A Game of Darts

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Outline

1. The “darts” game
2. Encoding games as permutations
3. Combinatorics of the 2- and 3-player games
4. A non-combinatorial approach (sorry!)
5. Potential threads to explore...

Acknowledgment

This talk is based on work begun with my advisor, Jeffrey Remmel.

So there's this dartboard...

The Game!

Imagine p players lined up to take turns throwing a dart at a dartboard. Player 1 is at the front, followed by Player 2, and so on, ending with Player p at the very back.

The players are all equally skilled.

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The players are all equally skilled.

The **golden rule** is: whenever it is your turn to throw, your dart **must** land closer to the center than any other previously thrown dart.

- If you **succeed**, you go to the back of the line to await your next turn.
- If you **don't**, you are eliminated.

Play continues for as long as it takes until one player remains. The last player remaining is the **winner**.

So there's this dartboard...

Example game with 3 players. ("Eliminating" throws are in red.)

- Player 1's first throw lands 10 inches from the center.
- Then Player 2 throws a dart that lands 12 inches away.
- Then Player 3 throws a dart that lands 7 inches away.
- Then Player 1 throws a dart that lands 6.3 inches away.
- Then Player 3 throws a dart that lands π inches away.
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Player 3 wins!

Encoding games as permutations

We can **encode** each game as a permutation, by replacing the "distances to the center" by the corresponding permutation with the same relative ordering.

The example game, with distances $10, 12, 7, 6.3, \pi, 8$, would then be encoded by the permutation $(5, 6, 3, 2, 1, 4)$.

Given any permutation $\sigma \in S_n$ there is a $\frac{1}{n!}$ chance that n random throws will have the relative ordering σ.

Encoding games as permutations

The final throw in any given game is a losing throw, therefore no game can be encoded in a permutation ending in 1.

Let's use $\overline{S}_n$ to denote the set of $n! - (n-1)!$ permutations of $[n]$ that do *not* end in 1.

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some permutations $\sigma \in \overline{S}_4$:

σ	# elim	winner
1234	3	1
2413	2	3
3214	1	1
4123	2	2

The number of eliminations in a game equals the number of values in its permutation that are **not** left-to-right minimal.

Terminology... and a claim!

Terminology

We use $w_{n,p,k}$ to denote the number of permutations of length n that correspond to p-player games in which player k is the winner.

We use $P_{p,k}$ to denote the **win probability** for player k in a p-player game. Note that $P_{p,k} = \sum_n \frac{w_{n,p,k}}{n!}$

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A Claim!

$$\sum_k w_{n,p,k} = (n-1) \begin{bmatrix} n+p-1 \\ n \end{bmatrix}$$

(Can be shown by setting up boundary conditions & recurrence relation...)

The 2-player game

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Some examples:

σ	winner
1 2	1
3 1 2	2
4 2 1 3	1
4 3 2 1 5	2
6 5 3 2 1 4	1

The 2-player game

It is not hard to see that

$$w_{n,2,1} = \begin{cases} n-1 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \quad \text{and} \quad w_{n,2,2} = \begin{cases} 0 & n \text{ is even} \\ n-1 & n \text{ is odd} \end{cases}$$

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$$= e^{-1}$$

Result — Niedermaier, Remmel (2007)

$P_{2,1} = 1 - e^{-1}$, and $P_{2,2} = e^{-1}$.

The 3-player game

Things are a bit more complicated for the 3-player game...

For instance, consider the set of permutations which describe a 3-player game in which Player 1 is the winner. One can show that

$$P_{3,1} = \sum_{a \geq 0} \sum_{b \geq 0} \frac{(3a+1)(3a+2b+2)}{(3a+2b+3)!} + \frac{(3a+2)(3a+2b+4)}{(3a+2b+5)!}$$
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We can use the numerators above to back out values for $w_{n,3,1}$:

$$\{w_{n,3,1}\}_{n\geq 3} = 2, 0, 12, 20, 18, 63, 80, 81, 180, 209, 216, 390, 434, \ldots$$

