Combinatorial Exploration

an algorithmic framework for enumeration

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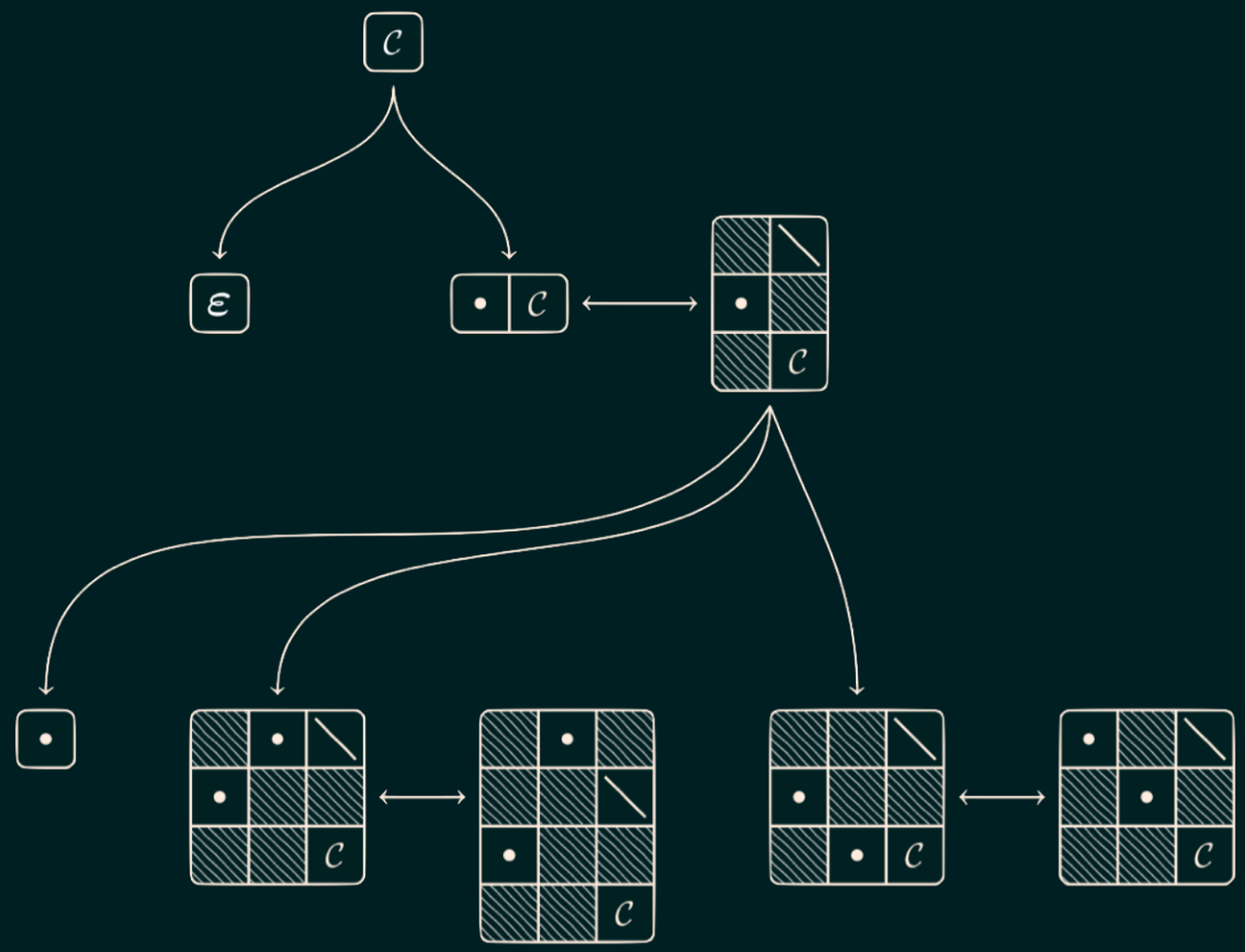
with:
Michael Albert
Christian Bean
Anders Claesson
Émile Nadeau
Henning Ulfarsson



Extra references:

- preprint: https://arxiv.org/abs/2202.07715
- Longer talk specifically about Combinatorial Exploration (domain-agnostic): https://vimeo.com/687277242
- PermPAL: https://permpal.com/

Mathematics > Combinatorics

[Submitted on 15 Feb 2022]

Combinatorial Exploration: An algorithmic framework for enumeration

Michael H. Albert, Christian Bean, Anders Claesson, Émile Nadeau, Jay Pantone, Henning Ulfarsson

Combinatorial Exploration is a new domain-agnostic algorithmic framework to automatically and rigorously study the structure of combinatorial objects and derive their counting sequences and generating functions. We describe how it works and provide an open-source Python implementation. As a prerequisite, we build up a new theoretical foundation for combinatorial decomposition strategies and combinatorial specifications.

We then apply Combinatorial Exploration to the domain of permutation patterns, to great effect. We rederive hundreds of results in the literature in a uniform manner and prove many new ones. These results can be found in a new public database, the Permutation Pattern Avoidance Library (PermPAL) at this https URL. Finally, we give three additional proofs-of-concept, showing examples of how Combinatorial Exploration can prove results in the domains of alternating sign matrices, polyominoes, and set partitions.

Subjects: **Combinatorics (math.CO)**

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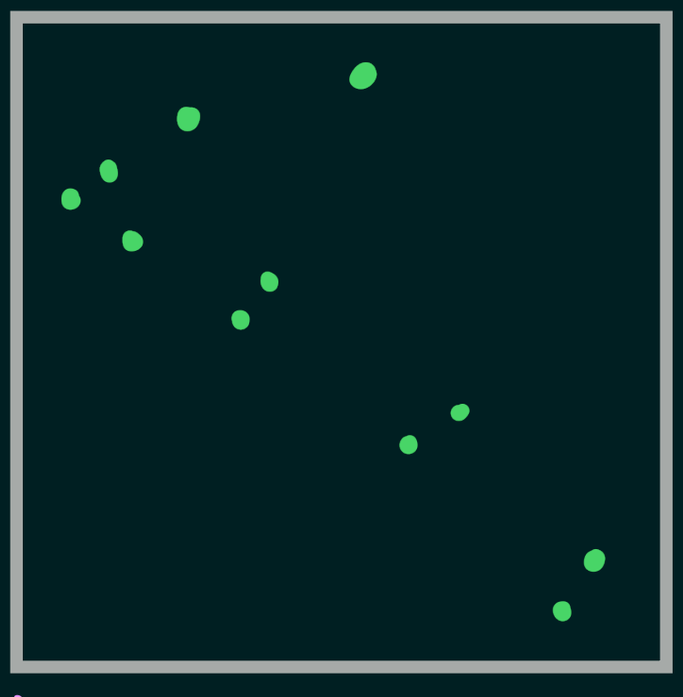

