Transport of Patterns by Burge transpose

Giulio Cerbai
Joint work with Anders Claesson

University of Iceland
giulio@hi.is
akc@hi.is

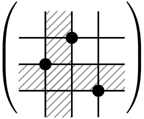
Motivation

(2+2)-free posets, ascent sequences and pattern avoiding permutations
M. Bousquet-Melou, A. Claesson, M. Dukes, S. Kitaev.

Ascent Sequences embody the combinatorial structures of:

- $(2+2)$-free posets;
- Stoimenow's matchings;
- Fishburn permutations:

$$\mathcal{F} = \mathrm{Sym}$$



- Fishburn trees (ongoing work, PP2023?).

Literature

Patterns on ascent sequences

- P. Duncan, E. Steingrimsson, *Pattern avoidance in ascent sequences*.
- Chen et al, *On 021-Avoiding Ascent Sequences*.
- T. Mansour, M. Shattuck, *Some enumerative results related to ascent sequences*.
- L. Pudwell, *Ascent sequences and the binomial convolution of Catalan numbers*.
- A. Baxter, L. Pudwell, *Ascent sequences avoiding pairs of patterns*.

Patterns on Fishburn permutations

- J. Gil, M. D. Weiner, *On pattern-avoiding Fishburn permutations*.

Motivation

Main Goal

To develop a general framework to handle pattern avoidance on several combinatorial objects.

Ascent sequences $\Longleftrightarrow$ Fishburn permutations

Transport of patterns

- Settings: **Endofunctions**.
- Operator: **Burge transpose**.

Endofunctions

- $[\mathrm{n}] = \{1, \dots, n\}$.
- $\mathrm{End}_n = \{x : [\mathrm{n}] \to [\mathrm{n}]\}$.
- Linear notation: $x = x_1 \cdots x_n$, where $x_i = x(i)$.

Endofunctions

- $[\mathrm{n}] = \{1, \ldots, n\}$.
- $\mathrm{End}_n = \{x : [\mathrm{n}] \to [\mathrm{n}]\}$.
- Linear notation: $x = x_1 \cdots x_n$, where $x_i = x(i)$.

Permutations: Sym_n

$\mathrm{Im}(x) = [\mathrm{n}]$ $\longrightarrow$ Each integer from 1 to n appears exactly once.

Cayley Permutations: Cay_n

$\mathrm{Im}(x) = [\mathrm{k}]$, for some k $\longrightarrow$ Each integer from 1 to k appears at least once.

Ascent sequences $\mathcal{A}$

An endofunction $x \in \mathrm{End}_n$ is an **ascent sequence** if:

- $x_1 = 1$;
- $x_{i+1} \leq |\mathrm{Asc}'(x_1 \cdots x_i)| + 2$, for each $i \leq n - 1$.

Ascent sequences $\mathcal{A}$

An endofunction $x \in \mathrm{End}_n$ is an **ascent sequence** if:

- $x_1 = 1$;
- $x_{i+1} \leq |\mathrm{Asc}'(x_1 \cdots x_i)| + 2$, for each $i \leq n-1$.

Example

- $x \ = \ 1$
- $|\mathrm{Asc}'(x_1 \cdots x_i)| = \{$
- $x_{i+1} \in \{1, 2,$

Ascent sequences $\mathcal{A}$

An endofunction $x \in \mathrm{End}_n$ is an **ascent sequence** if:

- $x_1 = 1$;
- $x_{i+1} \leq |\mathrm{Asc}'(x_1 \cdots x_i)| + 2$, for each $i \leq n - 1$.

Example

- $x \;=\; 12$
- $|\mathrm{Asc}'(x_1 \cdots x_i)| = \{1,$
- $x_{i+1} \in \{1, 2, 3,$

Ascent sequences $\mathcal{A}$

An endofunction $x \in \mathrm{End}_n$ is an **ascent sequence** if:

- $x_1 = 1$;
- $x_{i+1} \leq |\mathrm{Asc}'(x_1 \cdots x_i)| + 2$, for each $i \leq n-1$.

