Long paths, deep trees and dual cycles

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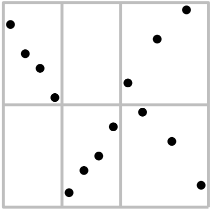
Monotone grid classes

Definition

A $k \times \ell$ matrix $\mathcal{M}$ of permutation classes with entries from the set $\{\mathrm{Av}(21), \mathrm{Av}(12), \emptyset\}$ is a monotone gridding matrix. The (monotone) grid class of $\mathcal{M}$, denoted by $\mathrm{Grid}(\mathcal{M})$, is a class of permutations that can be partitioned, by $\ell - 1$ horizontal and $k - 1$ vertical cuts, into a $k \times \ell$ array of cells (called gridding), where the cell in the i-th column and j-th row induces a pattern from $\mathcal{M}_{i,j}$.

$$\mathcal{M} = \begin{pmatrix} \mathrm{Av}(12) & & \mathrm{Av}(21) \\ & \mathrm{Av}(21) & \mathrm{Av}(12) \end{pmatrix}$$



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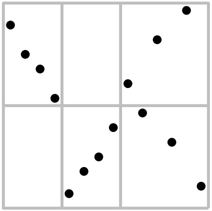

Definition

The cell graph of $\mathcal{M}$ is a graph whose vertices are the non-empty cells of $\mathcal{M}$, with two vertices being adjacent if they share a row or a column of $\mathcal{M}$ and all cells between them are empty.

Long paths

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A permutation class $\mathcal{C}$ has the long path property (LPP) if for every k the class $\mathcal{C}$ contains a monotone grid subclass whose cell graph is a path of length k.

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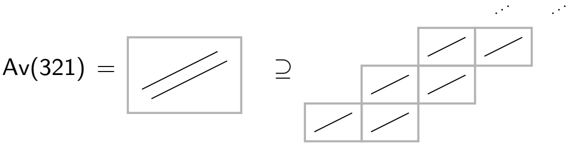
The class Av(321) has the LPP.



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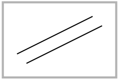
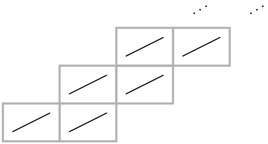
$\mathrm{Av}(321) =$  $\supseteq$ 

Definition

A permutation class $\mathcal{C}$ has the computable long path property if there exists an algorithm that given a k computes the description of a monotone grid subclass of $\mathcal{C}$ whose cell graph is a path of length k.

Deep trees

Definition

A c-subdivided binary tree of depth d is a graph obtained from a binary tree of depth d by replacing every edge by a path of length at most c. A permutation class $\mathcal{C}$ has the deep tree property (DTP) if there is a c such that for every d, the class $\mathcal{C}$ contains a monotone grid subclass whose cell graph is a c-subdivided binary tree of depth d.

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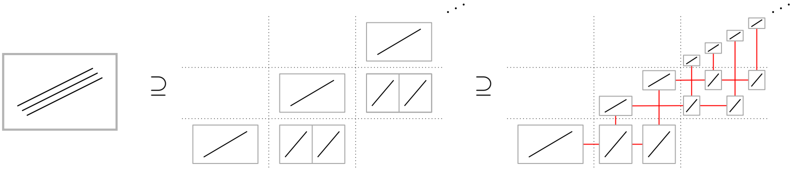
The class Av(4321) has the DTP.

Av(4321) $=$



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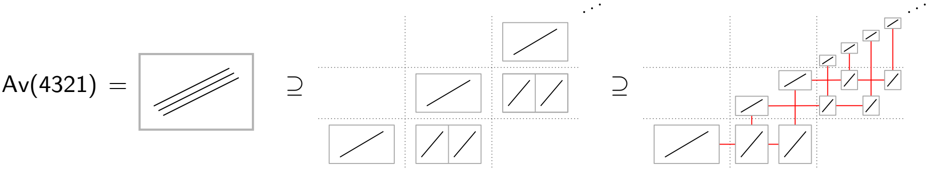

Definition

A permutation class $\mathcal{C}$ has the computable deep tree property, if there exists an algo...

