Restricted generating trees for weak orderings

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Valparaiso, IN, June 2022

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Joint work with D. Birmajer, D. Kenepp, and M. Weiner.

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- Generating trees and stopping conditions
- Stopping conditions of length 2 and 3
- Other stopping conditions

A *weak-ordering chain* in the variables $x_1, x_2, \ldots, x_n$ is an expression of the form

$$x_{i_1} \operatorname{op} x_{i_2} \operatorname{op} \cdots \operatorname{op} x_{i_n},$$

where op is either $<$ or $=$. Let $\mathcal{WOC}(n)$ denote the set of all weak-ordering chains in n variables. Every $w \in \mathcal{WOC}(n)$ corresponds to an ordered partition:

$$x_2 < x_4 = x_5 < x_1 < x_3 \quad \longleftrightarrow \quad \{\{2\}, \{4,5\}, \{1\}, \{3\}\}.$$

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Weak-ordering chains are counted by the Fubini numbers

$$f_0 = 1 \;\text{ and }\; f_n = \sum_{i=1}^{n} \binom{n}{i} f_{n-i} \;\text{ for } n \geq 1.$$

Every $w \in \mathcal{WOC}(n)$ can be recursively generated starting with x_1, and then inserting x_i (together with either $<$ or $=$) into an existing weak-ordering chain of length $i-1$.

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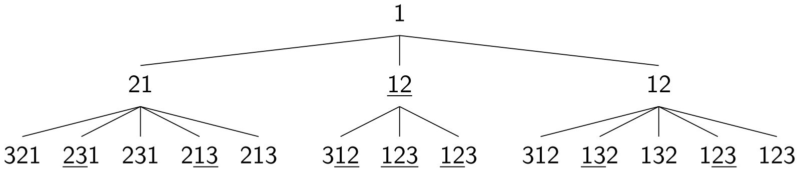
$$x_2 < x_4 = x_5 < x_1 < x_3$$

Generating tree

This insertion process generates a rooted labeled tree whose nodes are the weak-ordering chains.



ij is a shortcut for $x_i < x_j$ and $\underline{ij}$ represents $x_i = x_j$

Stopping conditions

Suppose we wish to stop the generating process as soon as we have a tie. In other words, suppose we do not allow nodes with $x_i = x_j$ to have descendants.

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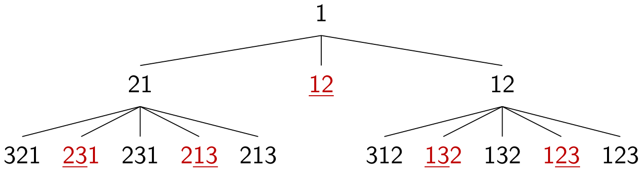

with only 11 leaves instead of 13. This is a generating tree of weak-ordering chains subject to the *stopping condition* $x_i = x_j$.

Given a stopping condition, how may leaves after n steps?

Counting strategy

Separate the *active leaves* (avoiding the stopping condition), from the *inactive leaves* (containing the stopping condition). Let a_n be the total number of active leaves and b_n be the total number of inactive leaves after n steps.

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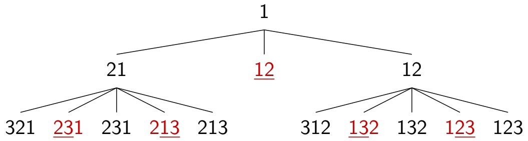
For the stopping condition $x_i = x_j$, we have

$$a_1 = 1, \quad b_1 = 0, \qquad a_2 = 2, \quad b_2 = 1, \qquad a_3 = 6, \quad b_3 = 5.$$



Stopping condition $x_i = x_j$

THEOREM $1, 3, 11, 47, 239, 1439, 10079, 80639, \ldots,$ [A020543]

If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_i = x_j$ with $i \neq j$, then $w_n = 2n! - 1$.

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Every permutation of $\{x_1, \ldots, x_n\}$ gives an active chain. So, $a_n = n!$. Every leaf that becomes inactive at level k is a descendant of an active node at level $k-1$, and each of these active chains generates $k-1$ inactive leaves (for $j \in \{1, \ldots, k-1\}$, replace x_j with $x_j = x_k$). Thus, there are $(k-1)a_{k-1}$ leaves becoming inactive at level k and so

$$b_n = \sum_{k=1}^{n} (k-1)(k-1)! = n! - 1.$$

Stopping condition $x_i < x_j$

THEOREM $1, 3, 9, 25, 65, 161, 385, 897, \ldots,$ [A002064]

If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_i < x_j$ with $i < j$, then $w_n = (n-1)2^{n-1} + 1$.

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Indices must appear in decreasing order $x_n \,\mathrm{op}\, x_{n-1} \,\mathrm{op} \cdots \mathrm{op}\, x_1$, and we can choose op to be either $<$ or $=$. Hence $a_n = 2^{n-1}$.

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Every active chain at level $k-1$ generates $k-1$ inactive leaves, obtained by replacing x_j with $x_j < x_k$ for $j \in \{1, \ldots, k-1\}$. So, there are $(k-1)a_{k-1}$ leaves becoming inactive at level k and so

$$b_n = \sum_{k=1}^{n} (k-1)2^{k-2} = (n-2)2^{n-1} + 1.$$

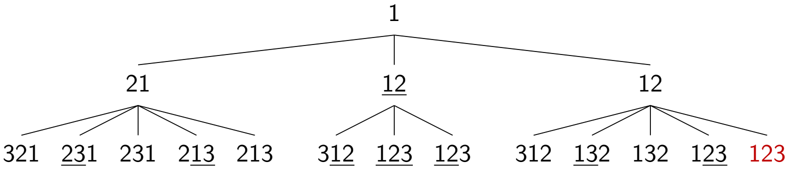
Stopping condition $x_i \leq x_j$

Theorem

If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_i \leq x_j$ with $i < j$, then $w_n = n^2 - n + 1$.