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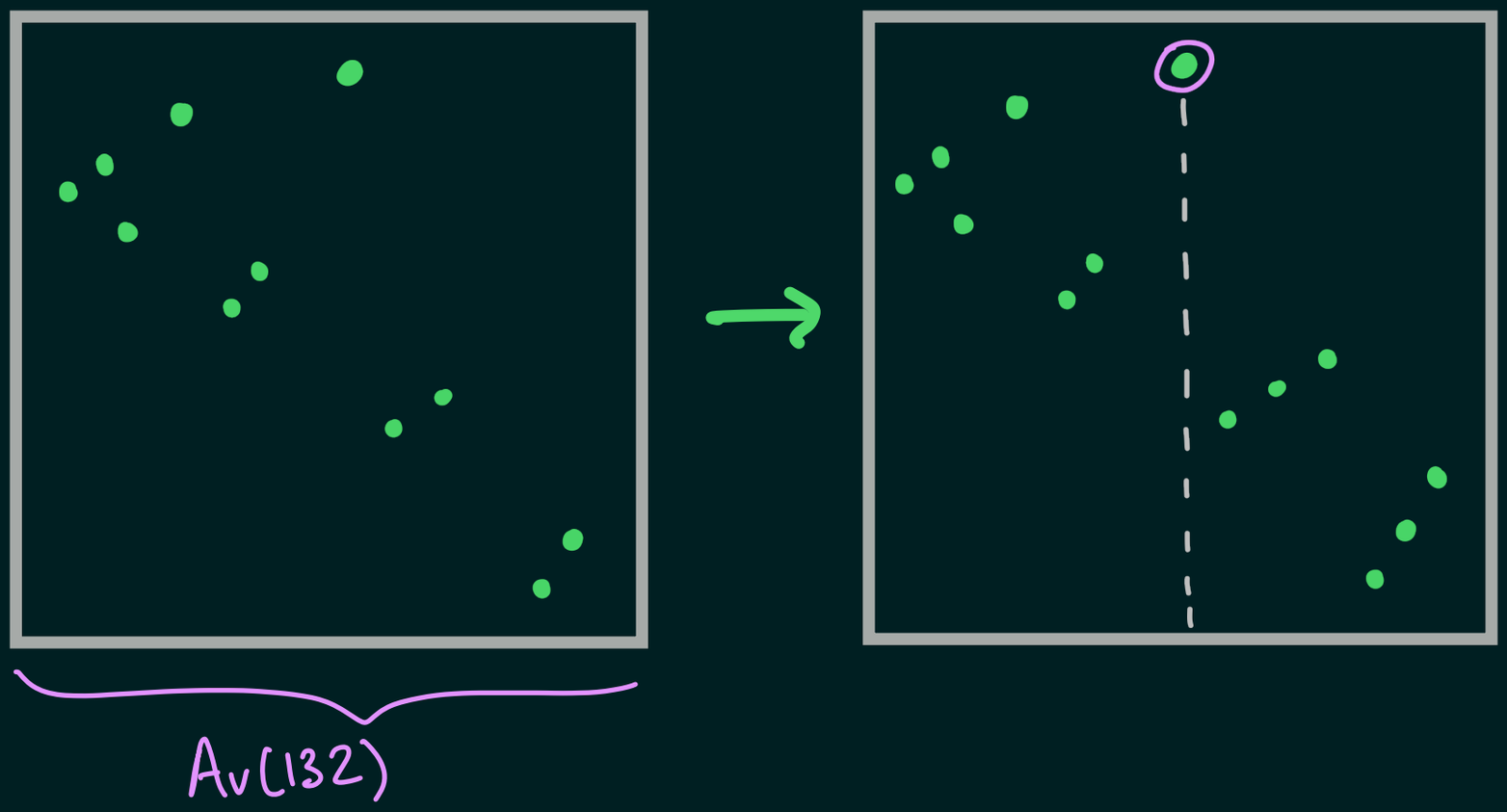
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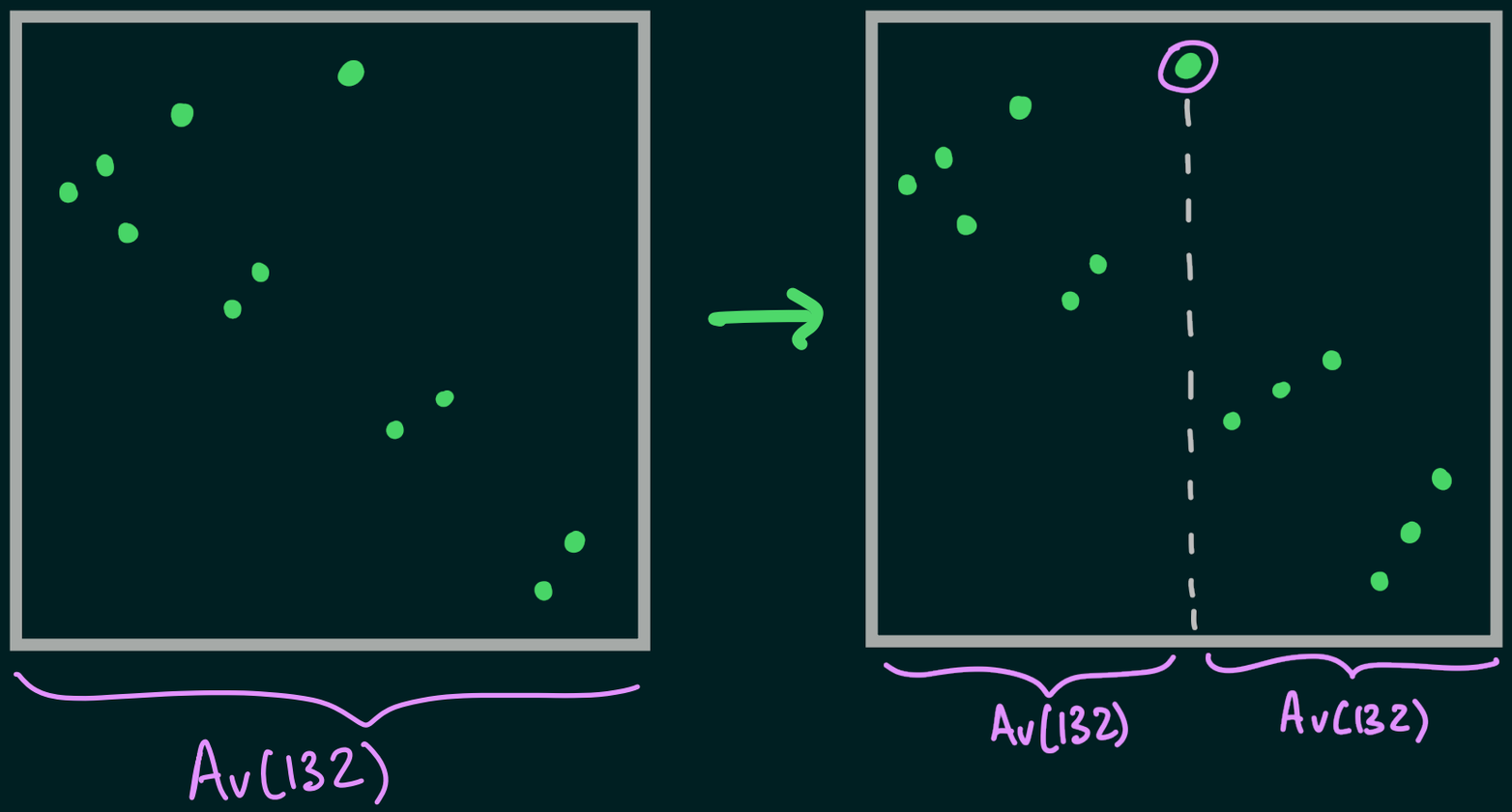
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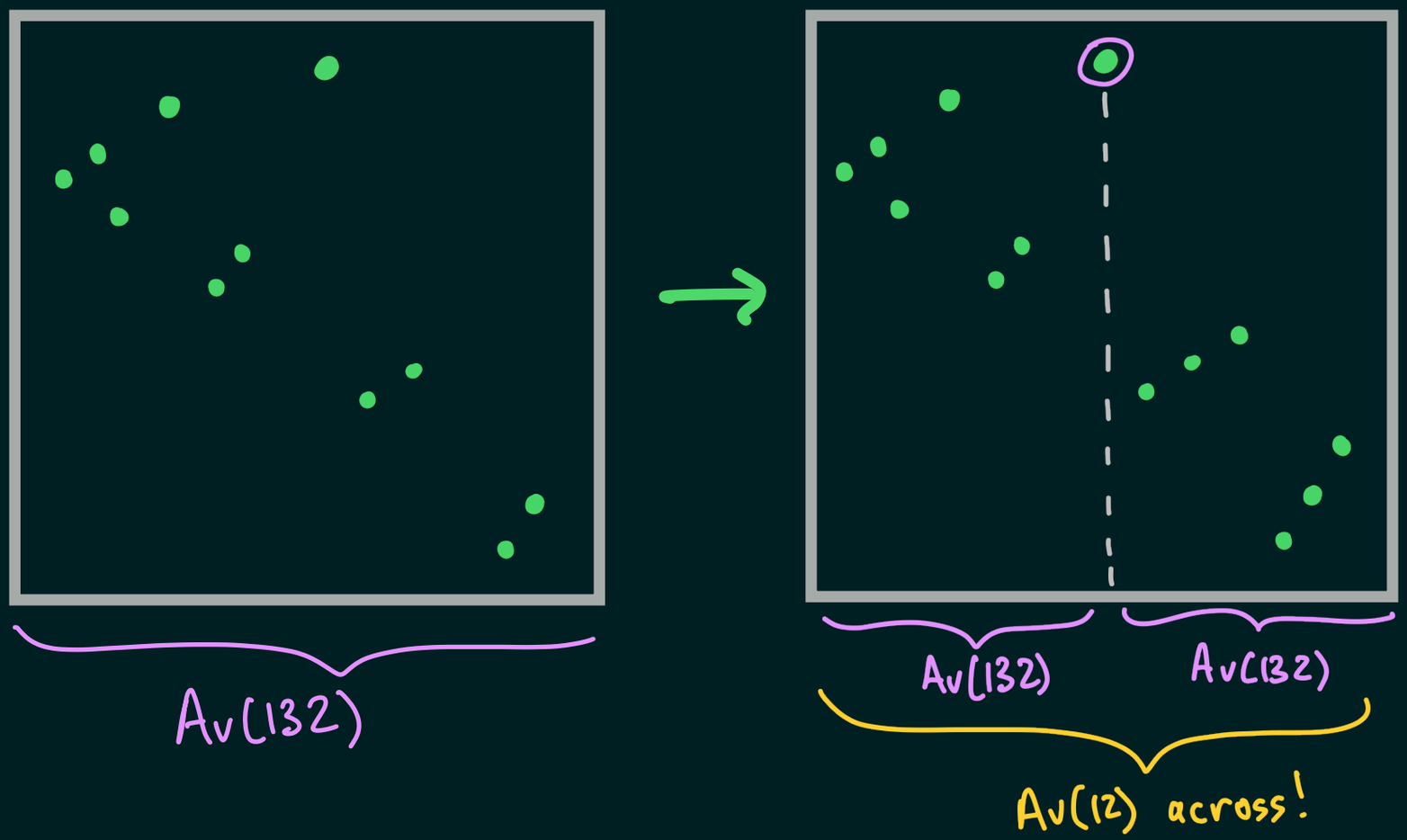
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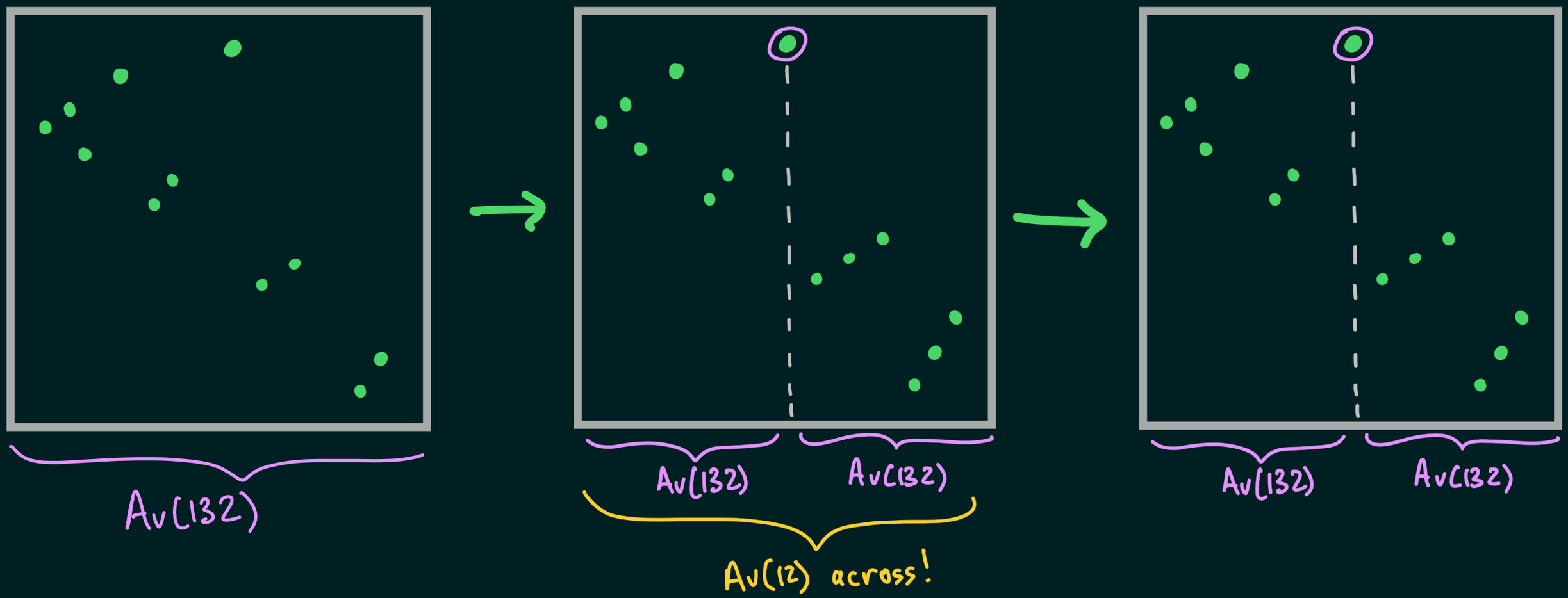
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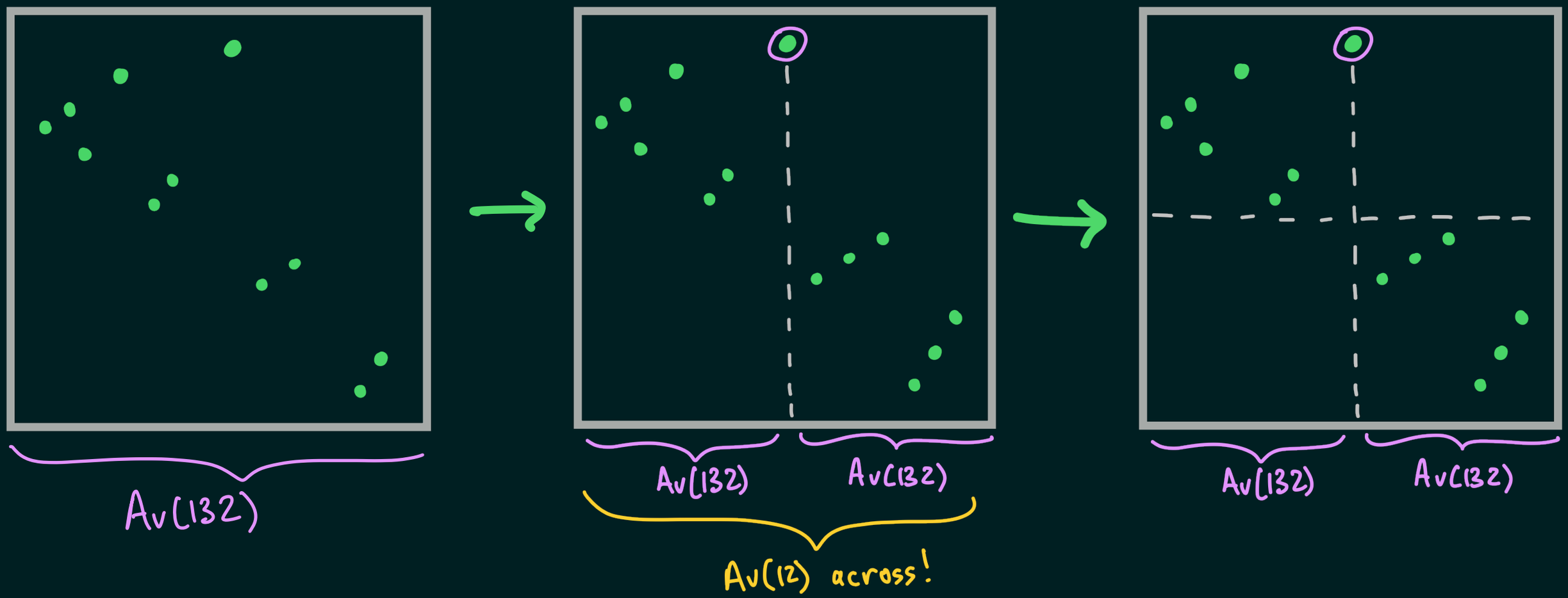
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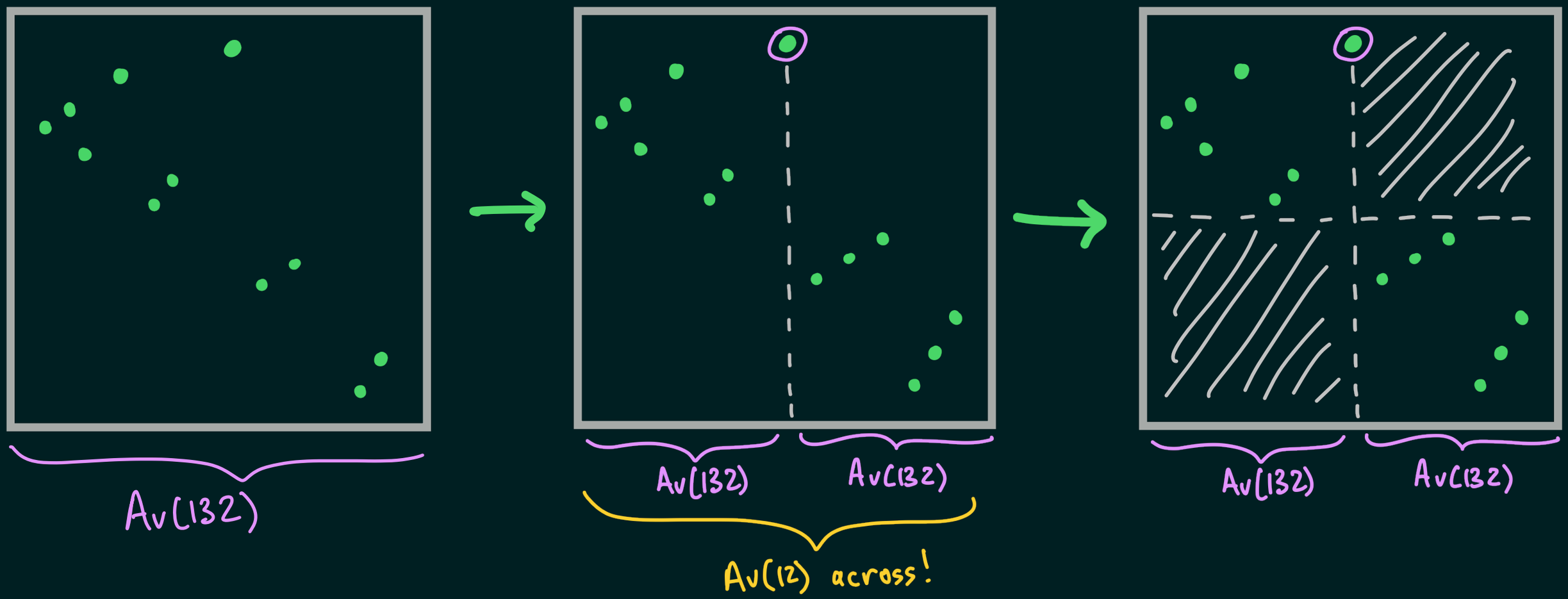
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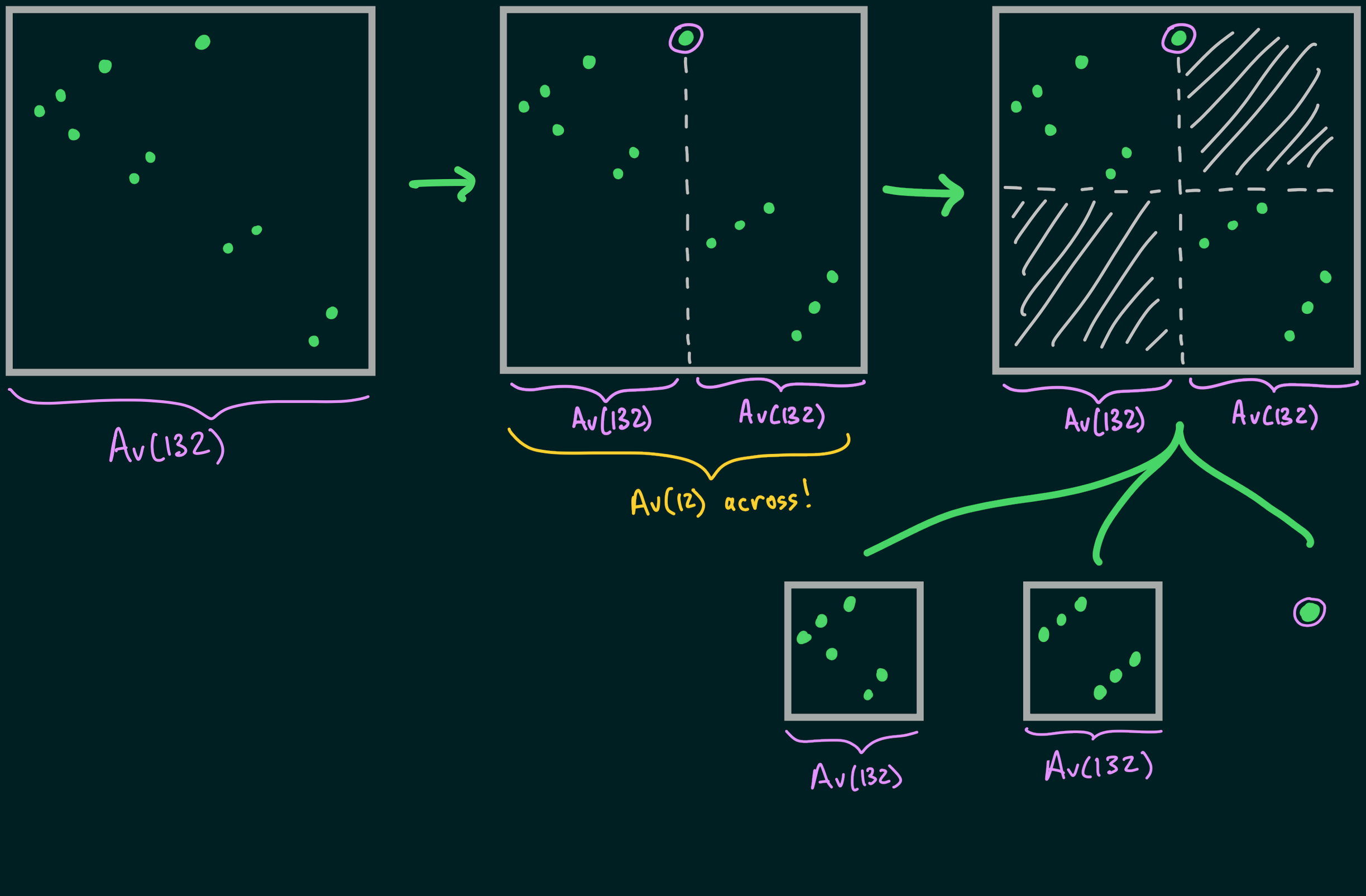
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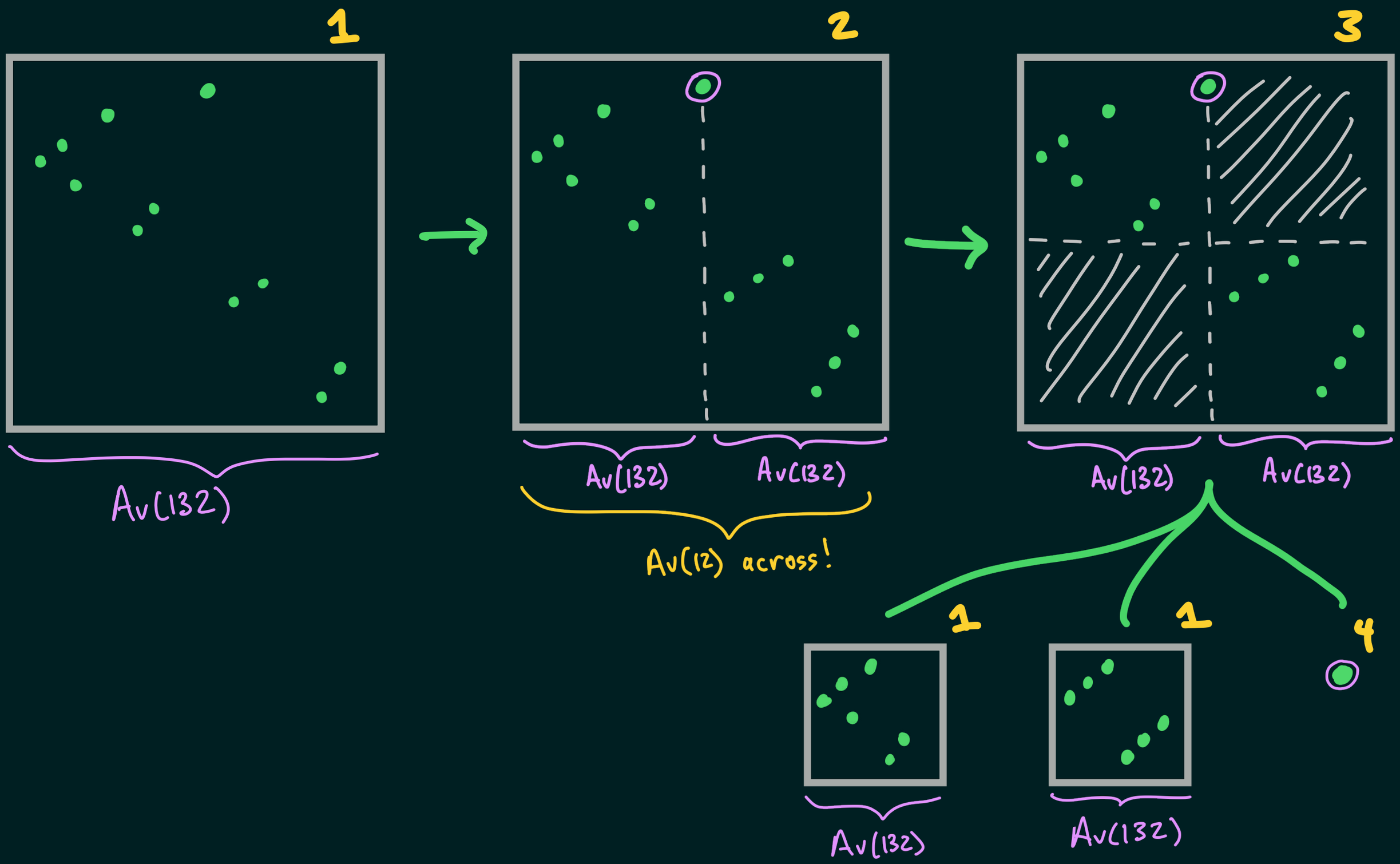
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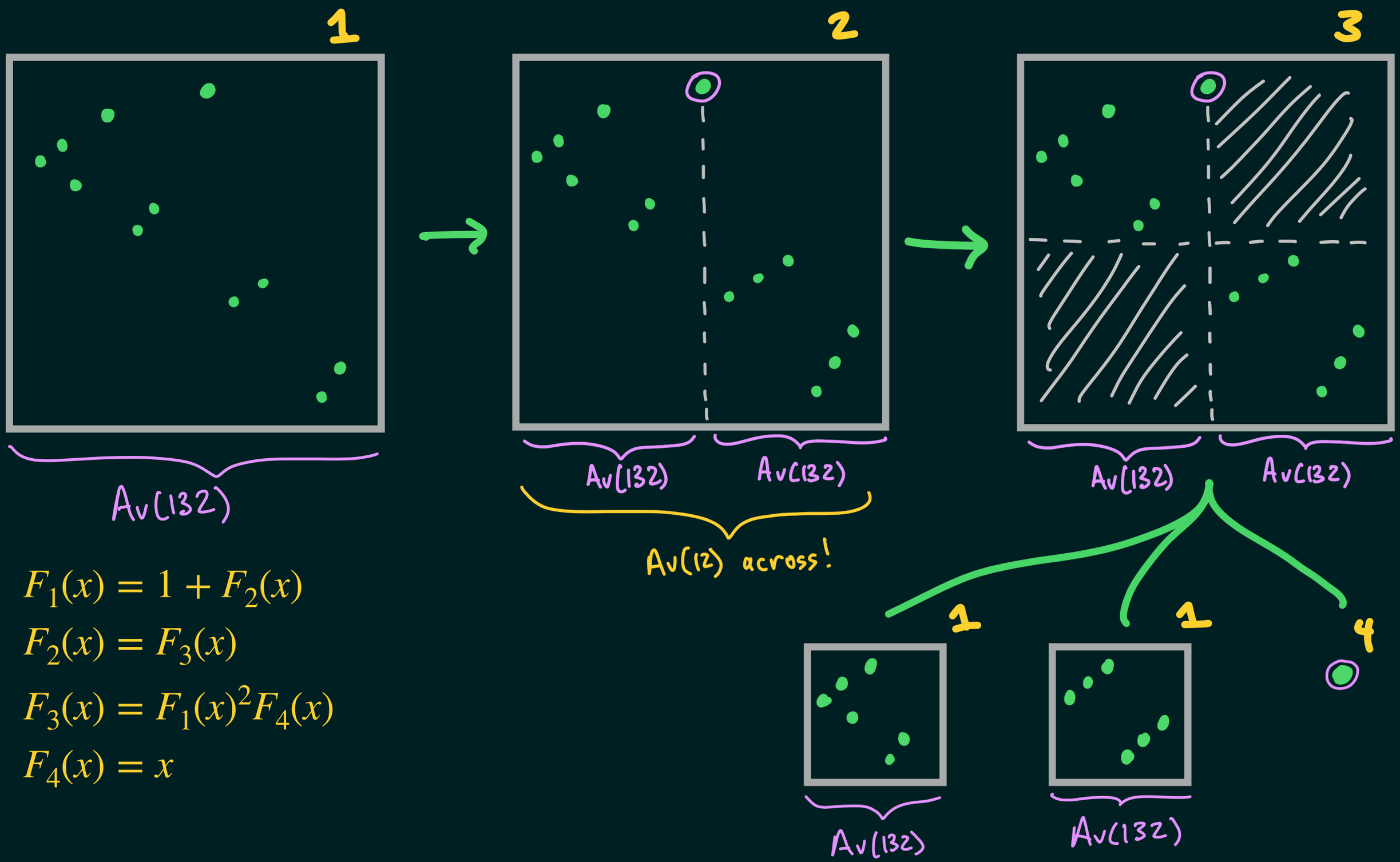
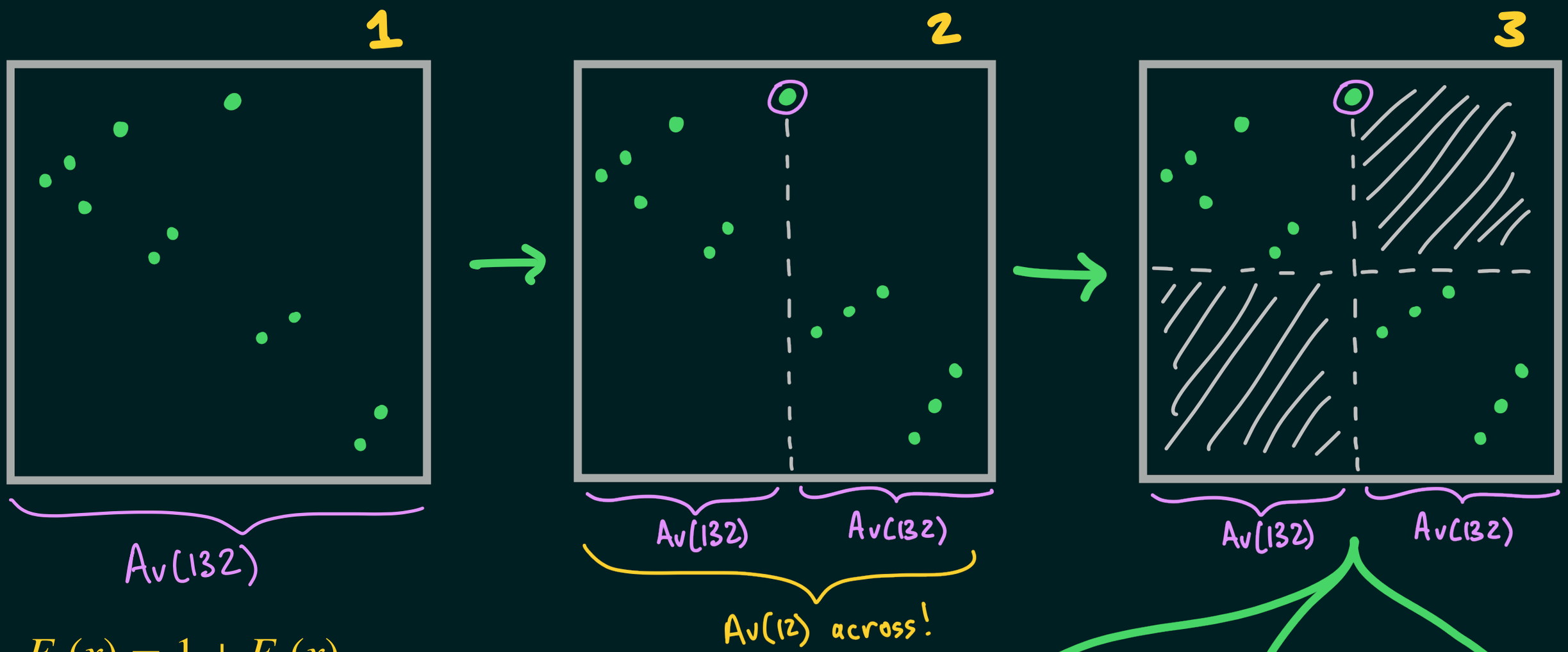


$$F_1(x) = 1 + F_2(x)$$
$$F_2(x) = F_3(x)$$
$$F_3(x) = F_1(x)^2 F_4(x)$$
$$F_4(x) = x$$