Dual cycles

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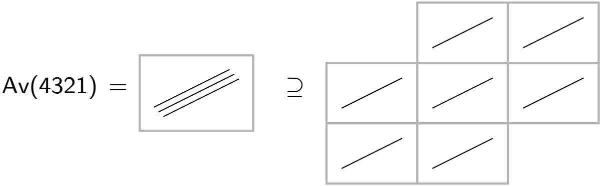

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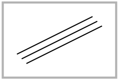
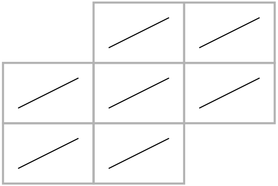
$\mathrm{Av}(4321) =$  $\supseteq$ 

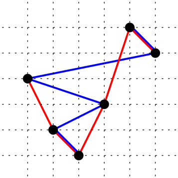
Proposition

The BP implies the DTP, and the DTP implies the LPP.

Tree-width

Definition

The incidence graph G_π of a permutation π is the graph whose vertices are the n entries $\pi_1, \ldots, \pi_n$, with two entries π_i and π_j connected by an edge if $|i - j| = 1$ or $|\pi_i - \pi_j| = 1$.



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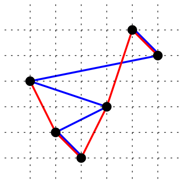

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Theorem (Berendsohn, Kozma and Marx, 2019)

Given a permutation π of length k and a permutation τ of length n, we can decide if τ contains π in time $O(n^{\mathrm{tw}(\pi)+1})$.

Tree-width lower bounds

Definition

The tree-width growth function of a permutation class $\mathcal{C}$ is defined as

$$\mathrm{tw}_{\mathcal{C}}(n) = \max\{\mathrm{tw}(\pi);\ \pi \in \mathcal{C} \wedge |\pi| = n\}.$$

Theorem (Berendsohn, 2019)

For a pattern σ, $\mathrm{tw}_{Av(\sigma)}(n) = \Omega(n/\log n)$ if $|\sigma| \geq 4$ and σ is not symmetric to one of 3412, 3142, 4213, 4123, 41352, 42153 and 42513.

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Theorem

For a permutation class $\mathcal{C}$ we have

- $\mathrm{tw}_{\mathcal{C}}(n) \in \Omega(\sqrt{n})$ *if $\mathcal{C}$ has the long path property,*
- $\mathrm{tw}_{\mathcal{C}}(n) \in \Omega(n/\log n)$ *if $\mathcal{C}$ has the deep tree property, and*
- $\mathrm{tw}_{\mathcal{C}}(n) \in \Theta(n)$ *if $\mathcal{C}$ has the bicycle property.*

Graph minors

Definition

A graph H is a minor of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges.

Fact

If H is a minor of G then we have $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

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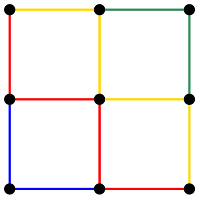
Definition
For a positive interger k, the $k \times k$ *grid graph* is the graph with vertex set $[k] \times [k]$ such that vertices (i, j) and (i', j') are joined with an edge if and only if $|i - i'| + |j - j'| = 1$.



Proposition (folklore)
The $k \times k$ grid graph has tree-width exactly k.

Lower bound for long paths

Goal: Find a permutation $\pi \in \mathcal{C}$ of length k^2 such that G_π contains the $k \times k$ grid graph as a minor.