Example

- $x \ = \ 121$
- $|\mathrm{Asc}'(x_1 \cdots x_i)| = \{1,$
- $x_{i+1} \in \{1, 2, 3,$

Ascent sequences $\mathcal{A}$

An endofunction $x \in \mathrm{End}_n$ is an **ascent sequence** if:

- $x_1 = 1$;
- $x_{i+1} \leq |\mathrm{Asc}'(x_1 \cdots x_i)| + 2$, for each $i \leq n - 1$.

Example

- $x \;=\; 1212$
- $|\mathrm{Asc}'(x_1 \cdots x_i)| = \{1, 3,$
- $x_{i+1} \in \{1, 2, 3, 4,$

Ascent sequences $\mathcal{A}$

An endofunction $x \in \mathrm{End}_n$ is an **ascent sequence** if:

- $x_1 = 1$;
- $x_{i+1} \leq |\mathrm{Asc}'(x_1 \cdots x_i)| + 2$, for each $i \leq n - 1$.

Example

- $x \; = \; 12124$
- $|\mathrm{Asc}'(x_1 \cdots x_i)| = \{1, 3, 4\}$
- $x_{i+1} \in \{1, 2, 3, 4, 5\}$

Modified ascent sequences $\hat{\mathcal{A}}$

- Originally defined as the bijective image $\hat{\mathcal{A}}$ of $\mathcal{A}$ through the map $\hat{}$.

Modified ascent sequences $\hat{A}$

- Originally defined as the bijective image $\hat{A}$ of $\mathcal{A}$ through the map $\hat{}$.

Alternative Description as subset of Cay

A Cayley permutation $x = x_1 \cdots x_n$ is a Modified Ascent Sequence if and only if:

1. $x_1 = 1$, and
2. $x_i < x_{i+1}$ if and only if x_{i+1} is the leftmost occurrence of that integer in x.

$$\boxed{\{\text{Ascent tops}\} \bigcup \{x_1\} \; = \; \{\text{Leftmost occurrences}\}}$$

Modified ascent sequences $\hat{\mathcal{A}}$

- Originally defined as the bijective image $\hat{\mathcal{A}}$ of $\mathcal{A}$ through the map $\hat{}$.

Alternative Description as subset of Cay

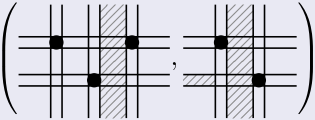
A Cayley permutation $x = x_1 \cdots x_n$ is a Modified Ascent Sequence if and only if:

1. $x_1 = 1$, and
2. $x_i < x_{i+1}$ if and only if x_{i+1} is the leftmost occurrence of that integer in x.

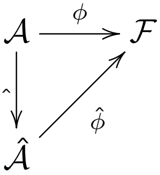
$$\boxed{\{\text{Ascent tops}\} \bigcup \{x_1\} \ = \ \{\text{Leftmost occurrences}\}}$$

In terms of patterns

$$\hat{\mathcal{A}} = \mathrm{Cay}$$



Summary



$\mathcal{A}$: **Ascent Sequences**

$\hat{\mathcal{A}}$: **Modified Ascent Sequences**

$\mathcal{F}$: **Fishburn permutations**

Outline

1. Provide a high-level description of $\hat{\phi}$ via the Burge transpose.
2. Transport of patterns on $[\mathrm{Cay}]$ and Sym.
3. Transport of patterns on $\hat{\mathcal{A}}$ and $\mathcal{F}$.

Definition of $\hat{\phi} : \hat{\mathcal{A}} \to \mathcal{F}$

Let $x = 1412231 \in \hat{\mathcal{A}}_n$.