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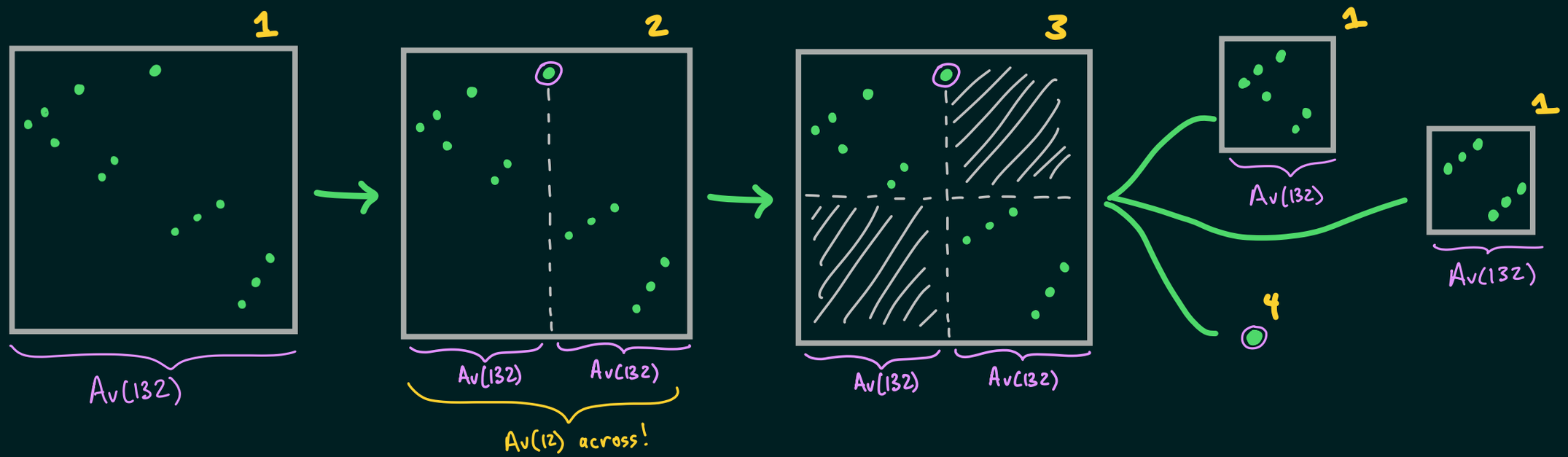
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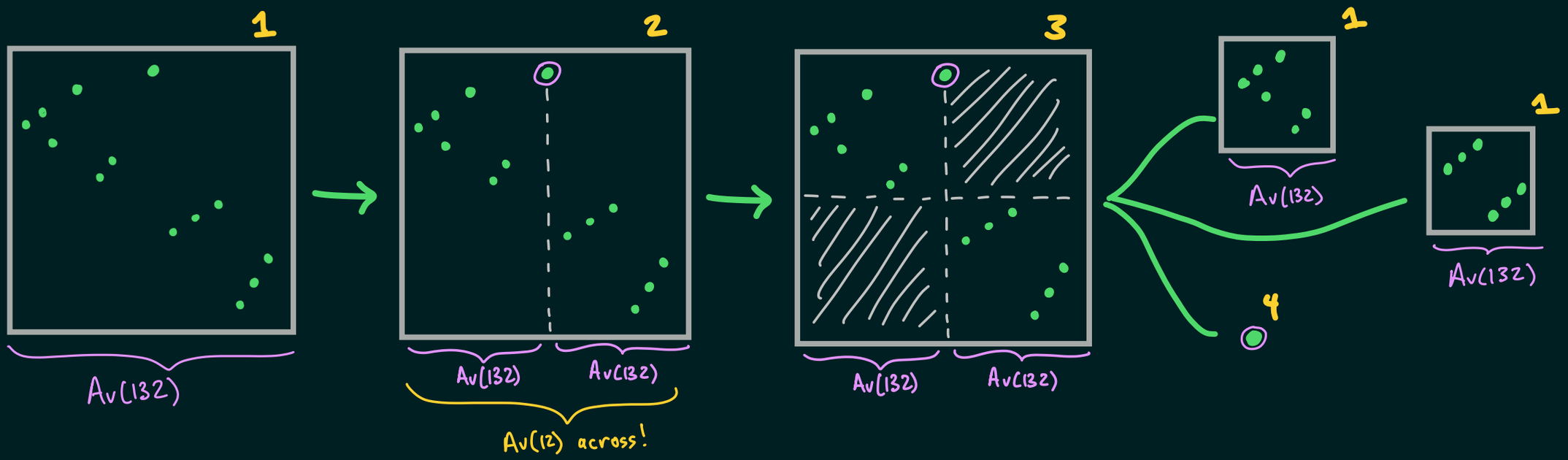
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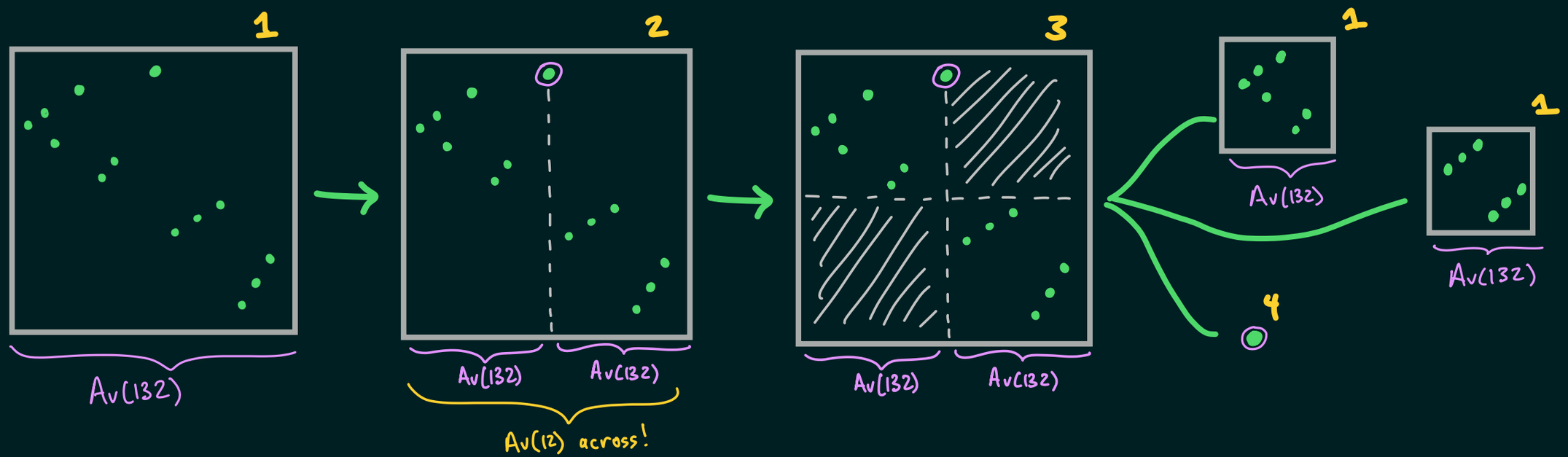
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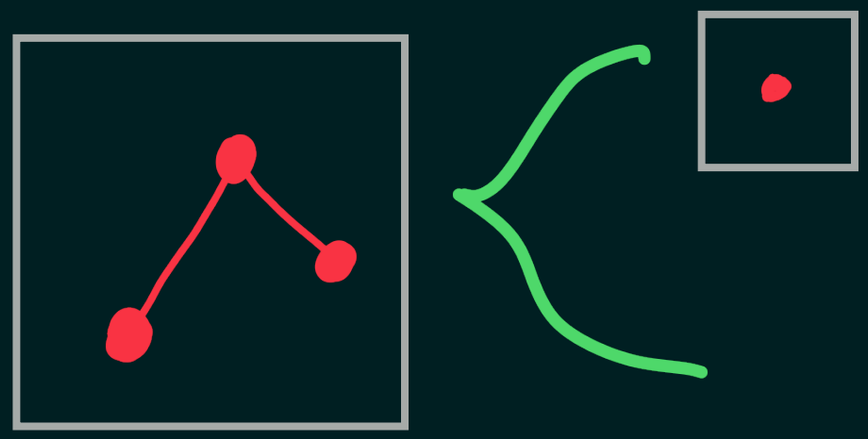

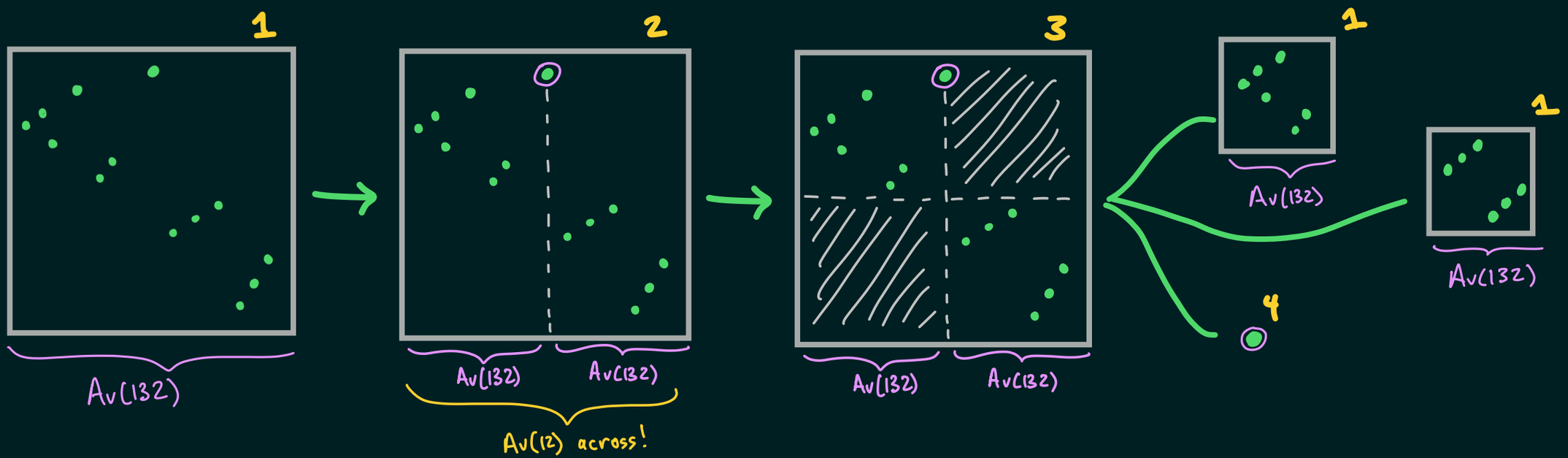
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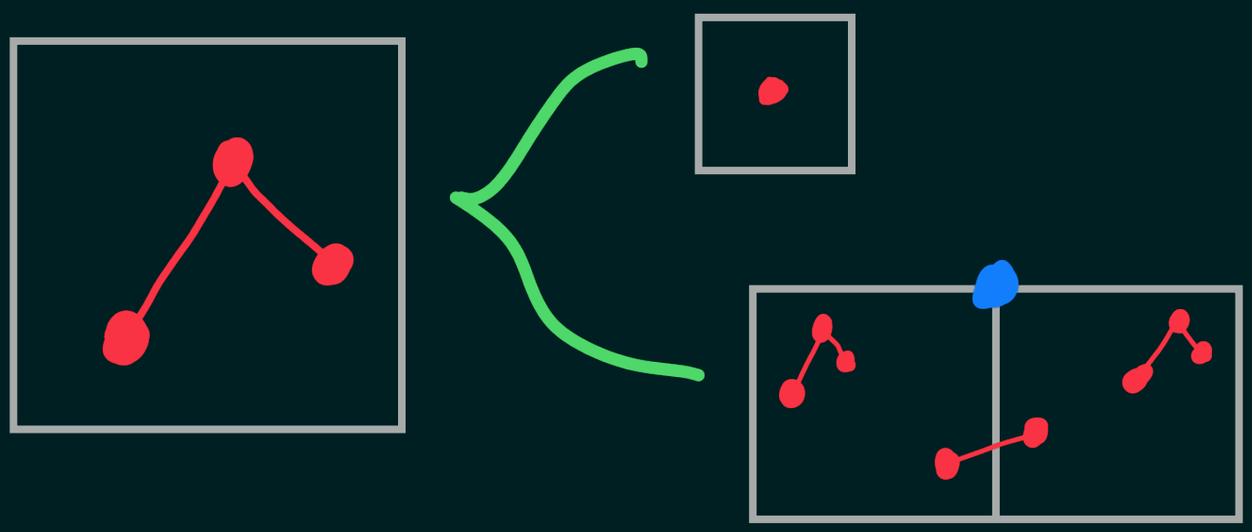

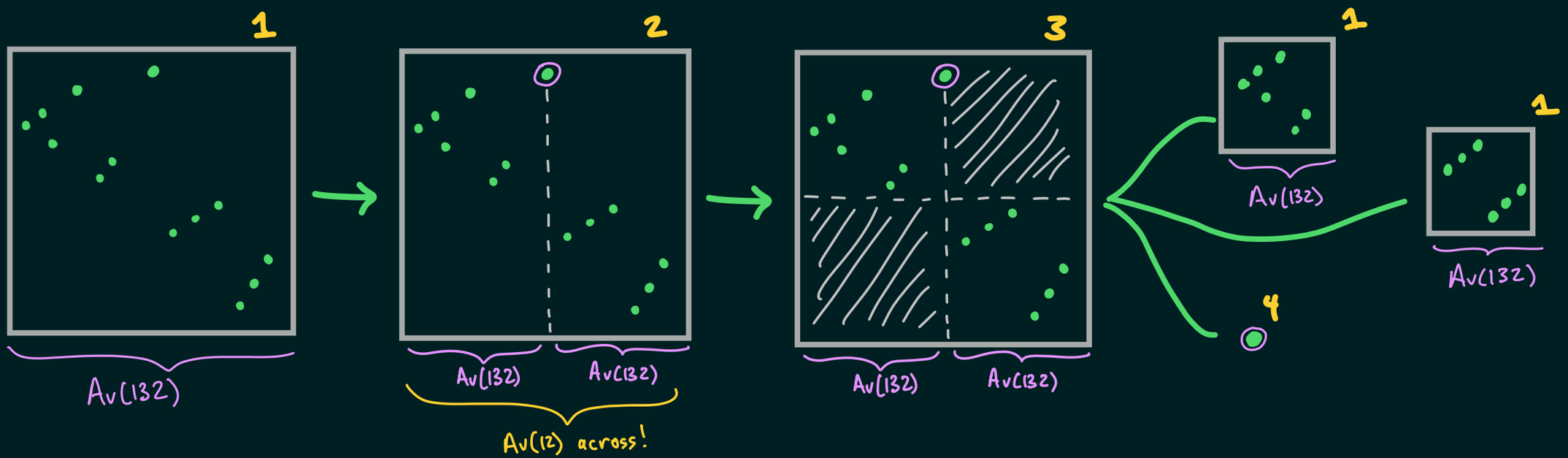
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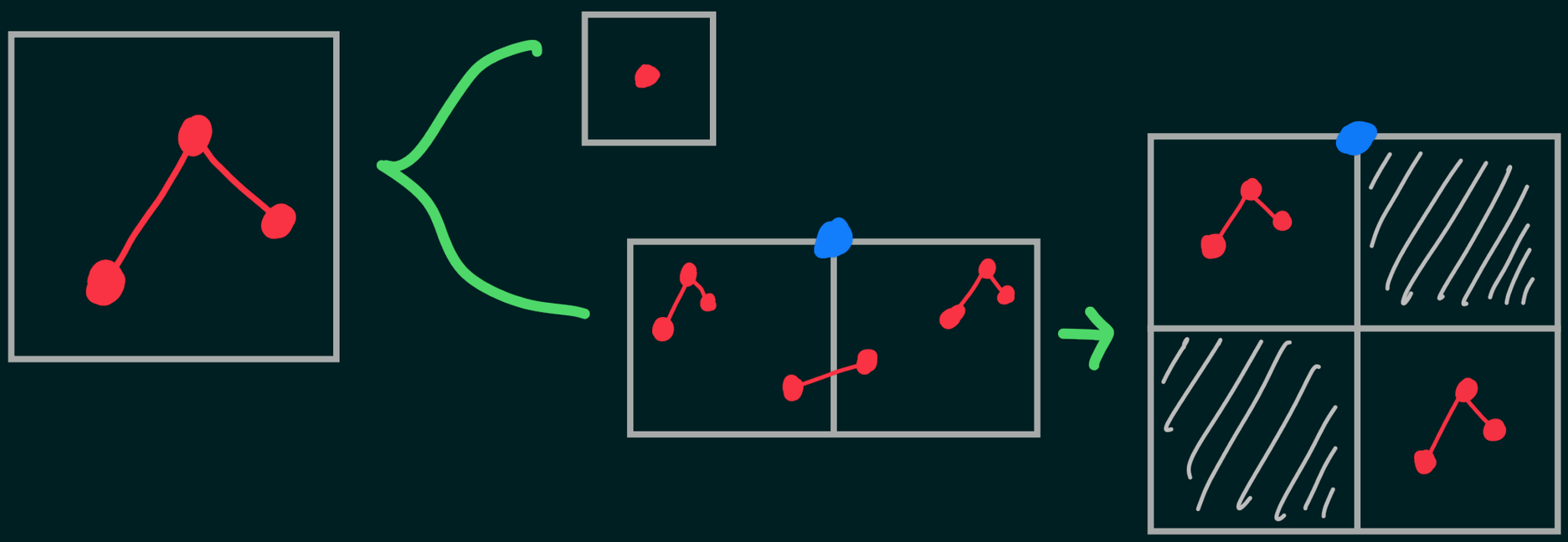

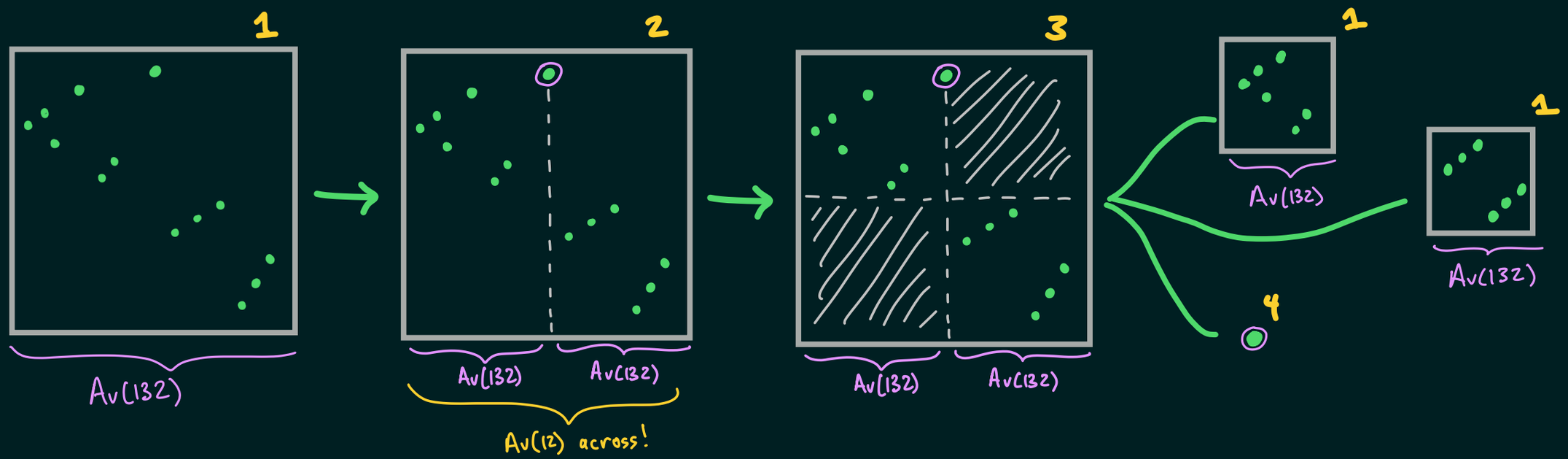
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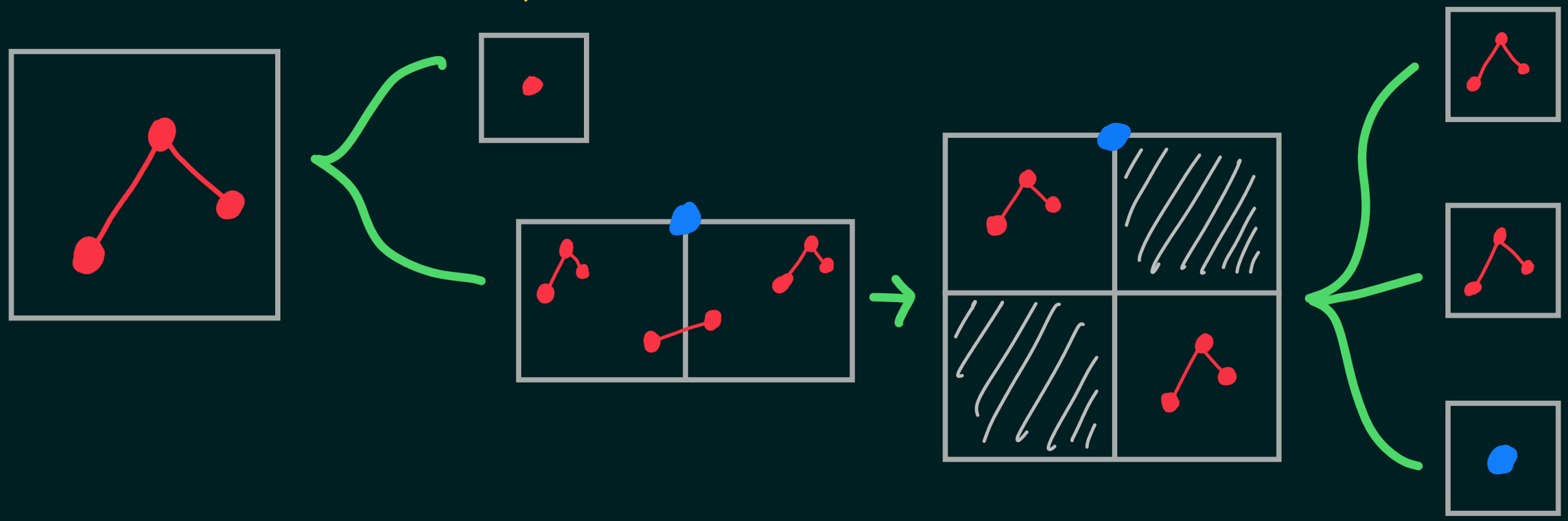