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$$\{w_{n,3,2}\}_{n \geq 3} = 0, 6, 0, 25, 30, 35, 88, 117, 110, 242, 264, 286, 476, \ldots$$
$$\{w_{n,3,3}\}_{n \geq 3} = 0, 3, 12, 5, 42, 49, 56, 126, 160, 154, 312, 338, 364, \ldots$$

Some values of $w_{n,p,k}$ (namely, for $n = 12$)

p \ k	1	2	3	4	5	6	7	8	9	10	11
2	11	0	0	0	0	0	0	0	0	0	0
3	209	242	154	0	0	0	0	0	0	0	0
4	3,410	3,597	4,081	3,432	0	0	0	0	0	0	0
5	46,255	38,918	34,111	37,631	42,735	0	0	0	0	0	0
6	261,789	266,860	303,017	319,704	314,633	269,500	0	0	0	0	0
7	1,689,688	1,736,658	1,609,784	1,403,072	1,060,796	1,086,822	1,335,785	0	0	0	0
8	6,845,058	5,374,600	3,989,348	2,442,528	3,193,014	4,289,340	5,301,274	6,151,068	0	0	0
9	12,993,640	7,300,656	2,652,408	7,977,552	10,723,284	12,078,000	12,726,120	12,996,720	13,056,120	0	0
10	11,292,336	0	19,958,400	19,958,400	18,295,200	16,632,000	15,190,560	13,970,880	12,937,320	12,054,240	0
11	0	39,916,800	19,958,400	13,305,600	9,979,200	7,983,360	6,652,800	5,702,400	4,989,600	4,435,200	3,991,680
12	39,916,800	0	0	0	0	0	0	0	0	0	0

p \ k	1	2	3	4	5	6	7	8	9	10	11
2	1	--	--	--	--	--	--	--	--	--	--
3	0.345455	0.4	0.254545	--	--	--	--	--	--	--	--
4	0.234848	0.247727	0.281061	0.236364	--	--	--	--	--	--	--
5	0.23168	0.194931	0.170854	0.188485	0.21405	--	--	--	--	--	--
6	0.150843	0.153765	0.174599	0.184214	0.181292	0.155286	--	--	--	--	--
7	0.170287	0.17502	0.162234	0.141402	0.106907	0.10953	0.13462	--	--	--	--
8	0.182116	0.142994	0.106139	0.064985	0.084952	0.11412	0.141043	0.163652	--	--	--
9	0.140465	0.078922	0.028673	0.08624	0.115922	0.130567	0.137573	0.140498	0.14114	--	--
10	0.080493	--	0.142266	0.142266	0.13041	0.118555	0.10828	0.099586	0.092219	0.085924	--
11	--	0.341417	0.170709	0.113806	0.085354	0.068283	0.056903	0.048774	0.042677	0.037935	0.034142
12	1	--	--	--	--	--	--	--	--	--	--

A continuous dartboard

Let's model the dartboard as the unit interval, and imagine each "dart" is a pull from $U[0,1]$. (Lower pulls are better, as they represent your distance to the center.)

Let us generalize $P_{p,k}$ to take a real-valued parameter. Define $P_{p,k}(x)$ to be the probability that the player currently in position k, out of p players in total, will eventually win the game, given that the current "score to beat" for the *person at the front of the line* is x.

Thus $P_{p,k} = P_{p,k}(1)$ for all p and k.

Note also that $P_{p,k}(0) = \begin{cases} 1 & \text{if } k = p \\ 0 & \text{otherwise} \end{cases}$

A continuous dartboard

The following system of equations arises:

$$
\begin{aligned}
P_{2,1}(x) &= \int_0^x P_{2,2}(u)\,\mathrm{d}u \\
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This is a solvable DE! We get

$$P_{2,1}(x) = 1 - e^{-x}$$
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Result — Niedermaier, Remmel (2007)

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P_{3,1}(x) &= \int_0^x P_{3,3}(u)\,\mathrm{d}u \\
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Wait for it...