Definition of $\hat{\phi} : \hat{\mathcal{A}} \to \mathcal{F}$

Let $x = 1412231 \in \hat{\mathcal{A}}_n$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 1 & 2 & 2 & 3 & 1 \end{pmatrix}$$

1. Write the integers from 1 to n above x.

Definition of $\hat{\phi} : \hat{\mathcal{A}} \to \mathcal{F}$

Let $x = 1412231 \in \hat{\mathcal{A}}_n$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 1 & 2 & 2 & 3 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 4 & 1 & 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

1. Write the integers from 1 to n above x.
2. Flip the biword.

Definition of $\hat{\phi} : \hat{\mathcal{A}} \to \mathcal{F}$

Let $x = 1412231 \in \hat{\mathcal{A}}_n$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 1 & 2 & 2 & 3 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 4 & 1 & 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 7 & 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}$$

1. Write the integers from 1 to n above x.
2. Flip the biword.
3. Sort the columns in increasing order w.r.t. the top entry.
4. Break ties by sorting in decreasing order w.r.t. the bottom entry.

Definition of $\hat{\phi} : \hat{\mathcal{A}} \to \mathcal{F}$

Let $x = 1412231 \in \hat{\mathcal{A}}_n$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 1 & 2 & 2 & 3 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 4 & 1 & 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 7 & 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}$$

1. Write the integers from 1 to n above x.
2. Flip the biword.
3. Sort the columns in increasing order w.r.t. the top entry.
4. Break ties by sorting in decreasing order w.r.t. the bottom entry.

$$\boxed{\hat{\phi}(1412231) \;=\; 7315462}$$

Burge biwords

$$\mathrm{Bur}_n = \left\{ \binom{u}{x} : x \in \mathrm{Cay}_n,\ u \in I_n,\ \mathrm{Des}(u) \subseteq \mathrm{Des}(x) \right\},$$

where I_n is the set of weakly increasing Cayley permutations.

Burge biwords

$$\mathrm{Bur}_n = \left\{ \binom{u}{x} : x \in \mathrm{Cay}_n,\ u \in I_n,\ \mathrm{Des}(u) \subseteq \mathrm{Des}(x) \right\},$$

where I_n is the set of weakly increasing Cayley permutations.

Burge Transpose

$$\binom{u}{x}^T = \binom{v}{y},$$

where $\binom{v}{y}$ is obtained by:

1. Flipping $\binom{u}{x}$;
2. Sorting the columns in increasing order w.r.t. the top entry;
3. Breaking ties by sorting in decreasing order w.r.t. the bottom entry.

Burge biwords

$$\mathrm{Bur}_n = \left\{ \binom{u}{x} : x \in \mathrm{Cay}_n,\ u \in I_n,\ \mathrm{Des}(u) \subseteq \mathrm{Des}(x) \right\},$$

where I_n is the set of weakly increasing Cayley permutations.

Burge Transpose

$$\binom{u}{x}^T = \binom{v}{y},$$

where $\binom{v}{y}$ is obtained by:

1. Flipping $\binom{u}{x}$;
2. Sorting the columns in increasing order w.r.t. the top entry;
3. Breaking ties by sorting in decreasing order w.r.t. the bottom entry.

Remark

T is an involution on Bur_n.

The map $\gamma : \mathrm{Cay}_n \to \mathrm{Sym}_n$

- Let $\mathfrak{i}_n = 12 \cdots n$.
- The Burge Transpose induces a map $\gamma : \ \mathrm{Cay}_n \to \mathrm{Sym}_n$ by:

$$\boxed{\begin{pmatrix} \mathfrak{i} \\ x \end{pmatrix}^T = \begin{pmatrix} v \\ \gamma(x) \end{pmatrix}}$$

The map $\gamma : \mathrm{Cay}_n \to \mathrm{Sym}_n$

- Let $\mathrm{i}_n = 12\cdots n$.
- The Burge Transpose induces a map $\gamma :\ \mathrm{Cay}_n \to \mathrm{Sym}_n$ by:

$$\boxed{\binom{\mathrm{i}}{x}^T = \binom{v}{\gamma(x)}}$$

γ on Modifed Ascent Sequences

If $x \in \hat{\mathcal{A}}_n$, then $\gamma(x) \;=\; \hat{\phi}(x)$.

$$\boxed{\gamma \text{ generalizes } \hat{\phi} :\ \hat{\mathcal{A}} \to \mathcal{F}}$$