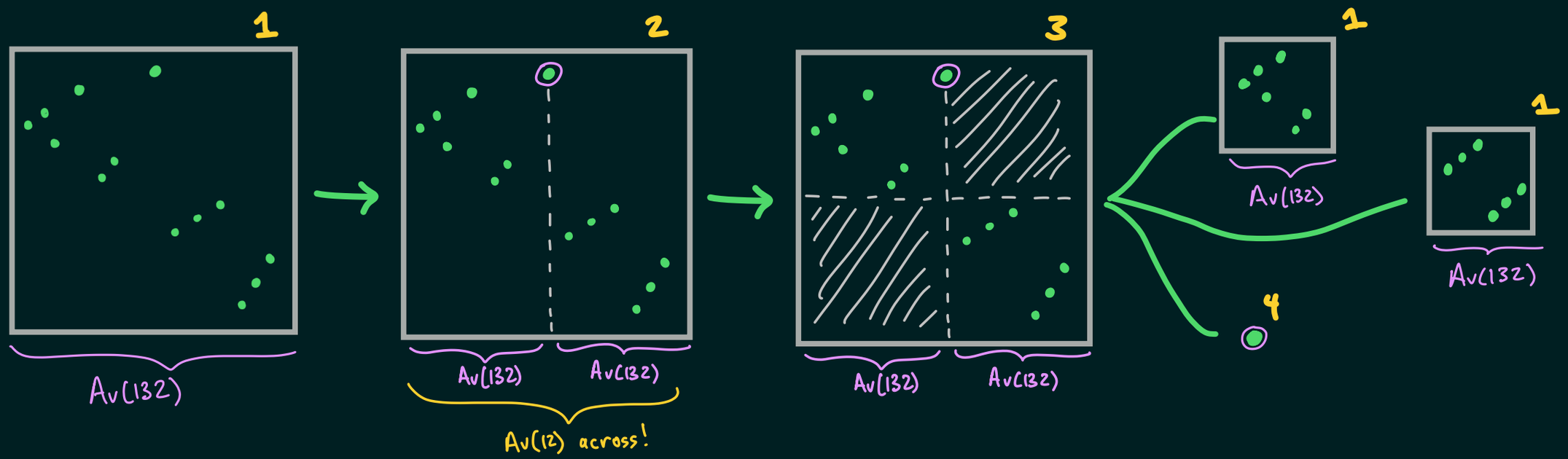

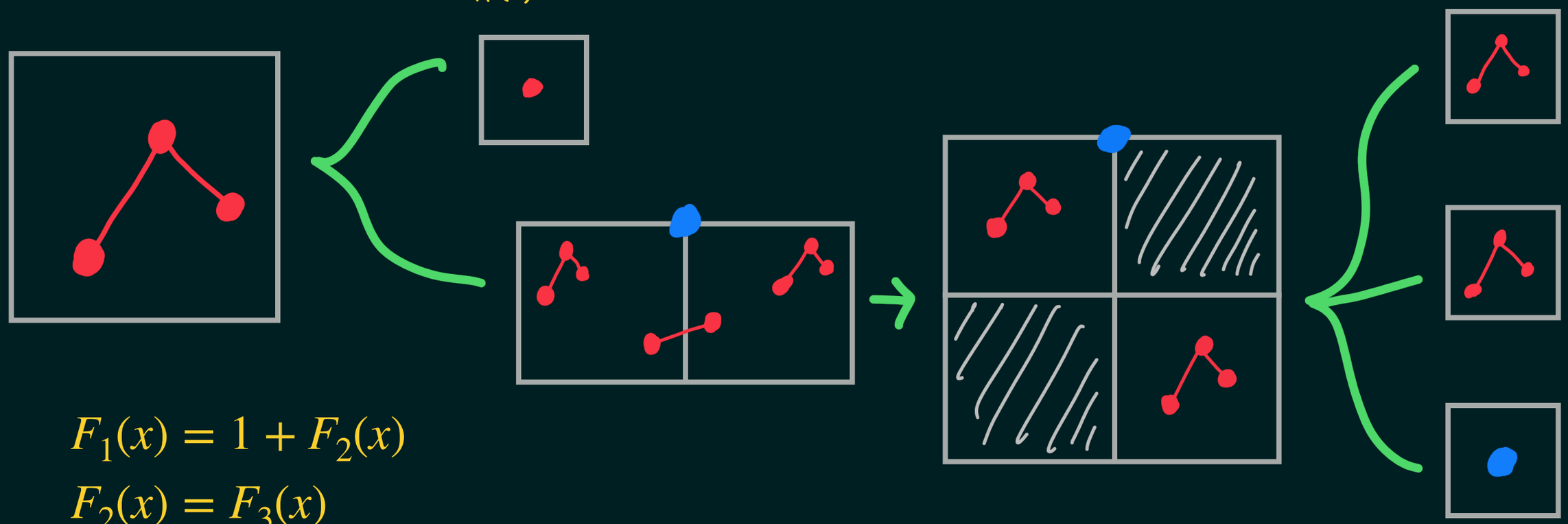

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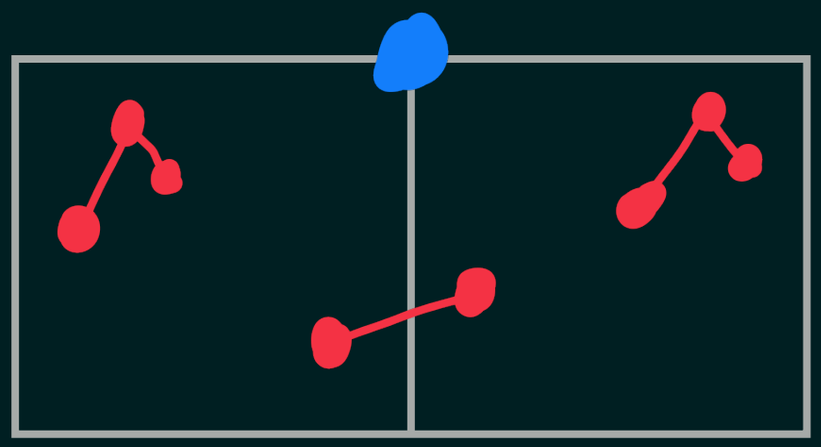
We call this a *tiling*.

It represents the set of permutations (really, a set of gridded permutations) that you can draw on top of the picture such that you

- Don’t make any of the red things (“obstructions”)
- Do make every blue thing (“requirements”)



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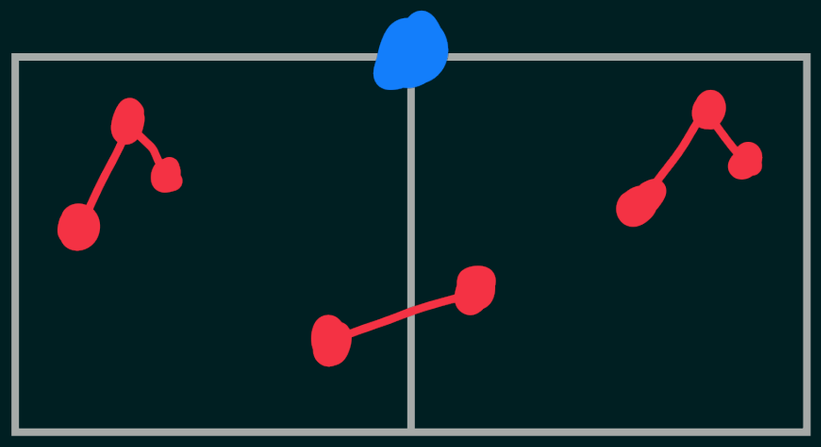

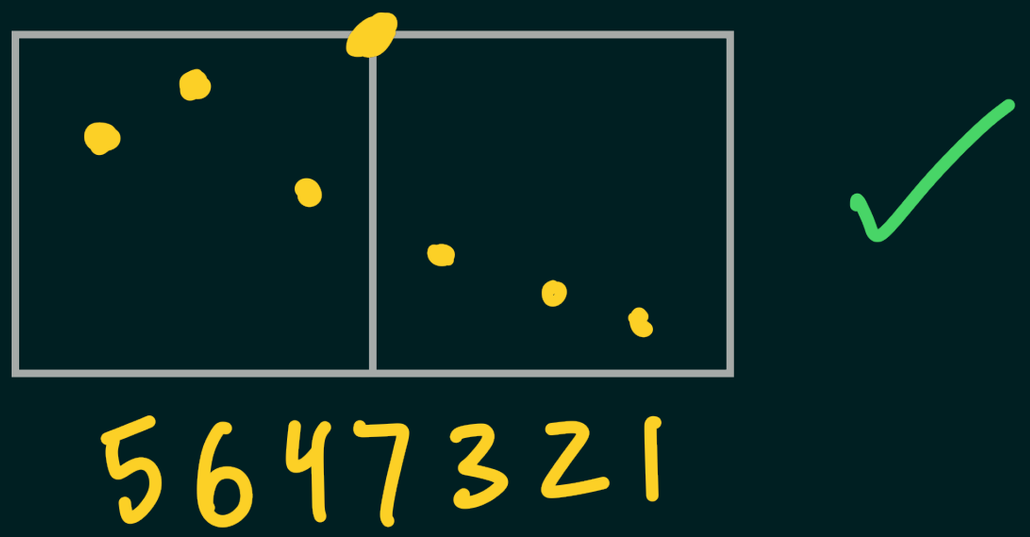

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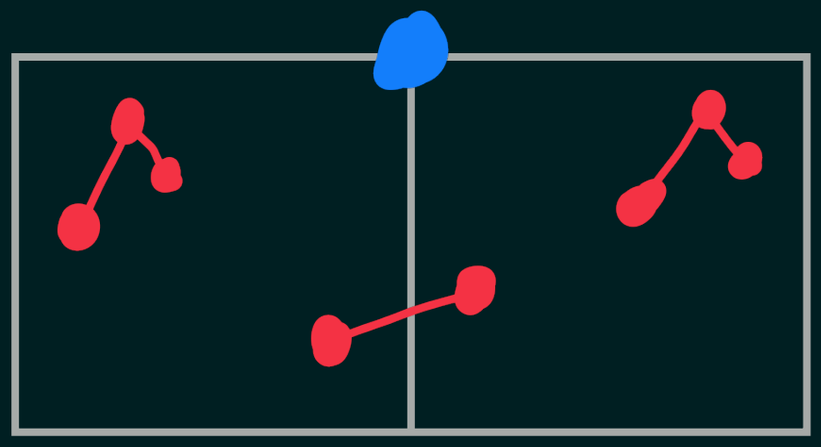
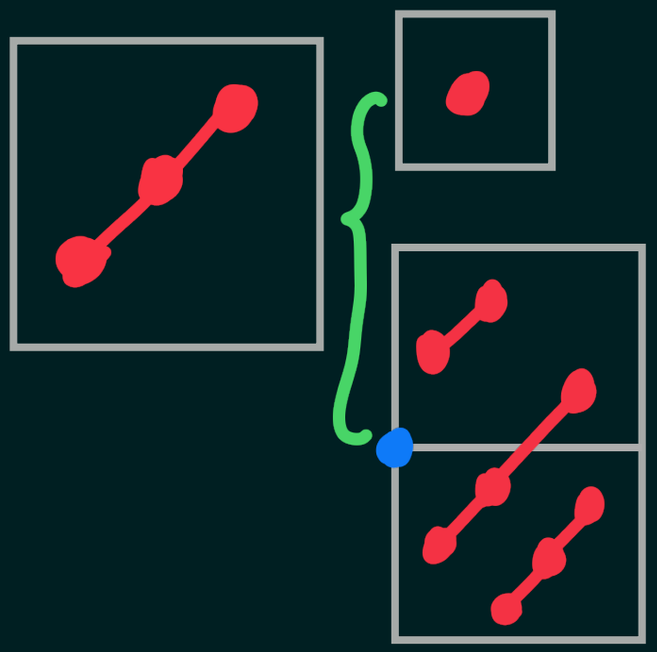
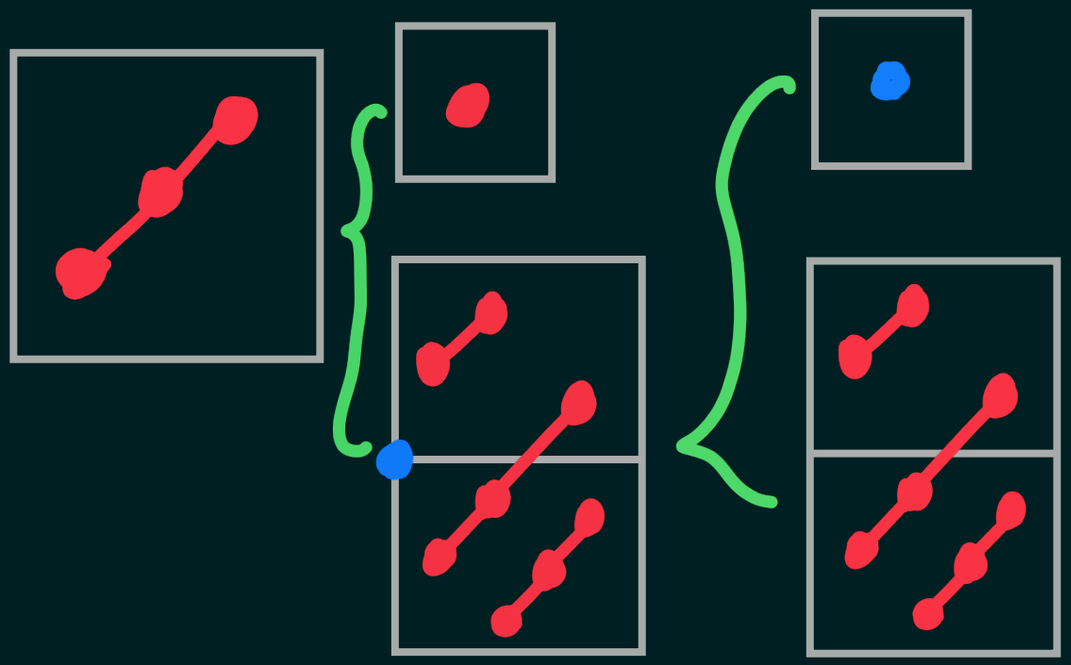
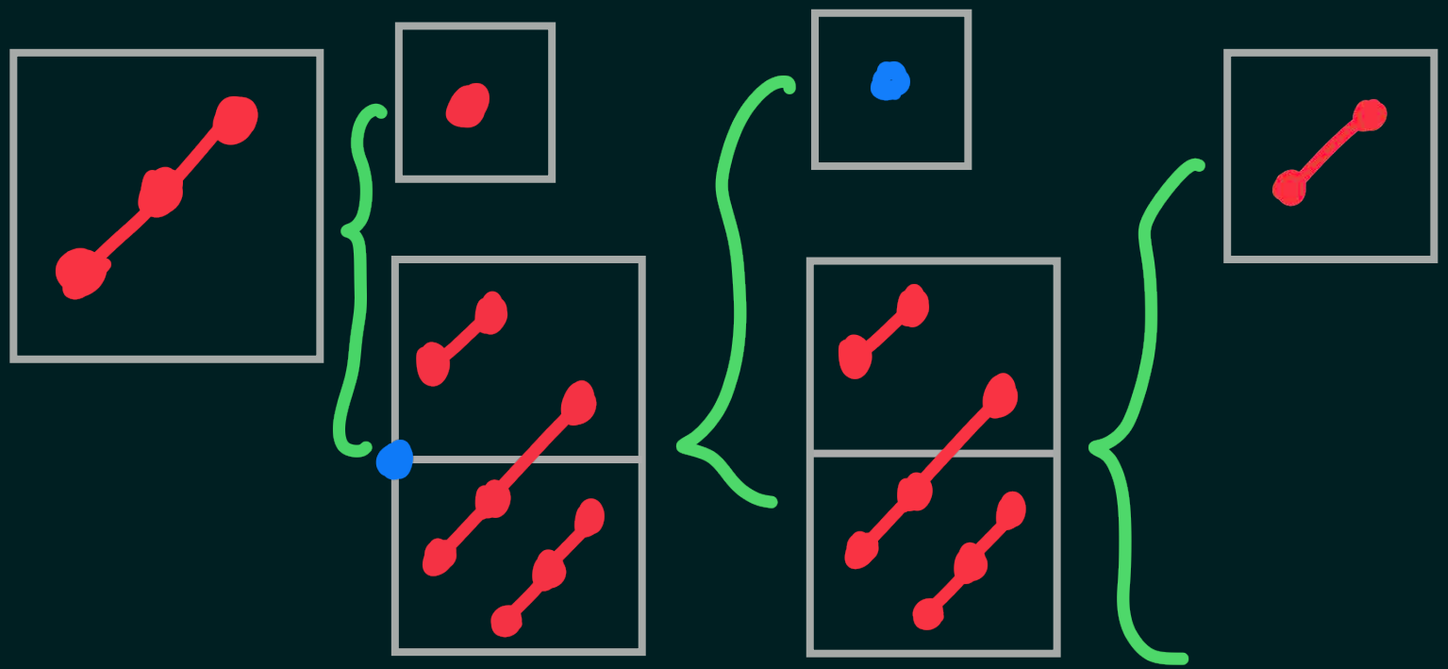
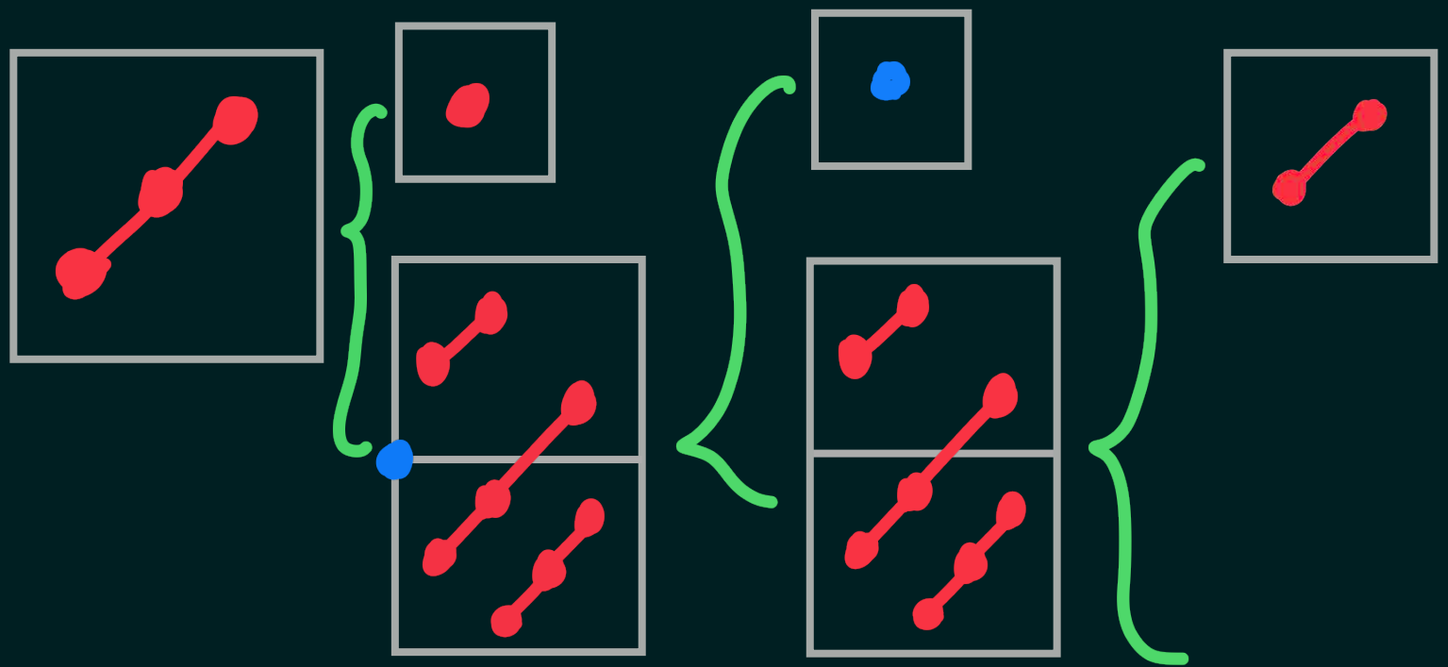
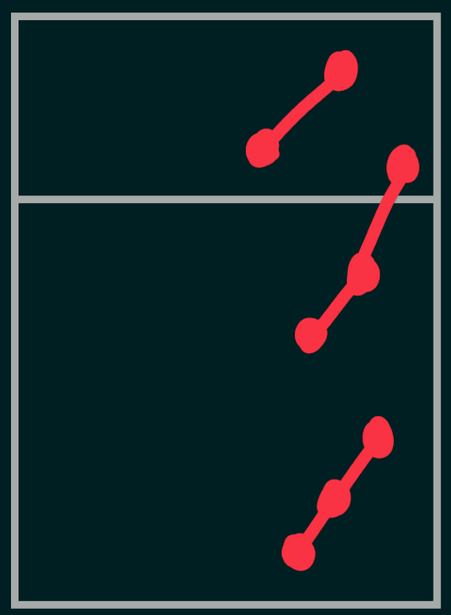