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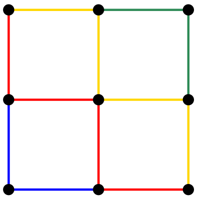

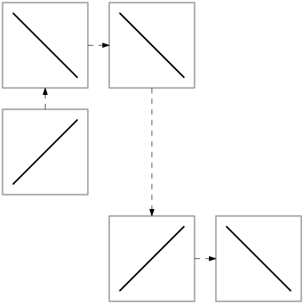

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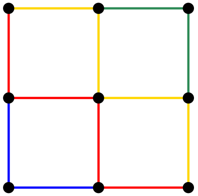
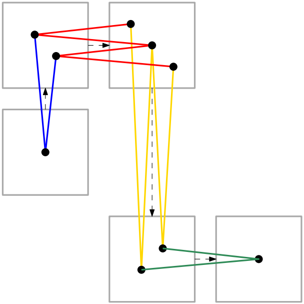
2. Map diagonals of the $k \times k$ grid to the cells on the path with two consecutive cells interleaved.





Lower bound for deep trees

Goal: For a given arbitrary graph G with m edges, find a permutation $\pi \in \mathcal{C}$ of length $O(m \log m)$ such that G_π contains G as a minor. The result follows from the existence of sparse graphs with linear tree-width.

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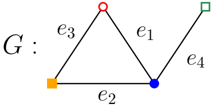

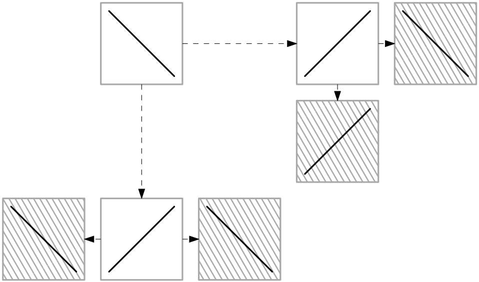

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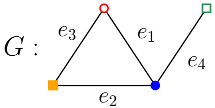
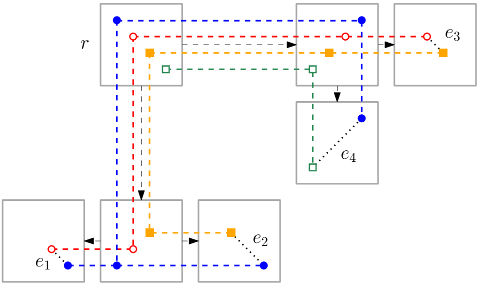
2. Map each leaf in $G_\mathcal{M}$ to a unique edge in G with two points representing its vertices. Each point is associated to a specific vertex of G.





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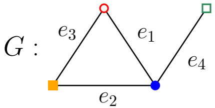
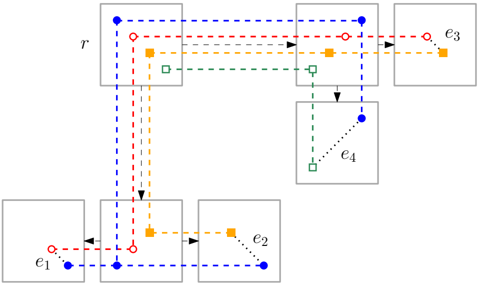
3. Contract the subgraphs induced by points associated to the same vertex.





Principal classes

σ	LPP of Av(σ)	DTP of Av(σ)	BP of Av(σ)
21, 312	✗	✗	✗
321, 3412, 3142, 4213, 4123, 41352	✓	✗	✗
all other	✓	✓	✓

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Corollary

For a pattern σ, $\mathrm{tw}_{Av(\sigma)}(n) = \Theta(n)$ if $|\sigma| \geq 4$ and σ is not symmetric to one of 3412, 3142, 4213, 4123, 41352.

Computational complexity

Can we also deduce lower bounds for the complexity of pattern matching or counting?

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Hypothesis (Exponential Time Hypothesis (ETH))

There exists $\delta > 0$ *such that* 3-SAT *cannot be solved in time* $O(2^{\delta n})$ *where* n *is the number of variables.*

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#PERMUTATION PATTERN MATCHING (#PPM)	
Input:	Permutations π of size k and τ of size n.
Question:	How many occurrences of π are contained in τ?