A continuous dartboard

Result Niedermaier (2021)

$$P_{3,1}(x) = 1 + e^{-x}(x-1) - \frac{2e^{-x/2}}{\sqrt{3}} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

$$P_{3,2}(x) = -e^{-x} + e^{-x/2}\left(\cos\left(\frac{\sqrt{3}x}{2}\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

$$P_{3,3}(x) = e^{-x}(2-x) + e^{-x/2}\left(\frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}x}{2}\right) - \cos\left(\frac{\sqrt{3}x}{2}\right)\right)$$

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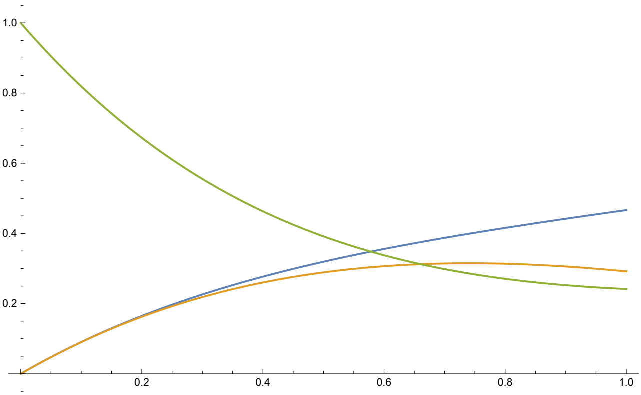
A continuous dartboard

The plots for $P_{3,k}(x)$:



At the far right ($x = 1$) we have the win probabilities:
$(P_{3,1}, P_{3,2}, P_{3,3}) \approx (0.4665, 0.2918, 0.2417)$.

4 players!

For 4 players we have

$$P_{4,1}(x) = \int_0^x P_{4,3}(u)\,\mathrm{d}u$$
$$P_{4,2}(x) = (1-x)\cdot P_{3,1}(x) + \int_0^x P_{4,1}(u)\,\mathrm{d}u$$
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Result Niedermaier (2021)

$$P_{4,1}(x) = 1 + \frac{5}{4}(\cos x - \sin x - e^{-x}) + 2xe^{-x} - \frac{3x^2e^{-x}}{4} - e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)(x+1) + \sqrt{3}e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)\left(x-\frac{1}{3}\right)$$

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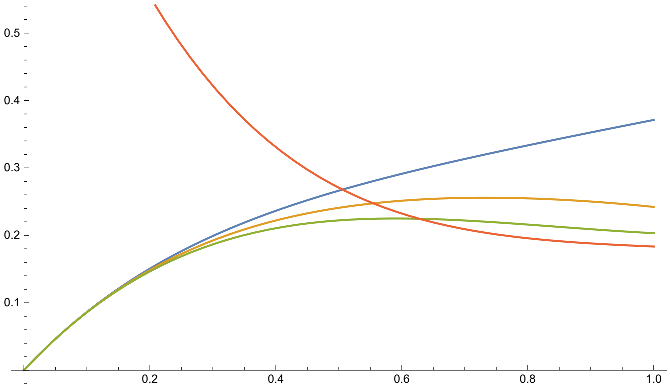

Win probabilities: $(P_{4,1}, P_{4,2}, P_{4,3}, P_{4,4}) \approx (0.3712, 0.2422, 0.2032, 0.1834)$.

Potential threads to explore...

- Connecting the winner of a game (permutation) to other areas in permutation statistics.
- Generating functions to count statistics on games, e.g.

$$\sum_{\sigma \in \overline{S}_n} p^{\#\text{ players}} \cdot q^{\text{winner}}$$

- Generating functions to count games by its "rank permutation" γ,

$$f_r(\gamma) = \sum_{\sigma \text{ w/ rk perm. } \gamma} r^{|\sigma|}$$

  - e.g., $f_r(12) = \frac{r^2+r^4}{(1-r^2)^2}$, and $f_r(132) = r^2\left(\frac{r^2}{1-r^2}\left(\frac{r}{1-r^3}\right)'\right)'$
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The End Niedermaier (2022)

THANKS!