A q-analogue and a symmetric function analogue of a result of Carlitz, Scoville and Vaughan

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Carlitz, Scoville, and Vaughan's result

- The Bessel function $J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z^2/4)^n}{n!n!}$.
- Let $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!n!}$, then $J_0(z) = f(z^2/4)$.
- Write $\frac{1}{f(z)} = \sum_{n=0}^{\infty} \omega_n \frac{z^n}{n!n!}$.

It follows that $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \omega_k = 0$, and ω_k's are the the coefficients of the reciprocal Bessel function.

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Given $\sigma \in \mathcal{S}_n$, a permutation of $[n] := \{1, 2, ..., n\}$, i is an *ascent* of σ if $\sigma(i) < \sigma(i+1)$. e.g. $\sigma = 213$, $\sigma(2) = 1 < \sigma(3) = 3$, 2 is an ascent.

C-S-V result (1976)

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- Let $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!n!}$, then $J_0(z) = f(z^2/2)$.
- Write $\frac{1}{f(z)} = \sum_{n=0}^{\infty} \omega_n \frac{z^n}{n!n!}$.
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- For example, $\omega_2 = 3$: $(12, 21)$, $(21, 12)$, $(21, 21)$.
- Carlitz, Scoville and Vaughan's result provided a combinatorial interpretation of the coefficient ω_k in the reciprocal Bessel function.
- Their proof uses generating functions.

Methods and Motivation

- $\omega_n = \#$ of pairs of permutations of $\mathcal{S}_n$ with no common ascent.
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- Obtain C-S-V result by counting the ω_n in 2 ways:
  1. $\#$ of decreasing maximal chains in $B_n \circ B_n$.
  2. The möbius number μ of the poset $B_n \circ B_n$
     Given P a poset,
     $\mu(s,s) = 1,$ for all $s \in P$,
     $$\mu(s,u) = -\sum_{s \le t < u} \mu(s,t), \text{ for all } s < u \text{ in } P.$$
  3. $\omega_n =$ ①'='② $\Longrightarrow$ C-S-V identity: $\sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 \omega_k = 0.$
     Rewrite C-S-V: $\omega_n = -\sum_{k=0}^{n-1}(-1)^k \binom{n}{k}^2 \omega_k.$

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- **Change B_n to its q-analogue poset $\longrightarrow$ q-analogue of C-S-V result**

The q-analogue of B_n

- Let $B_n :=$ all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion, also called Boolean Algebra.
- Let $B_n(q) :=$ all subspaces of $\mathbb{F}_q^n$, an n-dimensional vector space over $\mathbb{F}_q$, ordered by containment.
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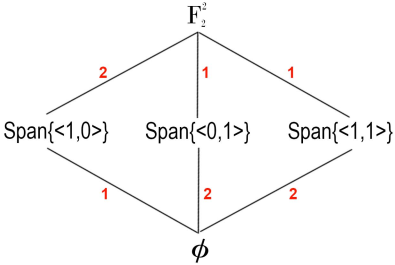
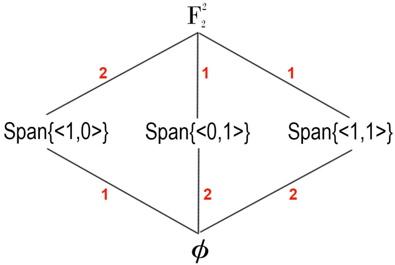


An edge labeling of $B_n(q)$

- An edge labeling of $B_2(2)$:



- A chain c is *increasing* if its label is strictly increasing and *decreasing* if its label is weakly decreasing (non-increasing).
- The left most chain of $B_2(2)$ is increasing with a label 12 and the other two chains are decreasing.
- $\mu(B_2(2)) = 2 = \#$ of decreasing maximal chains.

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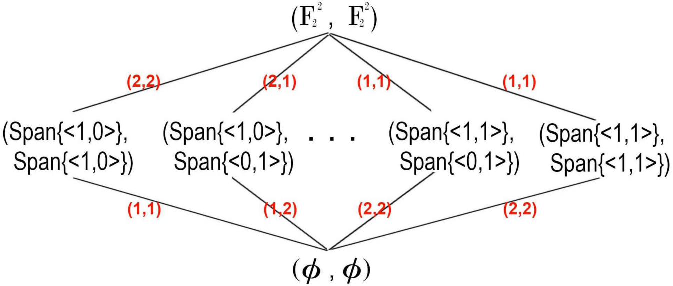
$B_n(q) \circ B_n(q) := \{(a, b) \in B_n(q) \times B_n(q) \text{ and } \dim(a) = \dim(b)\}$



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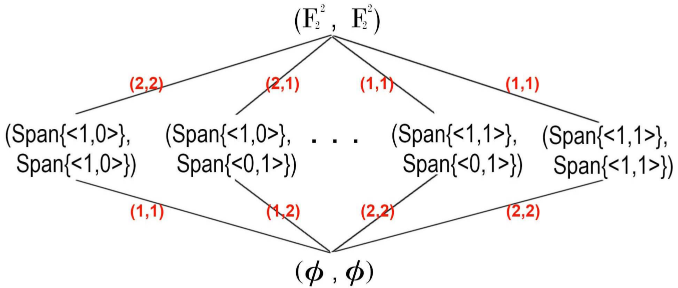

- The left most chain of $B_2(2) \circ B_2(2)$ is increasing with label $(12, 12)$.
- The labels of decreasing (non-increasing) chains in $B_n(q) \circ B_n(q)$ are pairs of permutations $(\sigma, \omega) \in \mathcal{S}_n \times \mathcal{S}_n$ with no common ascent. e.g. The second chain has label $(12, 21)$.

