

$$\lambda_n(\sigma) = (\cdots j-1 \cdots)(j\ a) \circ \pi_{N,j+1}.$$

Again, any values above n are excluded from the cycle notation. We can check that this defines an involution on $D_n \cup E_n$.

Proposition

Let $\sigma \in D_n \cup E_n$, with $\sigma \neq \pi_n$. If $\sigma \in D_n$, then $\lambda_n(\sigma) \in E_n$. If $\sigma \in E_n$, then $\lambda_n(\sigma) \in D_n$. Also, $\lambda_n(\lambda_n(\sigma)) = \sigma$.

Examples ($n = 5$):

$\lambda_5((12)(345))$: $j = 6$, $j-1 = 5$ which is not in a 2-cycle, so we send it to $(12)(3)(45)$.

$\lambda_5((12)(3)(45))$: First we change it to $(12)(36)(45)$, adding the 6 as a placeholder. Then $j = 7$, so $j-1 = 6$, which is in a 2-cycle. It gets sent to $(12)(345)$.

$\lambda_5((1234)(5))$: $j = 5$, so $j-1 = 4$, which is not in a 2-cycle. This gets sent to $(15)(234)$.

$\lambda_5((15)(234))$: $j = 6$, so $j-1 = 5$, which is in a 2-cycle. This gets sent to $(1234)(5)$.

An involution on S_n

From the recursive maps, we obtained a combinatorial description of α_n that was then extended to λ_n, which fixes π_n and exchanges the other elements of D_n and E_n. Using this, we can define a map on the entire symmetric group

$$\Lambda_n : S_n \to S_n, \quad \sigma \mapsto \begin{cases} \sigma & \text{if } \sigma \notin D_n \cup E_n \\ \lambda_n(\sigma) & \text{otherwise,} \end{cases}$$

which is an involution on S_n.

Nonderangements

Definition

Let $\overline{D}_n$ denote $S_n \setminus D_n$, the set of nonderangements of $[n]$. Similarly let $\overline{E}_n$ denote $S_n \setminus E_n$, the set of permutations of $[n]$ which do not have exactly one fixed point.

Having found the bijection $\alpha_n : D_n \to E_n(\pm\Pi_n)$, we can use the method of subtracting maps described in [1] to obtain a map

$$\overline{\alpha}_n : \overline{D}_n \to \overline{E}_n\ (\mp\Pi_n).$$

The map is given by subtracting α_n from the identity on S_n. We begin with $\sigma \in \overline{D}_n$. If $\sigma \in \overline{E}_n$ as well, that means σ has at least 2 fixed points. So σ is already in $\overline{E}_n$, so when applying the identity, σ lands where we want it to land. So we can just map σ to itself. Otherwise, $\sigma \notin \overline{E}_n$, which means $\sigma \in E_n$. In this case, we apply α_n^{-1} to obtain some permutation $\sigma' \in D_n \subseteq \overline{E}_n$.

Description of the nonderangement map

In summary, the description of $\overline{\alpha}_n$ is as follows:

$$\overline{\alpha}_n : \overline{D}_n \to \overline{E}_n\ (\mp\Pi_n), \quad \sigma \mapsto \begin{cases} \sigma & \text{if } \sigma \in \overline{E}_n \\ \alpha_n^{-1}(\sigma) & \text{if } \sigma \in E_n. \end{cases}$$

With the special case where $\sigma = \pi_n$, we only have that $\pi_n \in \overline{D}_n$ when n is odd. In this case, π_n gets sent to itself and we have $\overline{E}_n \cup \Pi_n$ as the image of $\overline{\alpha}_n$. If n is even, then $\pi_n \in \overline{E}_n$ since π_n is a derangement. However, there is nothing in $\overline{D}_n$ that maps to π_n, so in this case the image of $\overline{\alpha}_n$ is $\overline{E}_n \setminus \Pi_n$.

References

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