γ on classical permutations

If $x \in \mathrm{Sym}_n$, then $\gamma(x) \;=\; x^{-1}$.

$$\boxed{\gamma \text{ generalizes the permutation inverse}}$$

Transport on $[\mathrm{Cay}]$ and Sym

Equivalence Classes

Let $x, y \in \mathrm{Cay}_n$. Define the quotient set $[\mathrm{Cay}]$ by:

$$x \sim y \iff \gamma(x) = \gamma(y).$$

Transport on $[\mathrm{Cay}]$ and Sym

Equivalence Classes

Let $x, y \in \mathrm{Cay}_n$. Define the quotient set $[\mathrm{Cay}]$ by:

$$x \sim y \iff \gamma(x) = \gamma(y).$$

Pattern Containment on $[Cay]$

$[x] \geq [y]$ if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$.

Transport on $[\mathrm{Cay}]$ and Sym

Equivalence Classes

Let $x, y \in \mathrm{Cay}_n$. Define the quotient set $[\mathrm{Cay}]$ by:

$$x \sim y \iff \gamma(x) = \gamma(y).$$

Pattern Containment on $[Cay]$

$[x] \geq [y]$ if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$.

Transport Theorem on $[\mathrm{Cay}]$ and Sym

$$[x] \geq [y] \iff \gamma(x) \geq \gamma(y)$$

or

$$\gamma([\mathrm{Cay}][y]) = \mathrm{Sym}(\gamma(y)).$$

Transport Theorem with inverse permutation

We can rewrite:

$$\mathrm{Sym}(\gamma(y)) \;=\; \gamma([\mathrm{Cay}][y])$$

as:

$$\boxed{\mathrm{Sym}(p) \;=\; \gamma([\mathrm{Cay}][p^{-1}])}$$

Transport Theorem for sets of Patterns

- Let Σ be a set of patterns.
- Let $[\Sigma^{-1}] = \bigcup_{\sigma\in\Sigma} [\sigma^{-1}]$.

$$\boxed{\mathrm{Sym}(\Sigma) = \gamma\left([\mathrm{Cay}][\Sigma^{-1}]\right)}$$

A constructive procedure for $[p^{-1}]$

- Let $p = B_1 B_2 \cdots B_t$ be the decomposition of p into decreasing runs.
- Let $\ell(j) = |B_j|$.
- Let $I(p) = I_{\ell(1)} \oplus I_{\ell(2)} \oplus \cdots \oplus I_{\ell(t)}$.

$$\boxed{\{\mathrm{id}\} \times [p^{-1}] = (I(p) \times \{p\})^T}$$

A constructive procedure for $[p^{-1}]$

- Let $p = B_1 B_2 \cdots B_t$ be the decomposition of p into decreasing runs.
- Let $\ell(j) = |B_j|$.
- Let $I(p) = I_{\ell(1)} \oplus I_{\ell(2)} \oplus \cdots \oplus I_{\ell(t)}$.

$$\boxed{\{\mathrm{id}\} \times [p^{-1}] = (I(p) \times \{p\})^T}$$

Example: $p = 3142$

$$\big(I(3142) \times \{3142\}\big)^T =$$

A constructive procedure for $[p^{-1}]$

- Let $p = B_1 B_2 \cdots B_t$ be the decomposition of p into decreasing runs.
- Let $\ell(j) = |B_j|$.
- Let $I(p) = I_{\ell(1)} \oplus I_{\ell(2)} \oplus \cdots \oplus I_{\ell(t)}$.

$$\boxed{\{\mathrm{id}\} \times [p^{-1}] = (I(p) \times \{p\})^T}$$

Example: $p = 3142$

$$\left(I(3142) \times \{3142\}\right)^T = \left\{ \binom{1122}{3142}^T, \binom{1123}{3142}^T, \binom{1233}{3142}^T, \binom{1234}{3142}^T \right\}$$