Edge-Lexicographical labeling (EL-labeling)

An *EL-labeling* is an edge labeling that satisfies the following two conditions:

1. There is a unique increasing maximal chain in every closed interval,
2. and its label lexicographically precedes all other maximal chains in the same interval.

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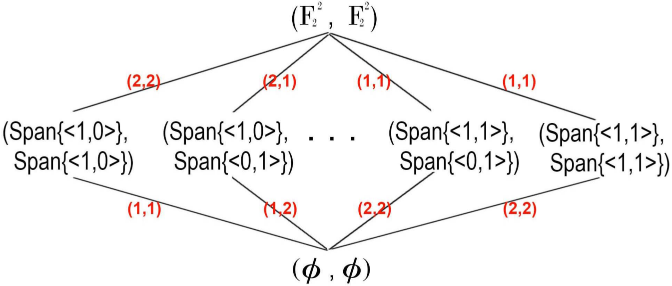

Decreasing maximal chains in an EL-labeling

Theorem (Björner, Wachs, and Hall)

Suppose P is a pure poset for which $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$ admits an EL-labeling. Then the number of decreasing maximal chains of $\hat{P}$ is $\mu(\hat{P})$, the möbius number of $\hat{P}$, without the '$-$' sign.

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- $[n]_q = q^{n-1} + q^{n-2} + ... + 1$ is the q-analogue of the natural number n
- $[n]_q! = [n]_q[n-1]_q...[2]_q[1]_q$
- The q-analogue of $\binom{n}{k}$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.
- For a permutation $\sigma \in \mathcal{S}_n$, the *inversion statistic* is defined by $inv(\sigma) := |\{(i,j) : 1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j)\}|$.

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- **Notation:** Let $\mathcal{D}_n$ be the set of all pairs of permutations $(\sigma, \tau) \in \mathcal{S}_n \times \mathcal{S}_n$ with no common ascent.

q-analogue to $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\omega_k = 0$, the C-S-V result

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Theorem 1 (L.): a q-analogue to C-S-V result

The Segre product $B_n(q) \circ B_n(q)$ admits an EL-labeling.
Let $W_n(q)$ be the total number of decreasing maximal chains of $B_n(q) \circ B_n(q)$. Then

$$\sum_{i=0}^{n}(-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q^2 W_i(q) = 0$$

and $W_i(q) = \sum_{(\sigma,\tau)\in\mathcal{D}_i} q^{inv(\sigma)+inv(\tau)}$.

Theorem 1 (L.): a q-analogue to C-S-V result

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and $W_i(q) = \sum_{(\sigma,\tau)\in\mathcal{D}_i} q^{inv(\sigma)+inv(\tau)}$.

- Equation (1) implies that $W_n(q)$ are the coefficients of the reciprocal of q-Bessel function $J_0^{(1)}(z;q)$
- Theorem 1 provides a combinatorial interpretation of those coefficients $W_n(q)$ and a method to compute $W_n(q)$.

q-analogue to C-S-V result

- We can get C-S-V's result by letting $q = 1$.
- Our q-analogue offers topological approach by using the top homology of $B_n(q) \circ B_n(q) \setminus \{\hat{0}, \hat{1}\}$.
- The number $(-1)^n W_n(q)$ is the reduced Euler characteristic of $B_n(q) \circ B_n(q)$.

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$W_n(q)$ are the signless reduced Euler characteristic of $\triangle(B_n(q) \circ B_n(q))$.

A further generalization

A well-known **symmetric function identity**: for $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0,$$

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- The *(Frobenius) characteristic ch* maps a representation of $\mathcal{S}_n$ to a symmetric function of degree n
- e_i is the characteristic of the representation of $\mathcal{S}_i$ on the top homology of B_i.
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- The *product Frobenius characteristic ch* maps a repr. of $\mathcal{S}_n \times \mathcal{S}_n$ to a symmetric function in two sets of variables $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$.

An analogue to $\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$

Theorem 2 (L.): a symmetric function analogue

Let $P_n := B_n \circ B_n - \{\hat{0}, \hat{1}\}$. The action of $\mathcal{S}_n \times \mathcal{S}_n$ on P_n induces a representation on $\widetilde{H}_{n-2}(P_n)$. Let $ch(\widetilde{H}_{n-2}(P_n))$ be the product Frobenius characteristic of this representation. Then

$$\sum_{i=0}^{n} (-1)^i ch(\widetilde{H}_{i-2}(P_i)) h_{n-i}(x) h_{n-i}(y) = 0. \tag{2}$$

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- Equation (2) contains symmetric functions with variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$.
- Specialize equation (2) by substituting (q^{i-1}) for both x_i and y_i. Denote the specialization of f by $ps(f)$.

The connection between two analogues

- $\widetilde{H}_{n-2}(P_n)$ is the top homology of $B_n \circ B_n$.
- $W_n(q)$ is the signless reduced Euler characteristic of $B_n(q) \circ B_n(q)$.

Specializing equation (2) $\Longrightarrow$

1. **Theorem 3 (L.)**: $ps(ch(\widetilde{H}_{n-2}(P_n))) = \frac{W_n(q)}{\prod_{i=1}^{n}(1-q^i)^2}$.
2. **Theorem 1 (L.)**: $\sum_{i=0}^{n}(-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q^2 W_i(q) = 0$ q-analogue to C-S-V result.

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Remark:

- Symmetric function analogue generalizes the q-analogue to C-S-V identity and the symmetric function identity.
- The symmetric function analogue a group theoretic result.

Thank You!