Theorem (Berendsohn, Kozma and Marx, 2019)

#PERMUTATION PATTERN MATCHING *cannot be solved in time* $f(k) \cdot n^{o(k/\log k)}$ *for any function* f *unless ETH fails.*

Counting patterns from a given class

$\mathcal{C}$-PATTERN #PERMUTATION PATTERN MATCHING ($\mathcal{C}$-PATTERN #PPM)

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Theorem (Berendsohn, 2019)

AV(654321)-PATTERN #PPM *cannot be solved in time* $f(k) \cdot n^{o(k/\log^4 k)}$ *for any function* f *unless ETH fails.*

Counting patterns from a given class

Theorem

- *If $\mathcal{C}$ has the computable LPP, then $\mathcal{C}$-*$\textsc{Pattern}$ $\#\mathrm{PPM}$ *cannot be solved in time* $f(k) \cdot n^{o(\sqrt{k})}$ *for any function* f *assuming ETH, and*
- *if $\mathcal{C}$ has the computable DTP, then $\mathcal{C}$-*$\textsc{Pattern}$ $\#\mathrm{PPM}$ *cannot be solved in time* $f(k) \cdot n^{o(k/\log^2 k)}$ *for any function* f *assuming ETH.*

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Corollary

For a pattern σ of length at least 4 and not symmetric to one of 3412, 3142, 4213, 4123, 41352, $\mathrm{Av}(\sigma)$-Pattern #PPM *cannot be solved in time* $f(k) \cdot n^{o(k/\log^2 k)}$ *under ETH.*

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As a byproduct, we also obtain asymptotic optimality of the tree-width based algorithm.

Theorem

If $\mathcal{C}$ has the computable LPP then $\mathcal{C}$-$\textsc{Pattern}$ PPM cannot be solved in time $f(\mathrm{tw}(\pi)) \cdot n^{o(\mathrm{tw}(\pi))}$ for any function f, unless ETH fails.

Trichotomy for monotone grid classes

Theorem (Jelínek, O. and Pekárek, 2020)

For a monotone gridding matrix $\mathcal{M}$ one of the following holds:

- *Either $G_{\mathcal{M}}$ is a forest, $\mathrm{tw}_{\mathsf{Grid}(\mathcal{M})}(n) = O(1)$ and $\mathsf{Grid}(\mathcal{M})$-Pattern PPM is polynomial-time solvable, or*
- *$G_{\mathcal{M}}$ contains a cycle, $\mathrm{tw}_{\mathsf{Grid}(\mathcal{M})}$ is unbounded and $\mathsf{Grid}(\mathcal{M})$-Pattern PPM is NP-complete.*

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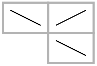
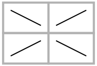
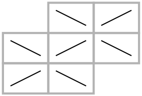
Theorem

# of cycles	*Example*	*LPP*	*BP*	$\mathrm{tw}_{Grid_{\mathcal{M}}}(n)$	$\mathsf{Grid}(\mathcal{M})$-Pattern #PPM *lower bound*
0		✗	✗	$\Theta(1)$	—
1		✓	✗	$\Theta(\sqrt{n})$	$f(k) \cdot n^{o(\sqrt{k})}$
≥ 2		✓	✓	$\Theta(n)$	$f(k) \cdot n^{o(k/\log^2 k)}$

Summary

- The LPP, DTP and BP imply lower bounds both for the tree-width growth function and the complexity of pattern counting.
- Av(σ) has the LPP for every σ of length at least 3 except for the symmetries of 312.
- Av(σ) has the BP for every σ of length at least 4 that is not symmetric to one of 3412, 3142, 4213, 4123, 41352.

Question

Is the LPP the only obstacle to bounded tree-width? In other words, is it true that a class $\mathcal{C}$ has unbounded tree-width if and only if it has the LPP?

Question

For $\sigma \in \{3412, 3142, 4213, 4123, 41352\}$, we have $\mathrm{tw}_{\mathsf{Av}(\sigma)}(n) = \Omega(\sqrt{n})$ but no upper bound other than the trivial $O(n)$. What are the actual tree-width growth functions?

Question

Can these properties be used to obtain results in other areas?

Thank you!