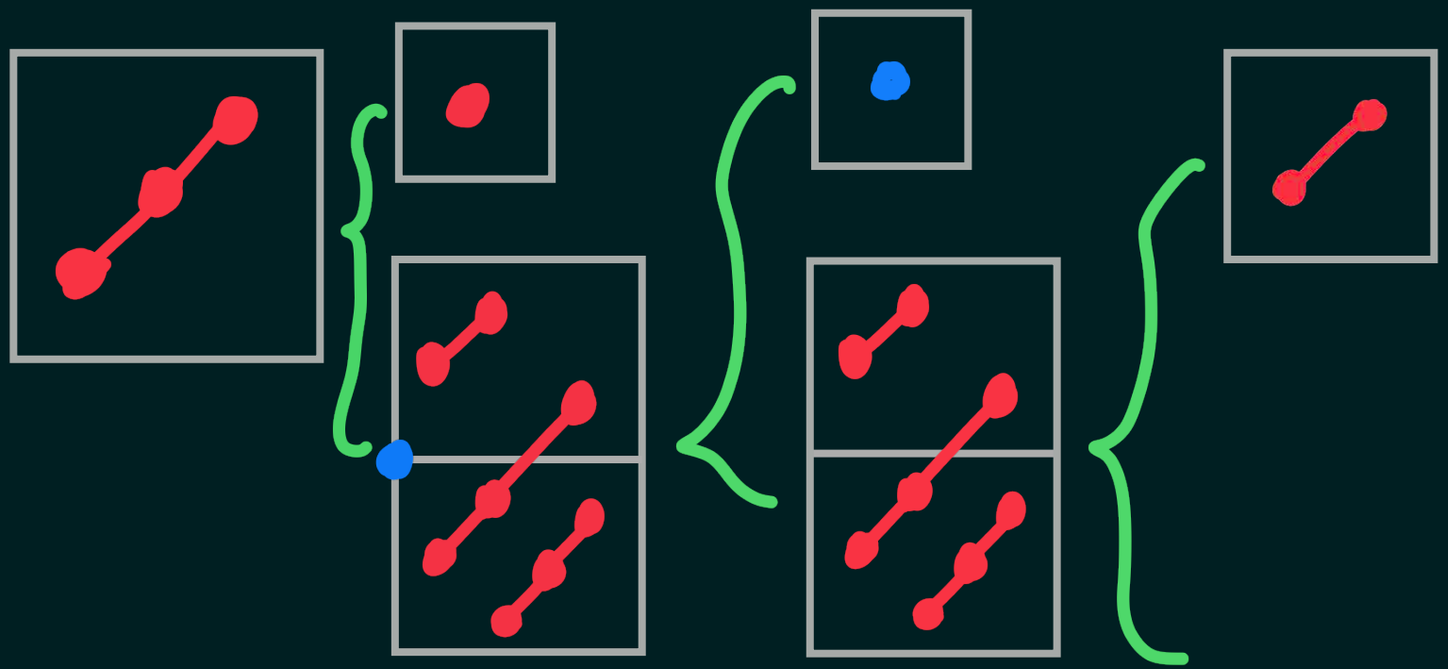


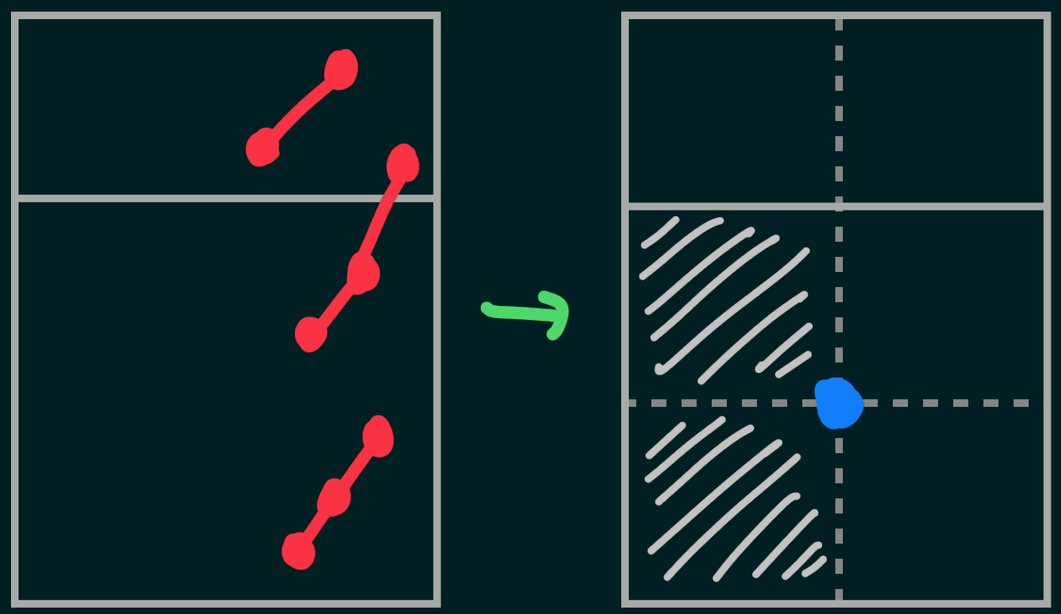


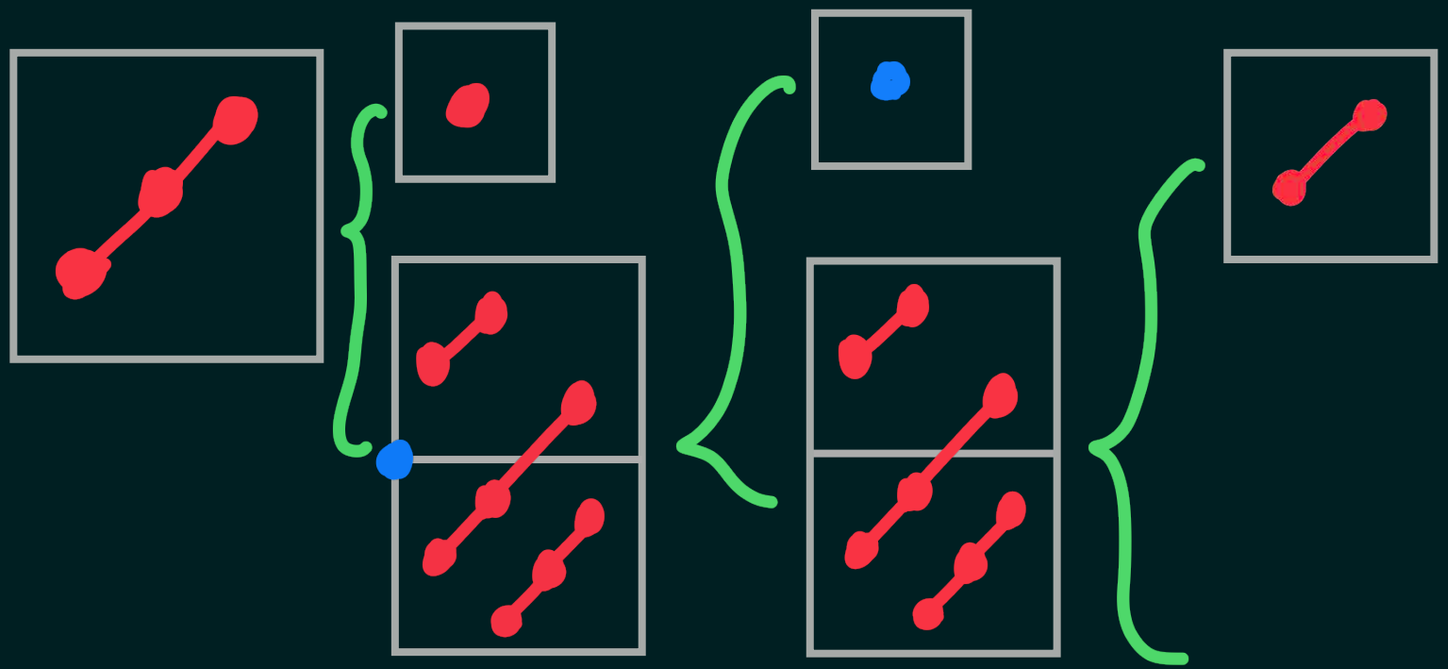


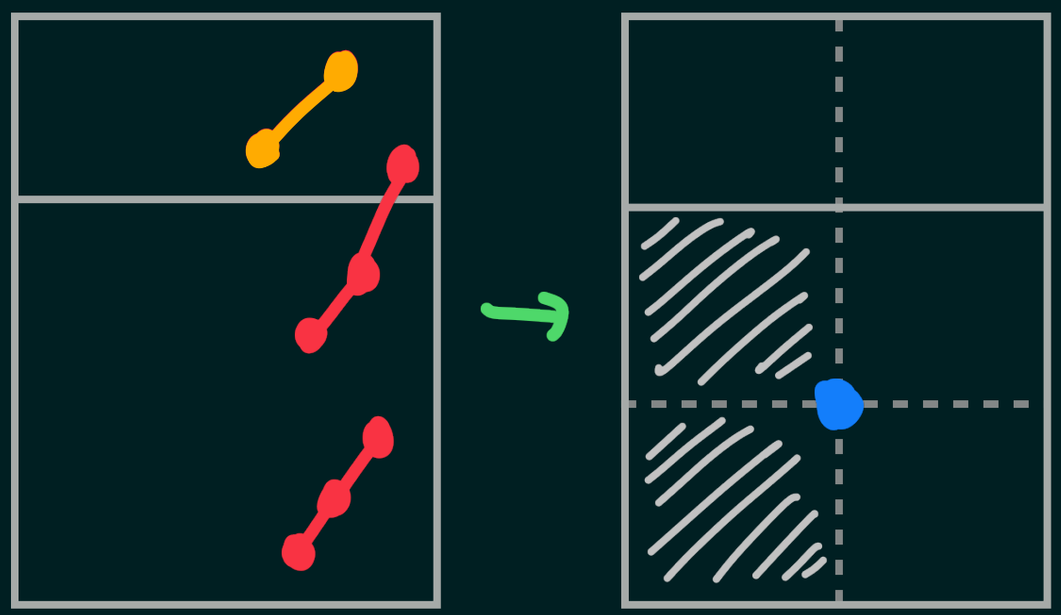


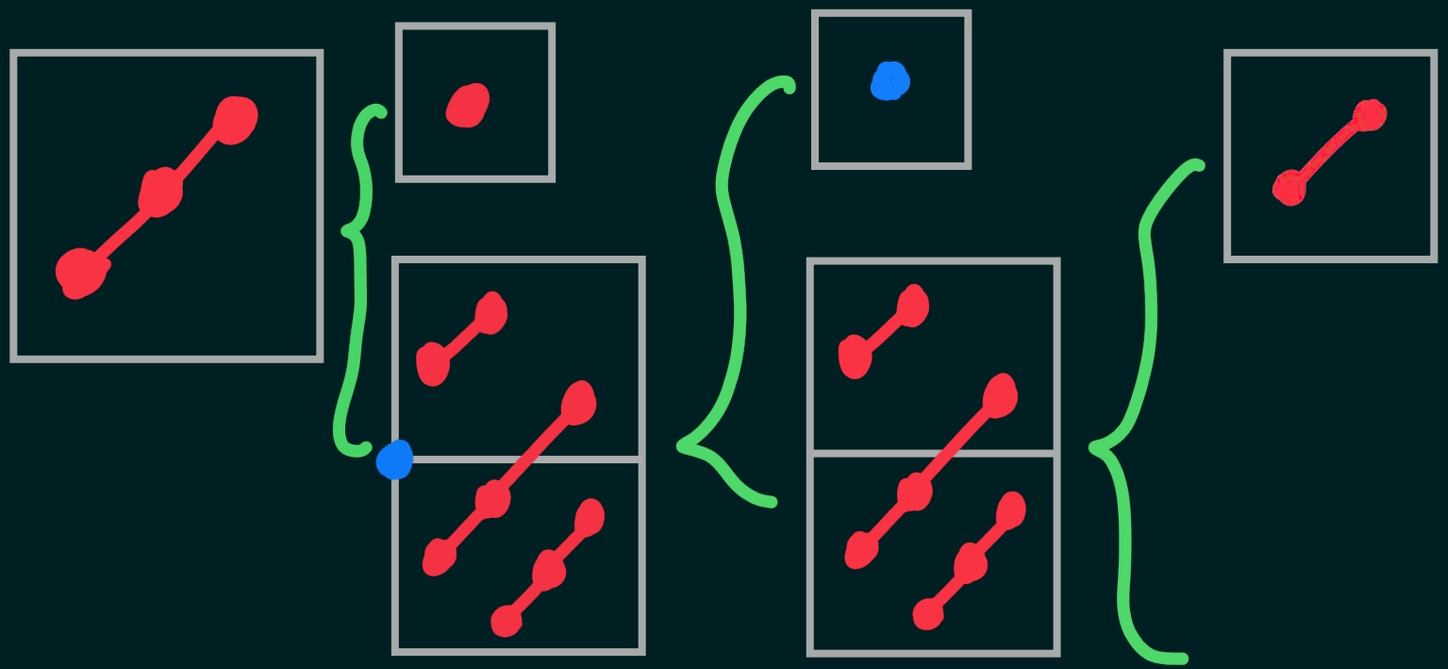


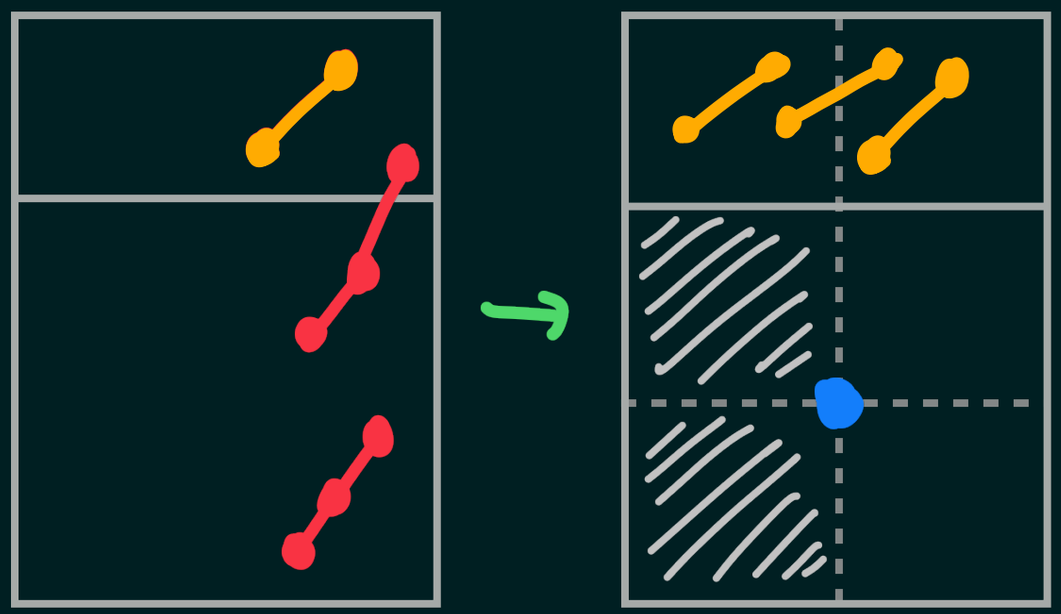


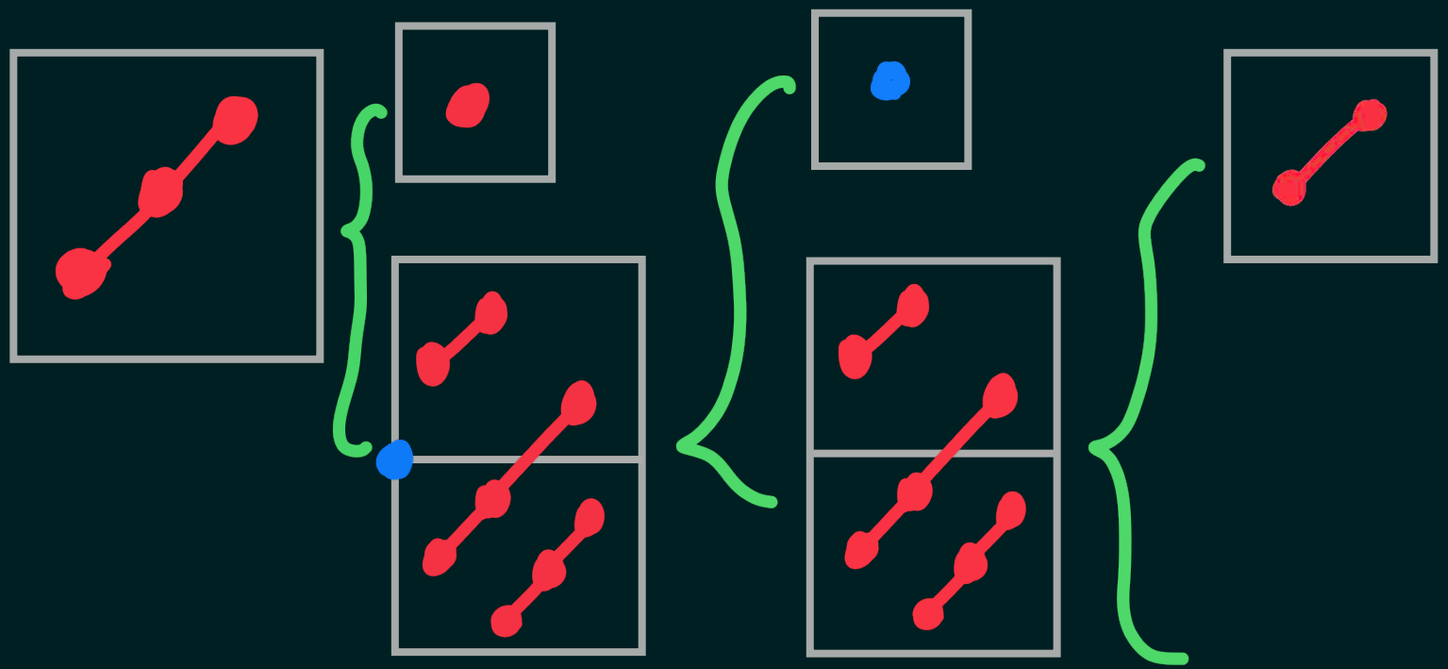