$\rightsquigarrow$ Central polygonal numbers $1, 3, 7, 13, 21, 31, 43, 57, \ldots,$ [A002061]

Consider the stopping condition is $x_i < x_j < x_k$ with $i < j < k$. In this case, the generating tree at level 3 looks like:



and the node with label 123 will have no descendants as the generating tree grows.

Passage to permutations

Given an ordered partition π of $[n]$, we let σ_π be the *underlined permutation* obtained by merging the parts of π and underlining the entries coming from the same block of π.

$$x_2 < x_4 = x_5 < x_1 < x_3 \longleftrightarrow \pi = 2 \,|\, 54 \,|\, 1 \,|\, 3 \longleftrightarrow \sigma_\pi = 2\,\underline{54}\,13,$$

$$x_2 = x_4 = x_6 < x_5 < x_1 = x_3 \longleftrightarrow \pi = 642 \,|\, 5 \,|\, 31 \longleftrightarrow \sigma_\pi = \underline{642}\,5\,\underline{31}.$$

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$$x_2 = x_4 = x_6 < x_5 < x_1 = x_3 \longleftrightarrow \pi = 642\,|\,5\,|\,31 \longleftrightarrow \sigma_\pi = \underline{642}\,5\,\underline{31}.$$

Let $\mathcal{V}_n(\sigma)$ be the set of chains projecting to σ. A descent in σ could come from $x_{\sigma(i)} < x_{\sigma(i+1)}$ or $x_{\sigma(i)} = x_{\sigma(i+1)}$ in the chain. If σ has d descents, $\mathcal{V}_n(\sigma)$ has 2^d elements, and if a chain contains an increasing subsequence $x_{i_1} < x_{i_2} < x_{i_3}$, then the projected permutation contains a 123-pattern.

Active leaves

The set of active chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_{i_1} < x_{i_2} < x_{i_3}$ is the union

$$\bigcup_{\sigma \in S_n(123)} \mathcal{V}_n(\sigma) = \bigcup_{d=0}^{n-1} \bigcup_{\sigma \in S_n^d(123)} \mathcal{V}_n(\sigma),$$

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M. Barnabei et al., The descent statistic on 123-avoiding permutations
Chen et al., Ordered partitions avoiding a permutation pattern of length 3

Connection to Dyck paths

The set of active chains in $\mathcal{WOC}(n)$ with stopping condition $x_{i_1} < x_{i_2} < x_{i_3}$ is in bijection with the set of Dyck paths of semilength n where valleys and triple down-steps come in 2 colors.

Inactive leaves

A leaf is inactive if the associated permutation has a 123 pattern. To count the elements that become inactive at level n, consider:

$$\mathcal{G}_n^d(123) = \{\sigma \in S_n \mid \sigma \text{ has a 123 pattern, } d \text{ descents, and } \sigma' \in S_{n-1}(123)\}$$

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If $g_{n,d} = |\mathcal{G}_n^d(123)|$, then

$$g_{n,d} = (d+1)e_{n-1,d} + (n-d)e_{n-1,d-1} - e_{n,d},$$

where $e_{n,d} = |S_n^d(123)|$.

Stopping condition $x_{i_1} < x_{i_2} < x_{i_3}$

THEOREM

If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_{i_1} < x_{i_2} < x_{i_3}$, then

$$w_n = \sum_{d=0}^{n-1} 2^d e_{n,d} + \sum_{j=3}^{n} \sum_{d=0}^{j-3} 2^d g_{j,d}.$$

$1, 3, 13, 69, 401, 2433, 15121, 95441, 609025, 3918273, \ldots,$

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$$w_n = \sum_{d=0}^{n-1} 2^{n-1-d} e_{n,d} + \sum_{j=3}^{n} \sum_{d=2}^{j-1} 2^{j-1-d} g_{j,d}.$$

1, 3, 13, 59, 269, 1227, 5613, 25771, 118765, 549227, . . . ,

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$$w_n = \sum_{d=0}^{n-1} 2^d N_{n,d} + \sum_{j=3}^{n} \sum_{d=1}^{j-2} 2^d \ell_{j,d},$$

where $\ell_{n,d} = |\mathcal{G}_n^d(213)|$ and $N_{n,v} = \frac{1}{n}\binom{n}{v}\binom{n}{v+1}$.

$1, 3, 13, 65, 341, 1827, 9913, 54273, 299209, 1658723, \ldots,$

Stopping condition $x_{i_1} \leq x_{i_2} < x_{i_3}$

$\mathcal{G}_n^d(213) = \{\sigma \in S_n \,|\, \sigma$ has a 213 pattern, d descents, and $\sigma' \in S_{n-1}(213)\}$

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Other stopping conditions

- $x_{i_1} = \cdots = x_{i_k}$
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} Done!

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- $x_{i_1} \leq x_{i_2} = x_{i_3}$
- $x_{i_1} < x_{i_2} > x_{i_3}$
- $x_{i_1} \leq x_{i_2} > x_{i_3}$
- $x_{i_1} \leq x_{i_2} \geq x_{i_3}$

(first three:) Done!

References

Thank you!

- M. Barnabei, F. Bonetti, and M. Silimbani, The descent statistic on 123-avoiding permutations, *Sém. Lothar. Combin.* **63** (2010), Art. B63a, 8 pp.
- W.Y.C. Chen, A.Y.L. Dai, and R.D.P. Zhou, Ordered partitions avoiding a permutation pattern of length 3, *European J. Combin.* **36** (2014), 416–424.