A constructive procedure for $[p^{-1}]$

- Let $p = B_1 B_2 \cdots B_t$ be the decomposition of p into decreasing runs.
- Let $\ell(j) = |B_j|$.
- Let $I(p) = I_{\ell(1)} \oplus I_{\ell(2)} \oplus \cdots \oplus I_{\ell(t)}$.

$$\boxed{\{\mathrm{id}\} \times [p^{-1}] = (I(p) \times \{p\})^T}$$

Example: $p = 3142$

$$\begin{aligned}(I(3142) \times \{3142\})^T &= \left\{ \binom{1122}{3142}^T, \binom{1123}{3142}^T, \binom{1233}{3142}^T, \binom{1234}{3142}^T \right\} \\ &= \left\{ \binom{1234}{1212}, \binom{1234}{1312}, \binom{1234}{2313}, \binom{1234}{2413} \right\}.\end{aligned}$$

A constructive procedure for $[p^{-1}]$

- Let $p = B_1 B_2 \cdots B_t$ be the decomposition of p into decreasing runs.
- Let $\ell(j) = |B_j|$.
- Let $I(p) = I_{\ell(1)} \oplus I_{\ell(2)} \oplus \cdots \oplus I_{\ell(t)}$.

$$\boxed{\{\mathrm{id}\} \times [p^{-1}] = (I(p) \times \{p\})^T}$$

Example: $p = 3142$

$$\begin{aligned}(I(3142) \times \{3142\})^T &= \left\{ \begin{pmatrix}1122\\3142\end{pmatrix}^T, \begin{pmatrix}1123\\3142\end{pmatrix}^T, \begin{pmatrix}1233\\3142\end{pmatrix}^T, \begin{pmatrix}1234\\3142\end{pmatrix}^T \right\} \\ &= \left\{ \begin{pmatrix}1234\\1212\end{pmatrix}, \begin{pmatrix}1234\\1312\end{pmatrix}, \begin{pmatrix}1234\\2313\end{pmatrix}, \begin{pmatrix}1234\\2413\end{pmatrix} \right\}.\end{aligned}$$

$$\Longrightarrow \quad [3142^{-1}] = \{1212, 1312, 2313, 2413\}$$

1324???

- $[1324^{-1}] = \{1223, 1324\}$.

$$\implies \boxed{\mathrm{Sym}(1324) = \gamma\left([\mathrm{Cay}][1223, 1324]\right)}$$

- $[4231^{-1}] = \{2121, 3121, 3231, 4231\}$.

$$\implies \boxed{\mathrm{Sym}(4231) = \gamma\left([\mathrm{Cay}][2121, 3121, 3231, 4231]\right)}$$

Remark

- $\gamma : \hat{\mathcal{A}} \to \mathcal{F}$ is bijective.
- $\hat{\mathcal{A}}$ is a set of representatives for classes $[x] \in [\mathrm{Cay}]$ s.t. $\gamma(y) \in \mathcal{F}$.

Transport Theorem for $\mathcal{F}$ and $\hat{\mathcal{A}}$

$$\mathcal{F}(p) = \gamma\left(\hat{\mathcal{A}}([p^{-1}])\right)$$

and

$$\mathcal{F}\left(\gamma(y)\right) = \gamma\left(\hat{\mathcal{A}}([y])\right).$$

Corollary

For each $n \geq 1$:

$$|\mathcal{F}_n(p)| \;=\; |\hat{\mathcal{A}}_n([p^{-1}])|$$

Transport of a single pattern $\mathcal{A} \longleftrightarrow \mathcal{F}$

$[p^{-1}]$	p	Transport
1	1	$\mathcal{A}(1) \longleftrightarrow \mathcal{F}(1)$
11,21	21	$\mathcal{A}(11) \longleftrightarrow \mathcal{F}(21)$
12	12	$\mathcal{A}(12) \longleftrightarrow \mathcal{F}(12)$
112,213	213	$\mathcal{A}(112) \longleftrightarrow \mathcal{F}(213)$
121,231	312	$\mathcal{A}(121) \longleftrightarrow \mathcal{F}(312)$
122,132	132	$\mathcal{A}(122) \longleftrightarrow \mathcal{F}(132)$
212,312	231	$\mathcal{A}(212) \longleftrightarrow \mathcal{F}(231)$
123	123	$\mathcal{A}(123) \longleftrightarrow \mathcal{F}(123)$
1212,2313,1312,2413	3142	$\mathcal{A}(1212) \longleftrightarrow \mathcal{F}(3142)$
1123,2134	2134	$\mathcal{A}(1123) \longleftrightarrow \mathcal{F}(2134)$
2312,3412	3412	$\mathcal{A}(312) \longleftrightarrow \mathcal{F}(3412)$
1232,1342	1423	$\mathcal{A}(132) \longleftrightarrow \mathcal{F}(1423)$

A fruitful instance of transport

$\mathcal{A}(212, 231)$ is in bijection with $\mathrm{Sym}(231, 4123)$ (Baxter, Pudwell).

A fruitful instance of transport

$\mathcal{A}(212, 231)$ is in bijection with $\mathrm{Sym}(231, 4123)$ (Baxter, Pudwell).

We transport the pair $(231, 4123)$:

1. $[231^{-1}] = \{212, 312\}$ and $[4123^{-1}] = \{2341, 1231\}$.
2. $\mathcal{F}(231, 4123) \; = \; \gamma\left(\hat{\mathcal{A}}(212, 312, 2341, 1231)\right)$.

A fruitful instance of transport

$\mathcal{A}(212, 231)$ is in bijection with $\mathrm{Sym}(231, 4123)$ (Baxter, Pudwell).

We transport the pair $(231, 4123)$:

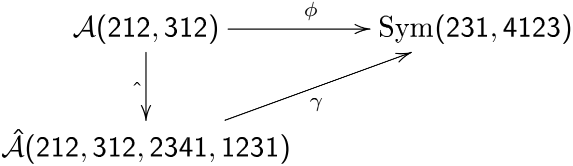
1. $[231^{-1}] = \{212, 312\}$ and $[4123^{-1}] = \{2341, 1231\}$.
2. $\mathcal{F}(231, 4123) \ = \ \gamma\left(\hat{\mathcal{A}}(212, 312, 2341, 1231)\right).$
3. $\hat{\mathcal{A}}(212, 312, 2341, 1231)$ is the modified set of $\mathcal{A}(212, 231)$.
4. $\mathcal{F}(231, 4123) = \mathrm{Sym}(231, 4123)$.

A fruitful instance of transport

$\mathcal{A}(212, 231)$ is in bijection with $\mathrm{Sym}(231, 4123)$ (Baxter, Pudwell).

We transport the pair $(231, 4123)$:

1. $[231^{-1}] = \{212, 312\}$ and $[4123^{-1}] = \{2341, 1231\}$.
2. $\mathcal{F}(231, 4123) \;=\; \gamma\left(\hat{\mathcal{A}}(212, 312, 2341, 1231)\right)$.
3. $\hat{\mathcal{A}}(212, 312, 2341, 1231)$ is the modified set of $\mathcal{A}(212, 231)$.
4. $\mathcal{F}(231, 4123) = \mathrm{Sym}(231, 4123)$.



Transport of RGF

- Restricted growth functions RGF are representatives for $[\mathrm{Cay}]$.
- $\gamma(\mathrm{RGF}) = \mathrm{Sym}(23\text{-}1)$.

Transport theorem for RGF

$$\mathrm{Sym}(23\text{-}1, \sigma) = \gamma\big(\mathrm{RGF}[\sigma^{-1}]\big) \quad \text{and} \quad \gamma\big(\mathrm{RGF}[y]\big) = \mathrm{Sym}\big(23\text{-}1, \gamma(y)\big)$$

Examples

[3-noncrossing set partitions]

1. $\gamma\big(\mathrm{RGF}(3123)\big) = \mathcal{F}(2341)$.
2. $\gamma\big(\mathrm{RGF}(1323)\big) = \mathcal{F}(1342)$.

Reference

Transport of patterns by Burge transpose, G. Cerbai, A. Claesson.

Thanks!