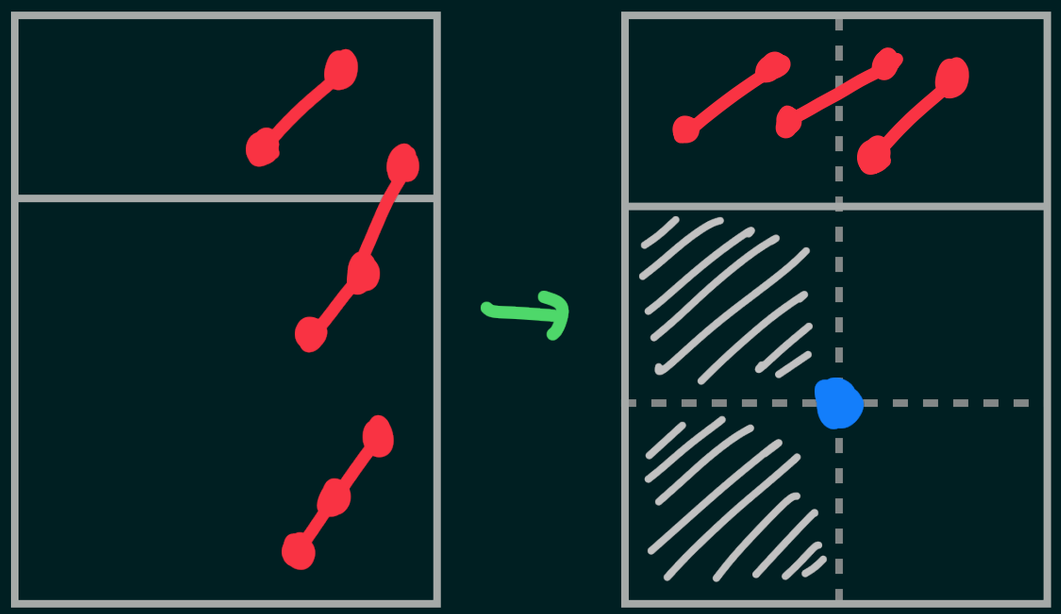


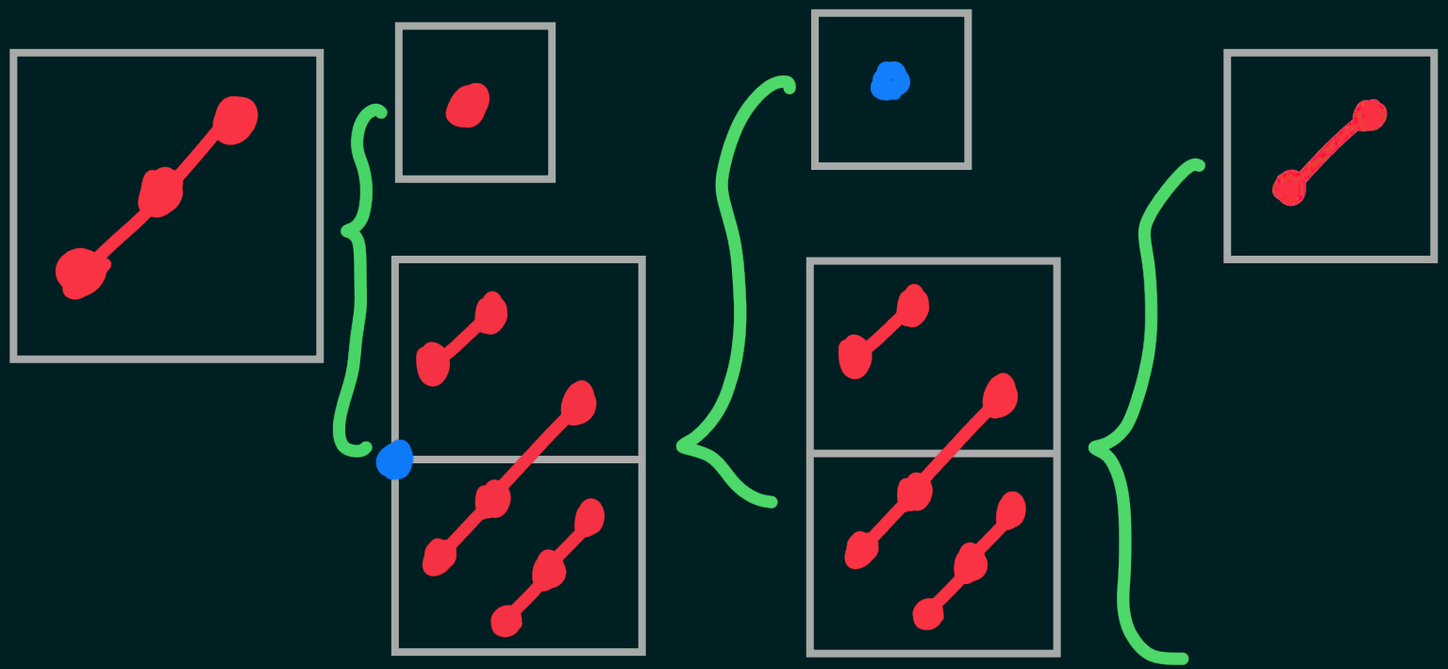


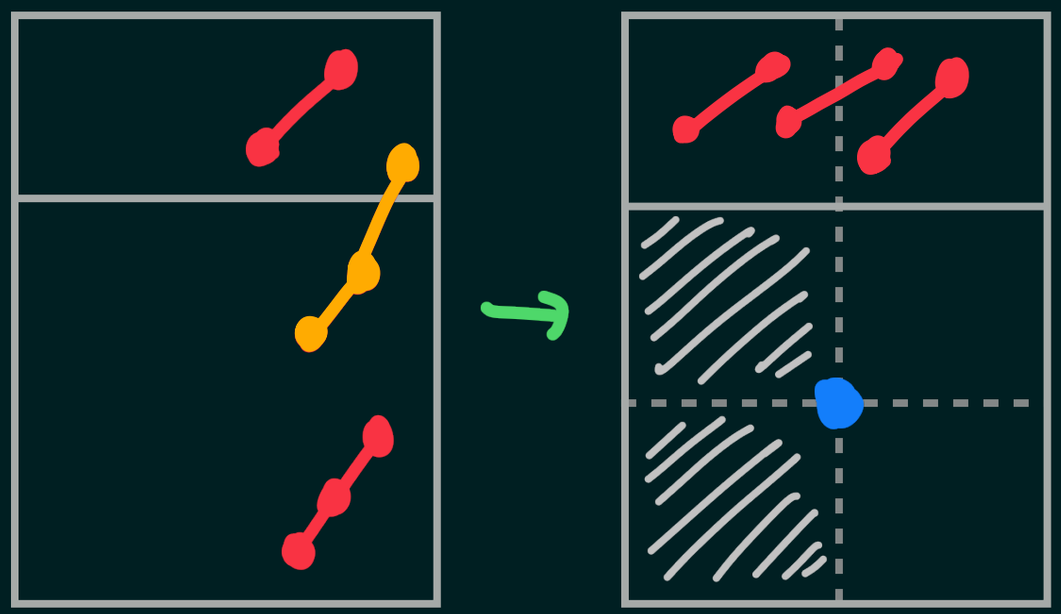


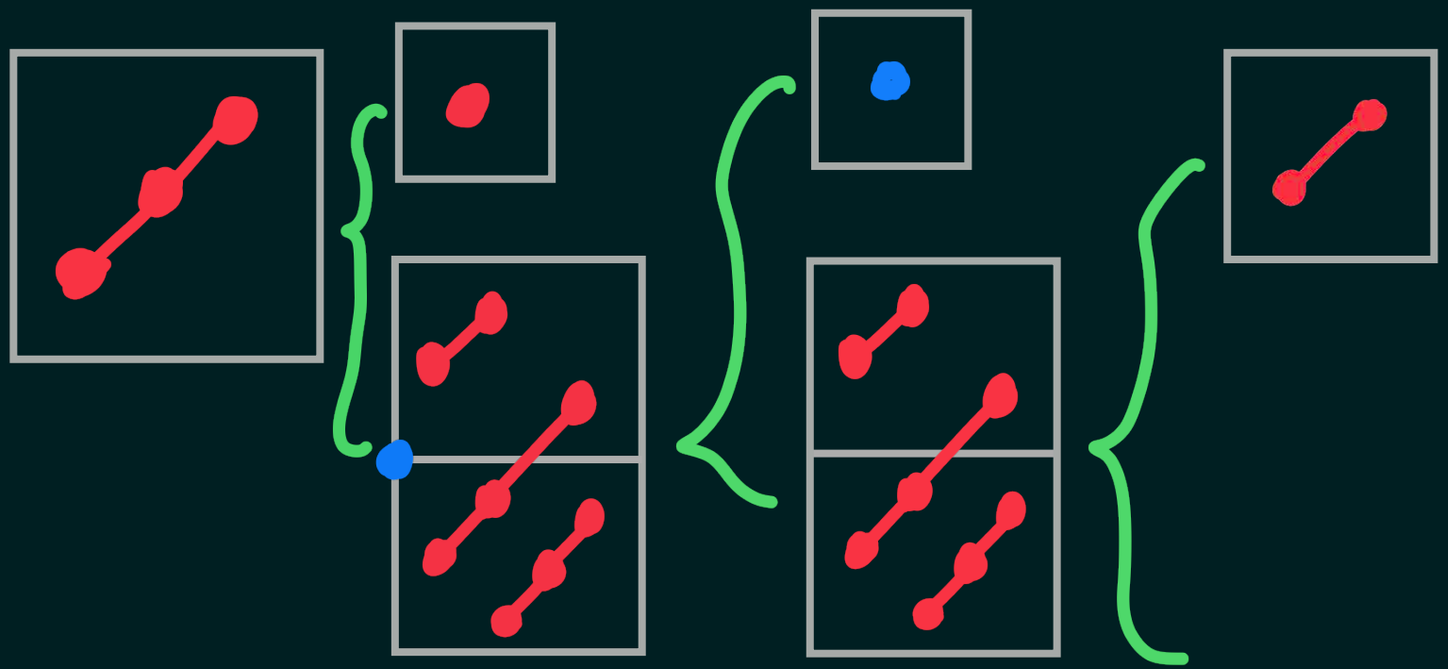


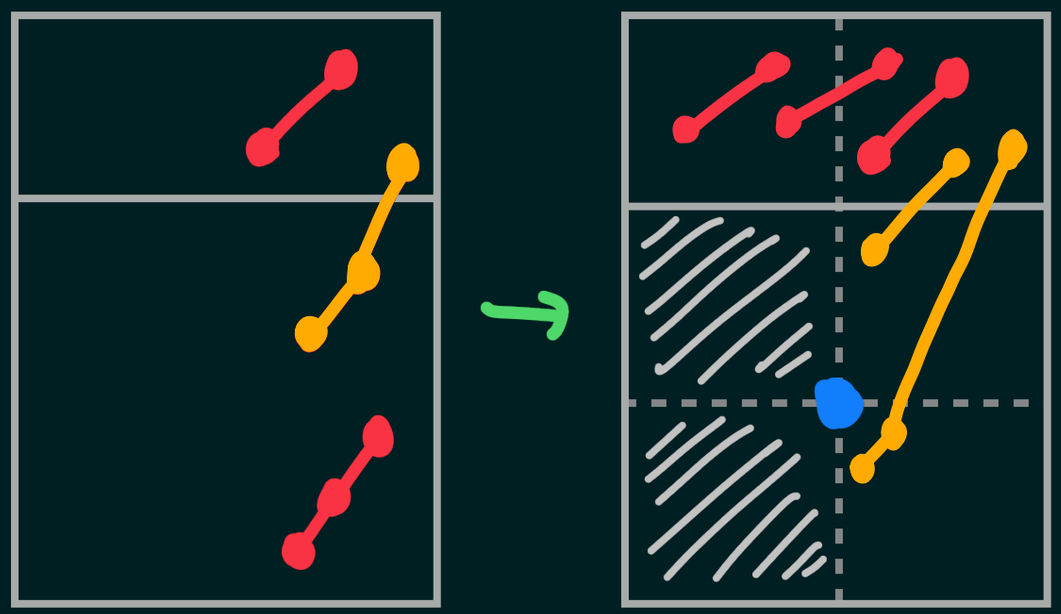


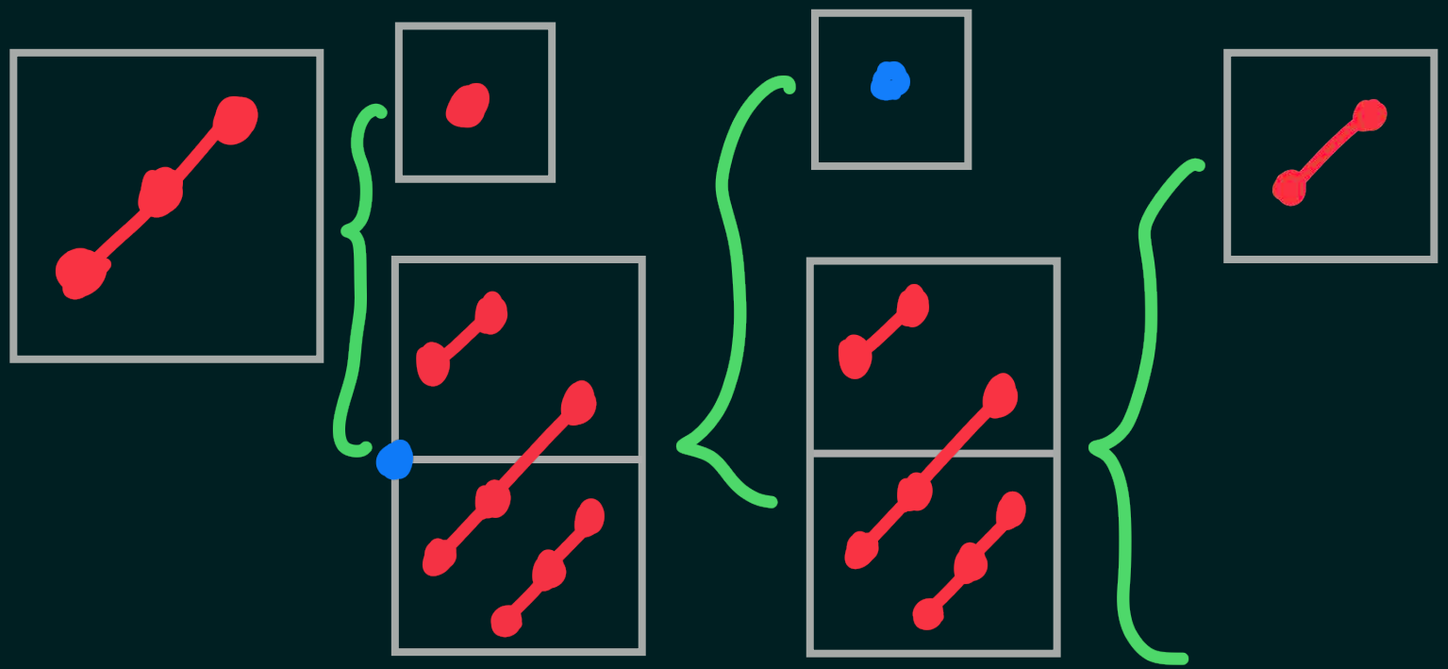


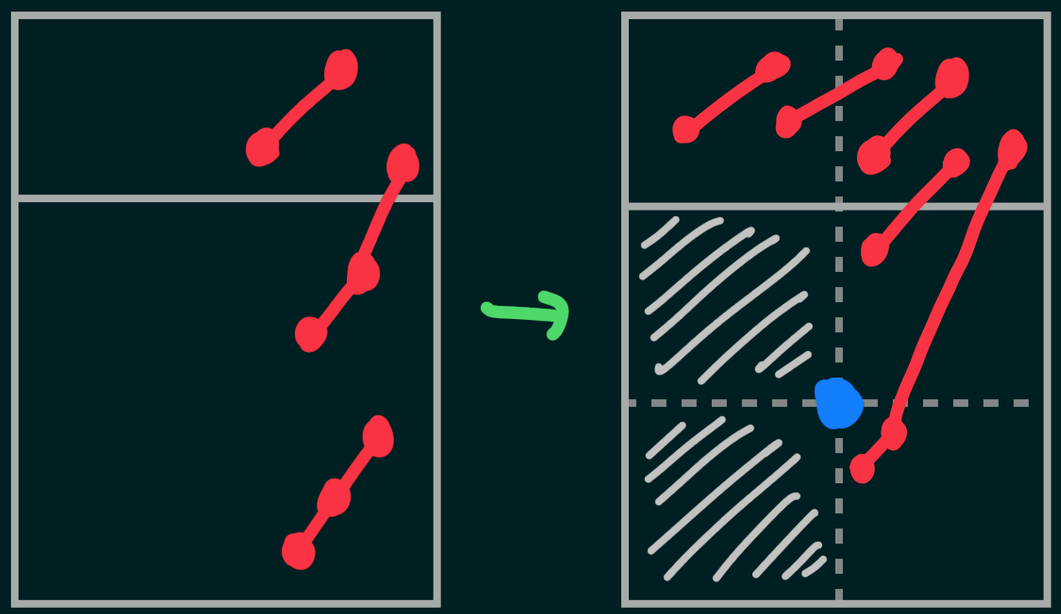


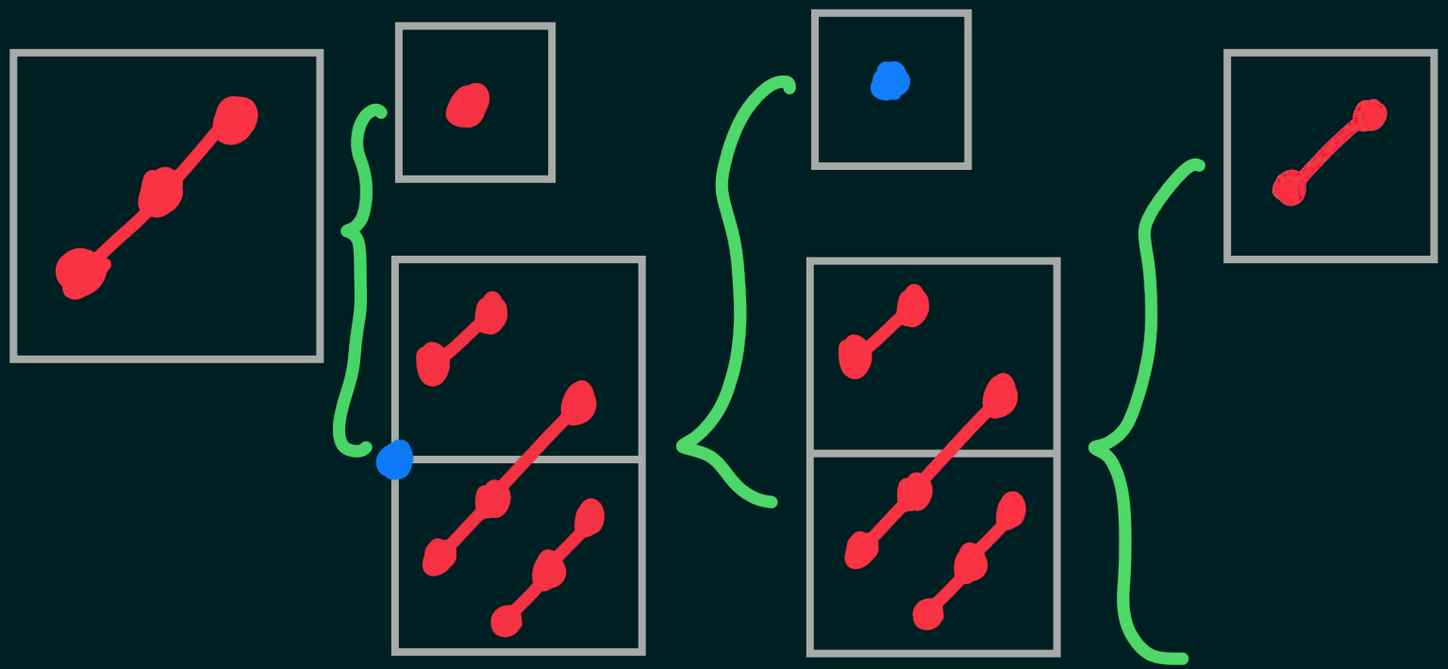


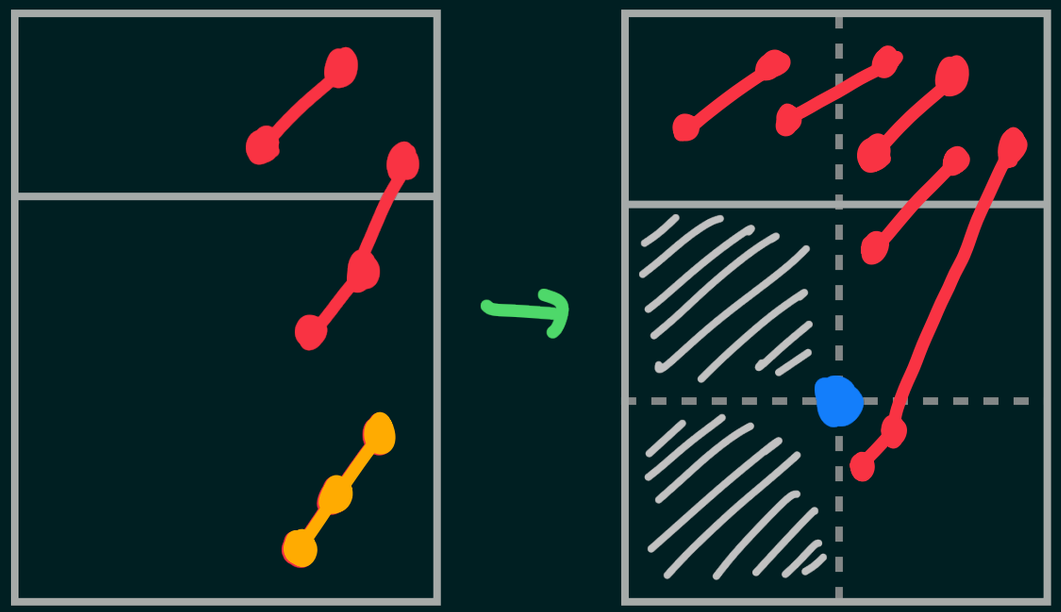


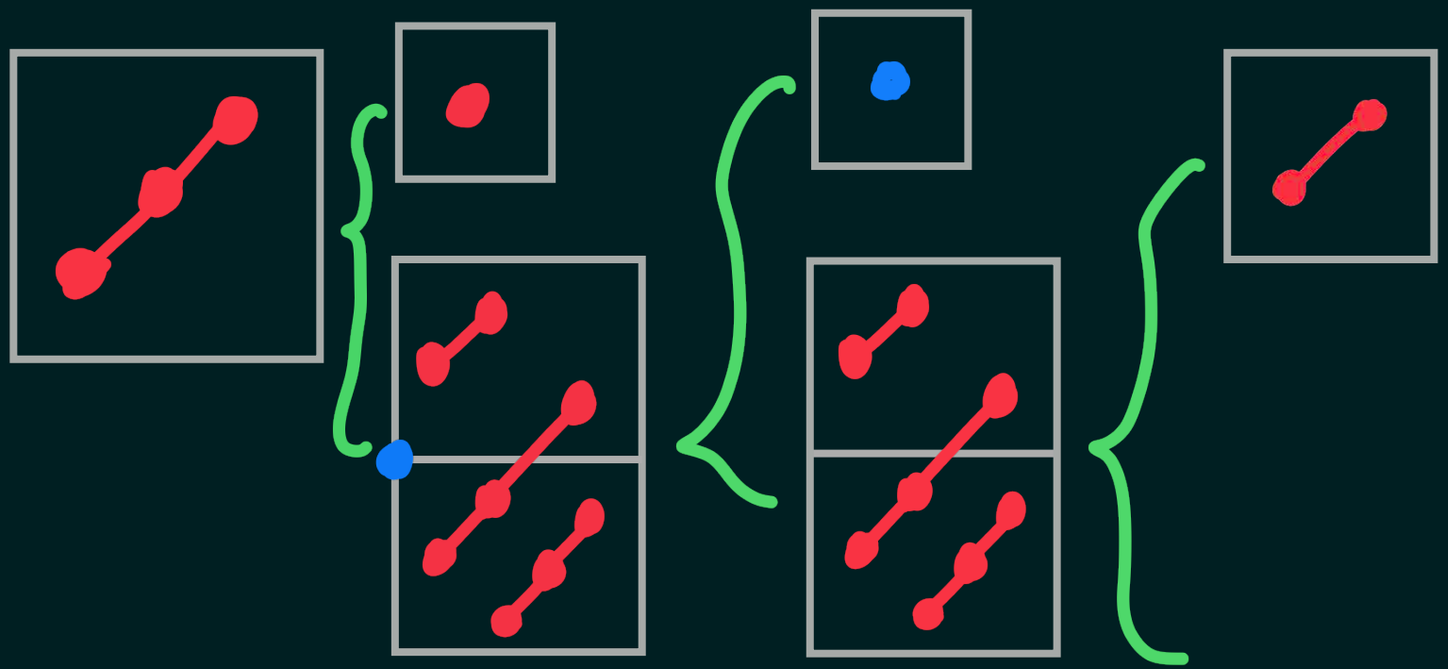


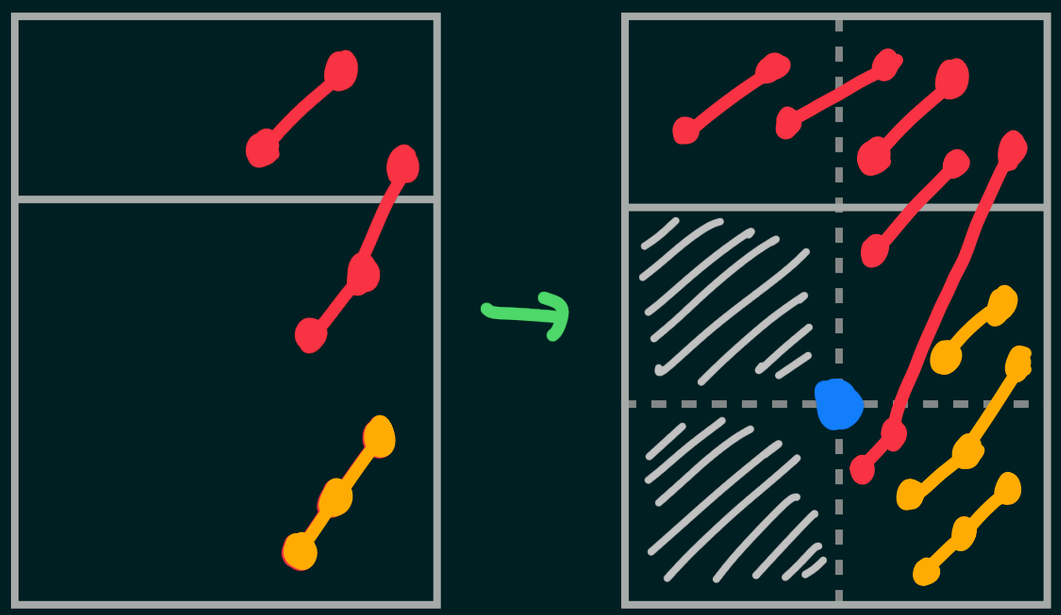


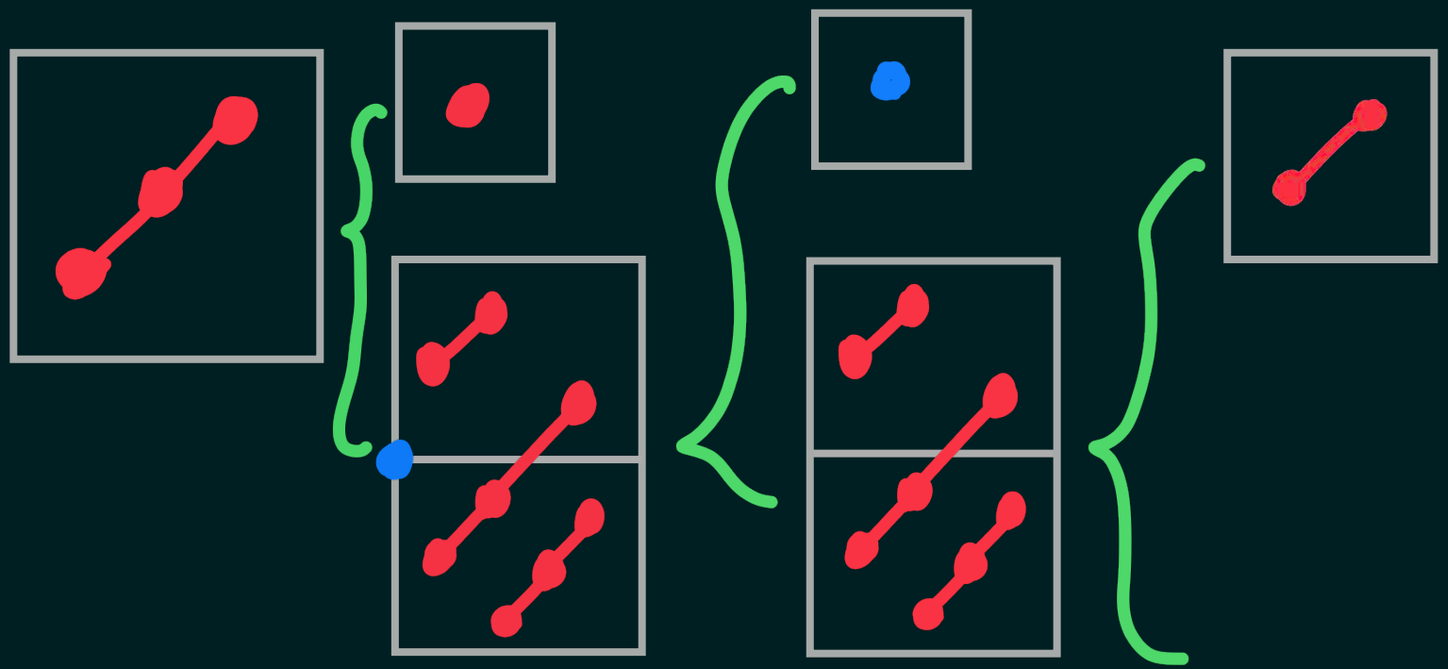


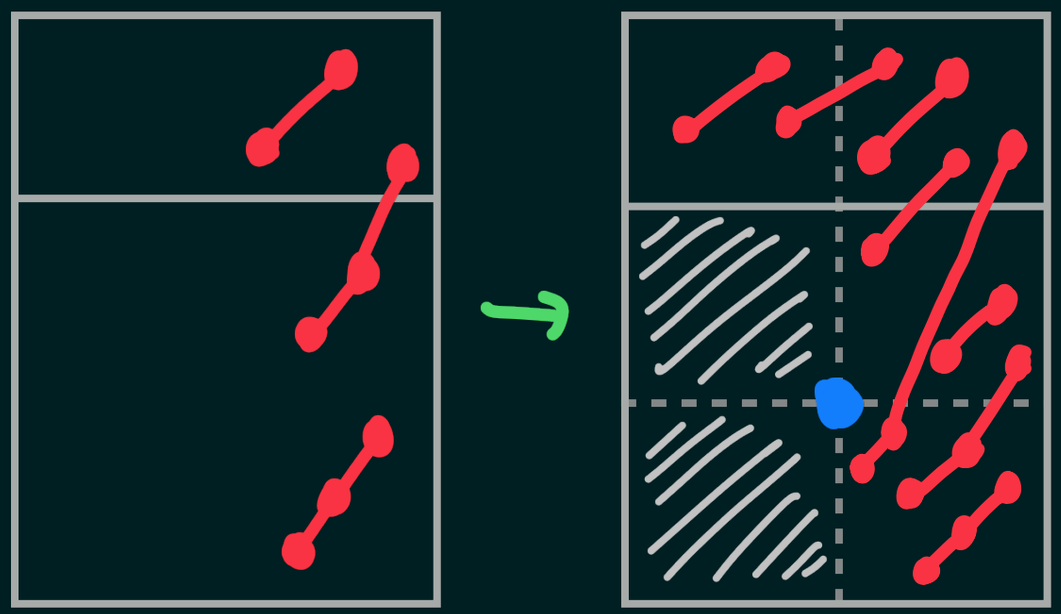


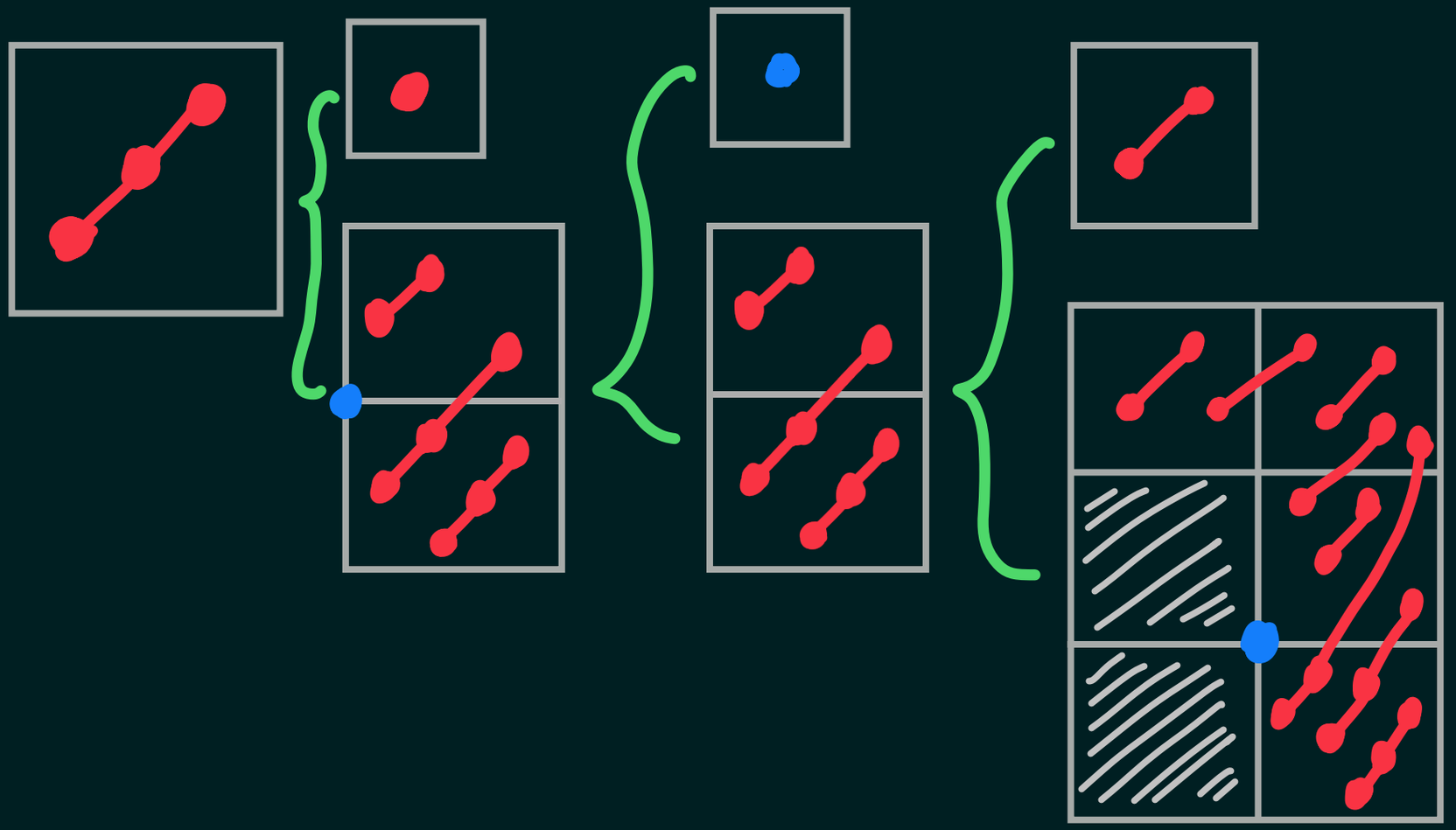


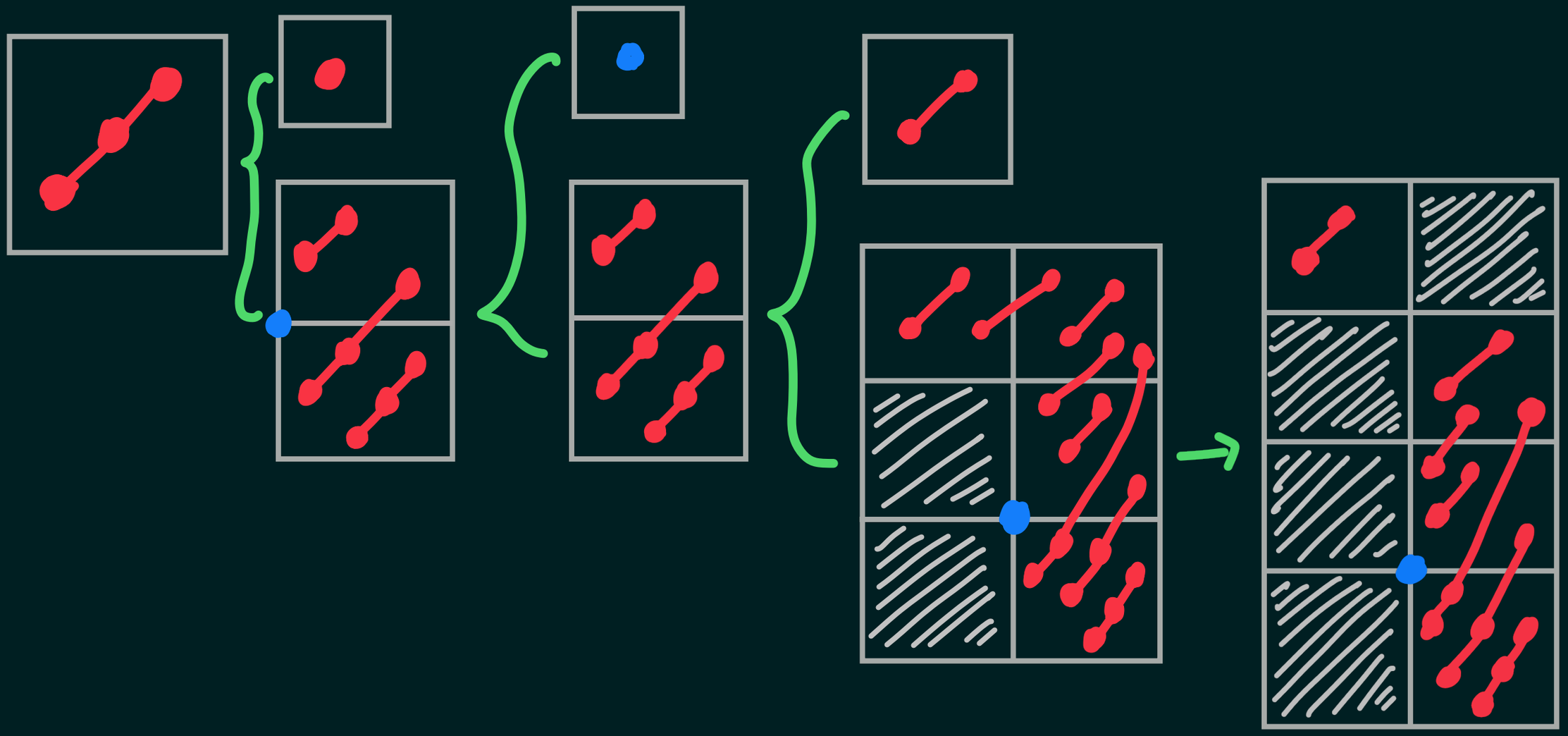


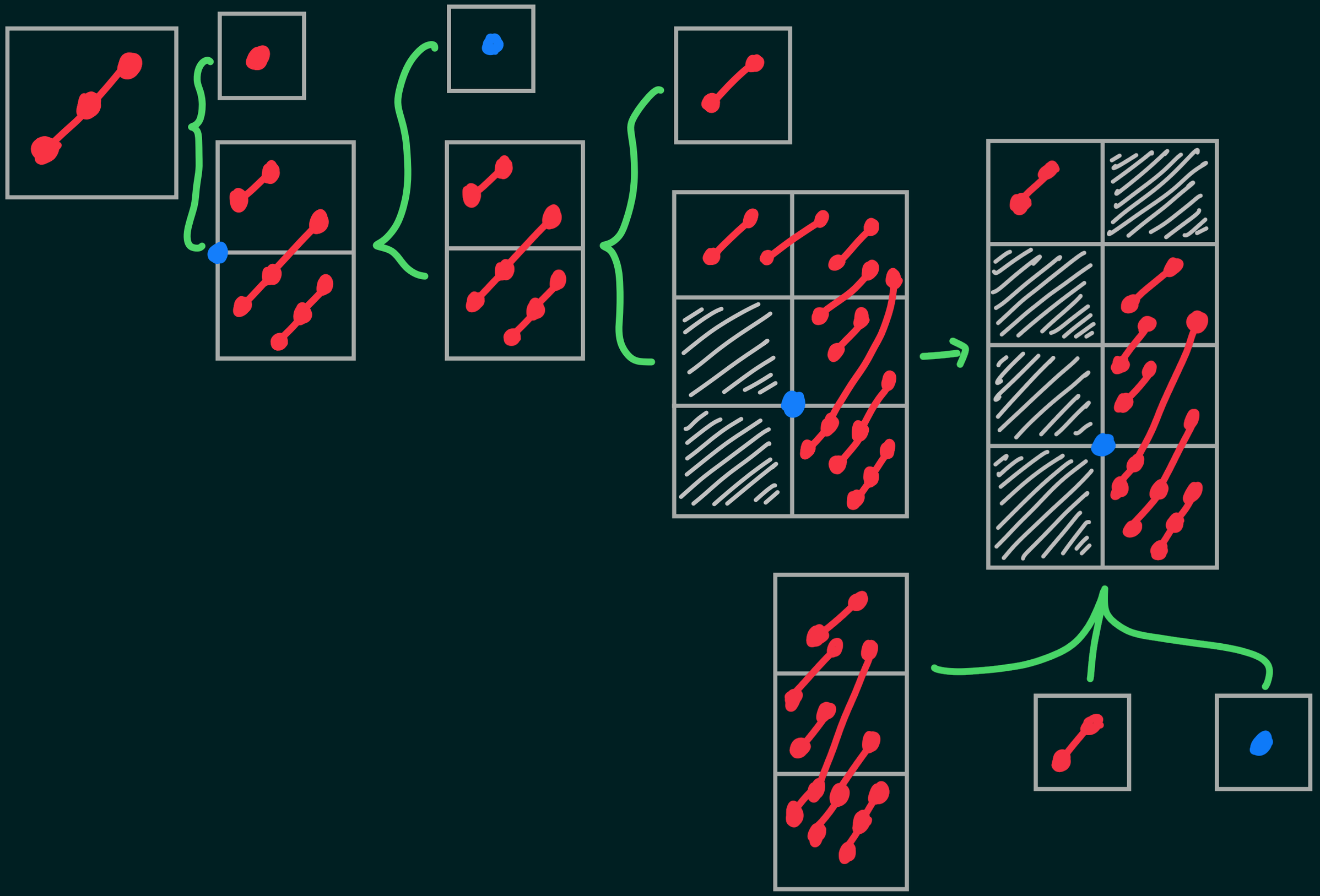


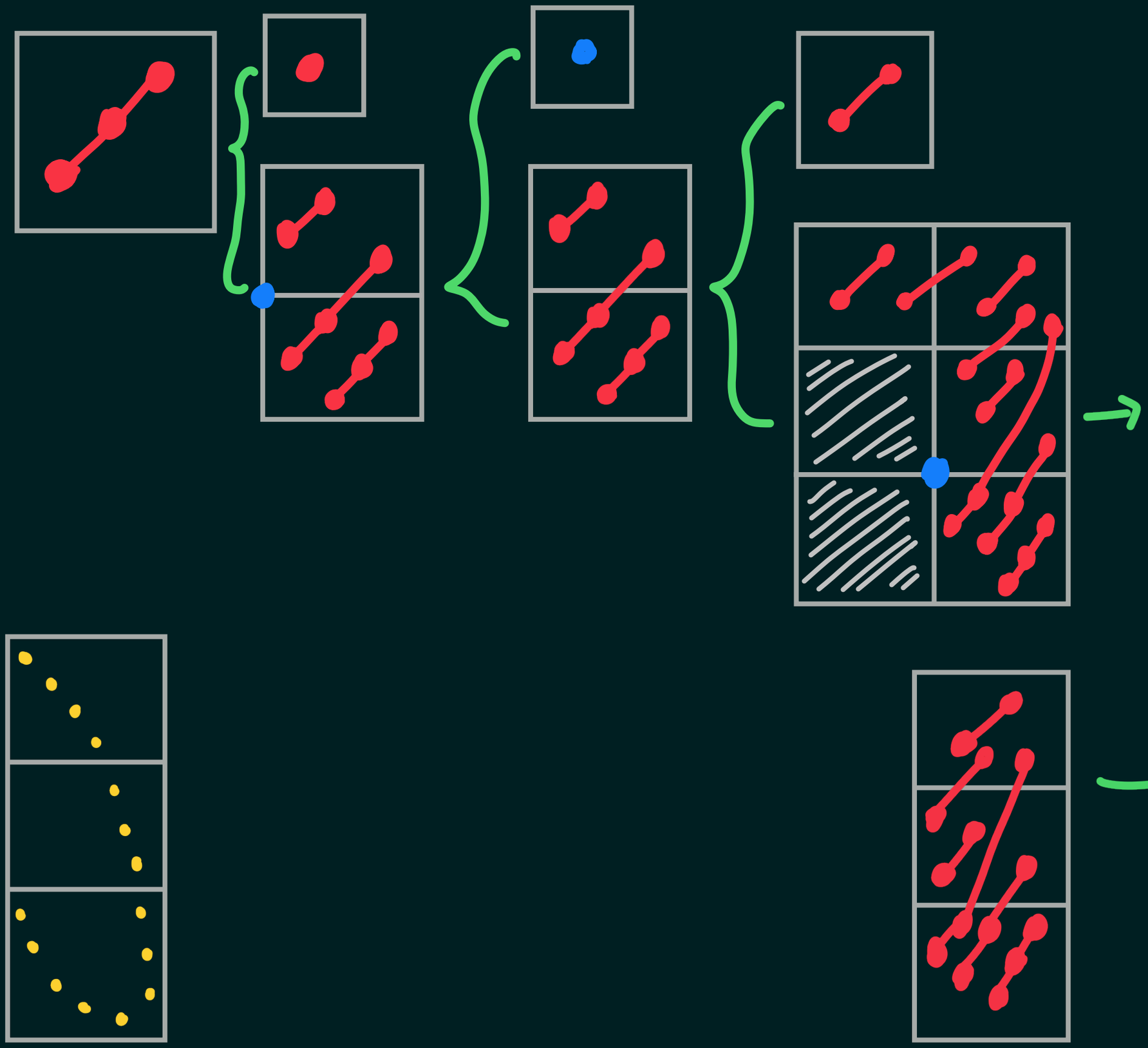


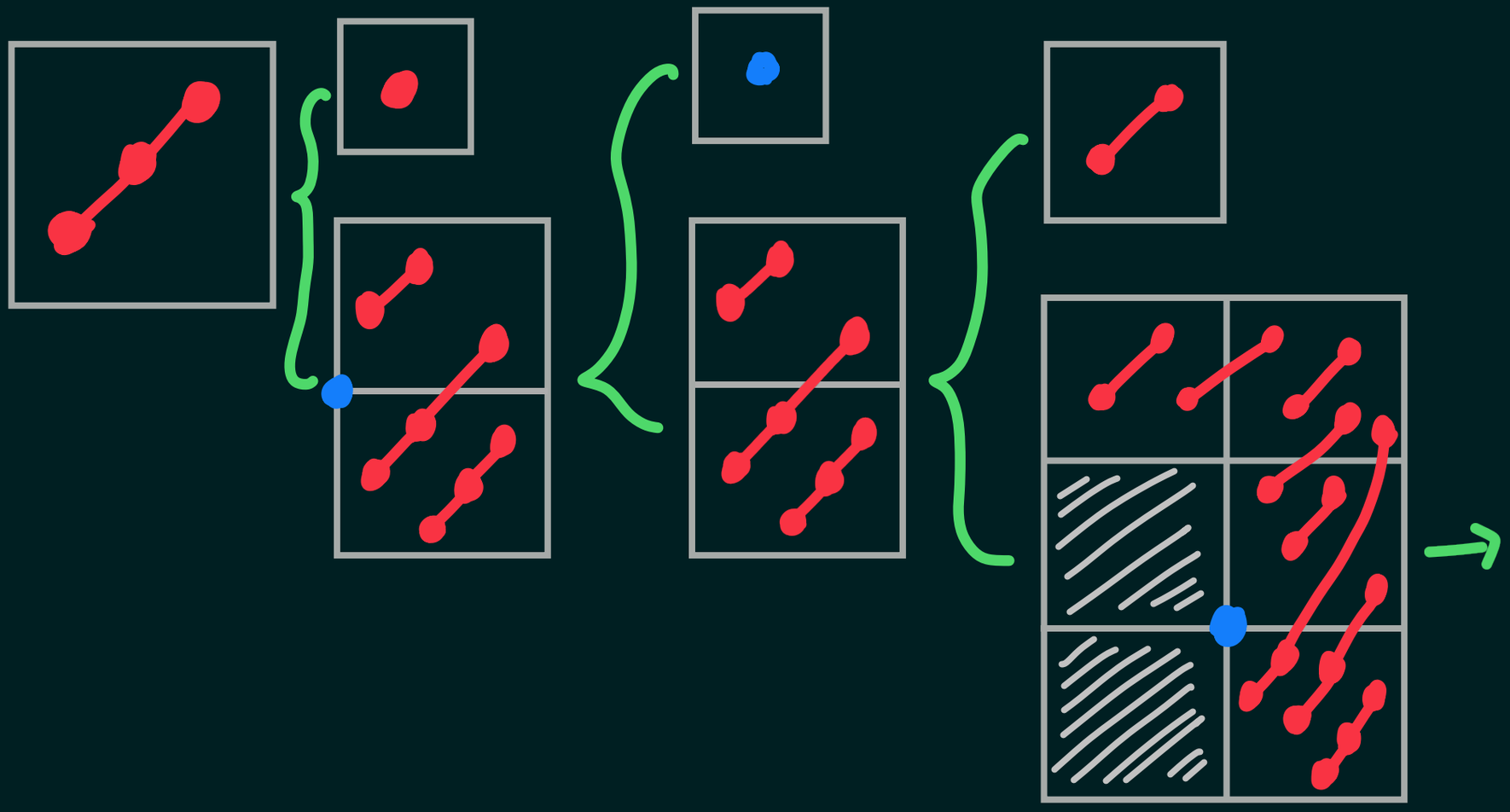


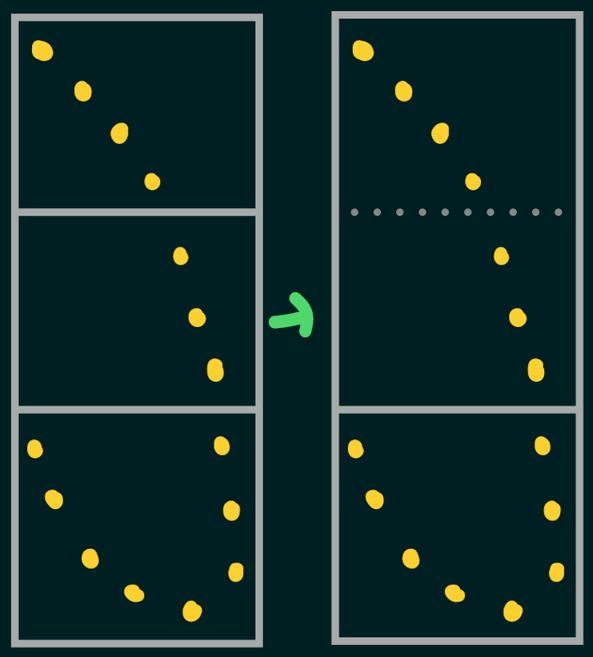


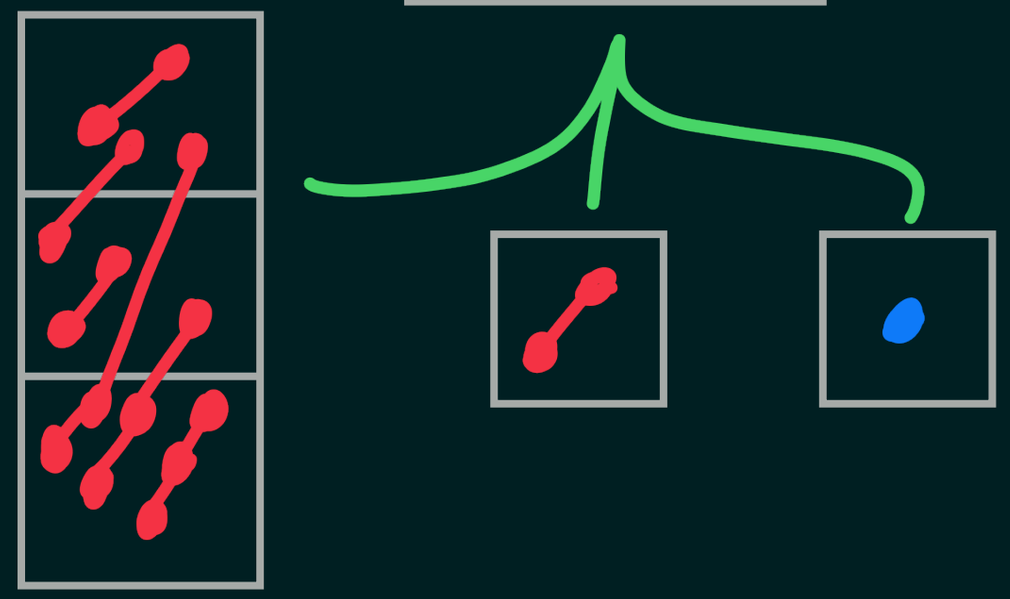


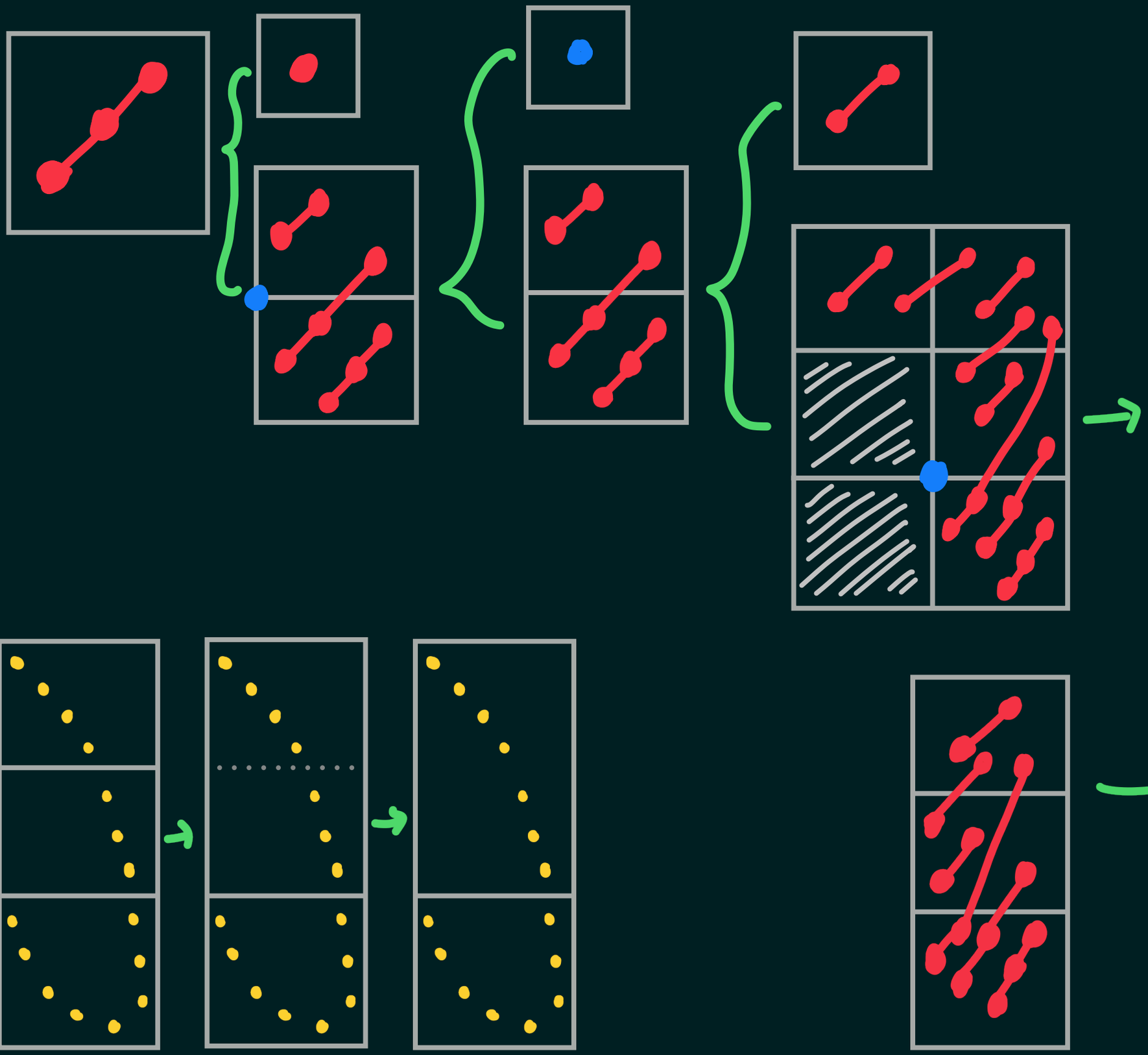


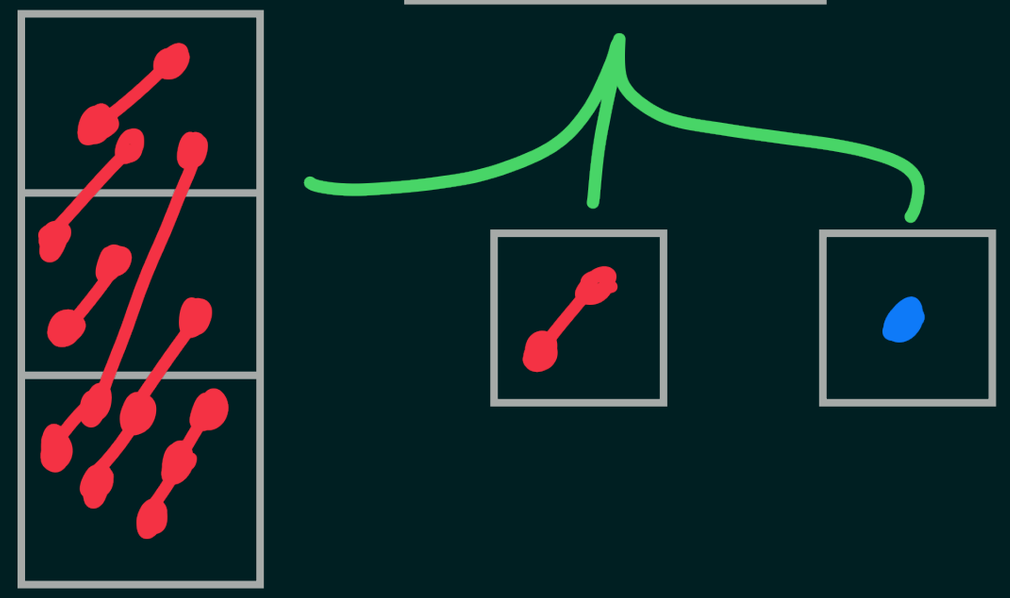


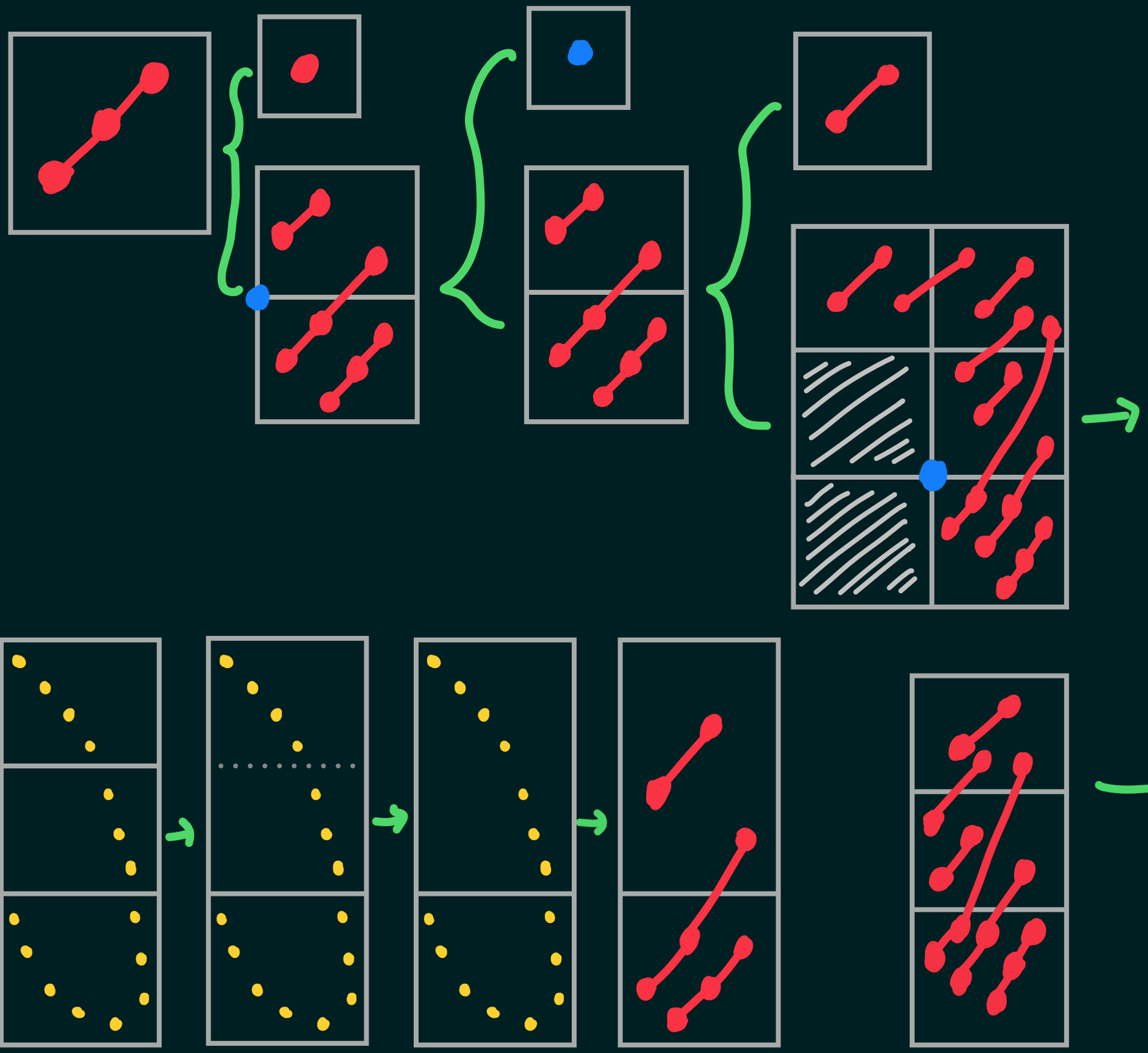


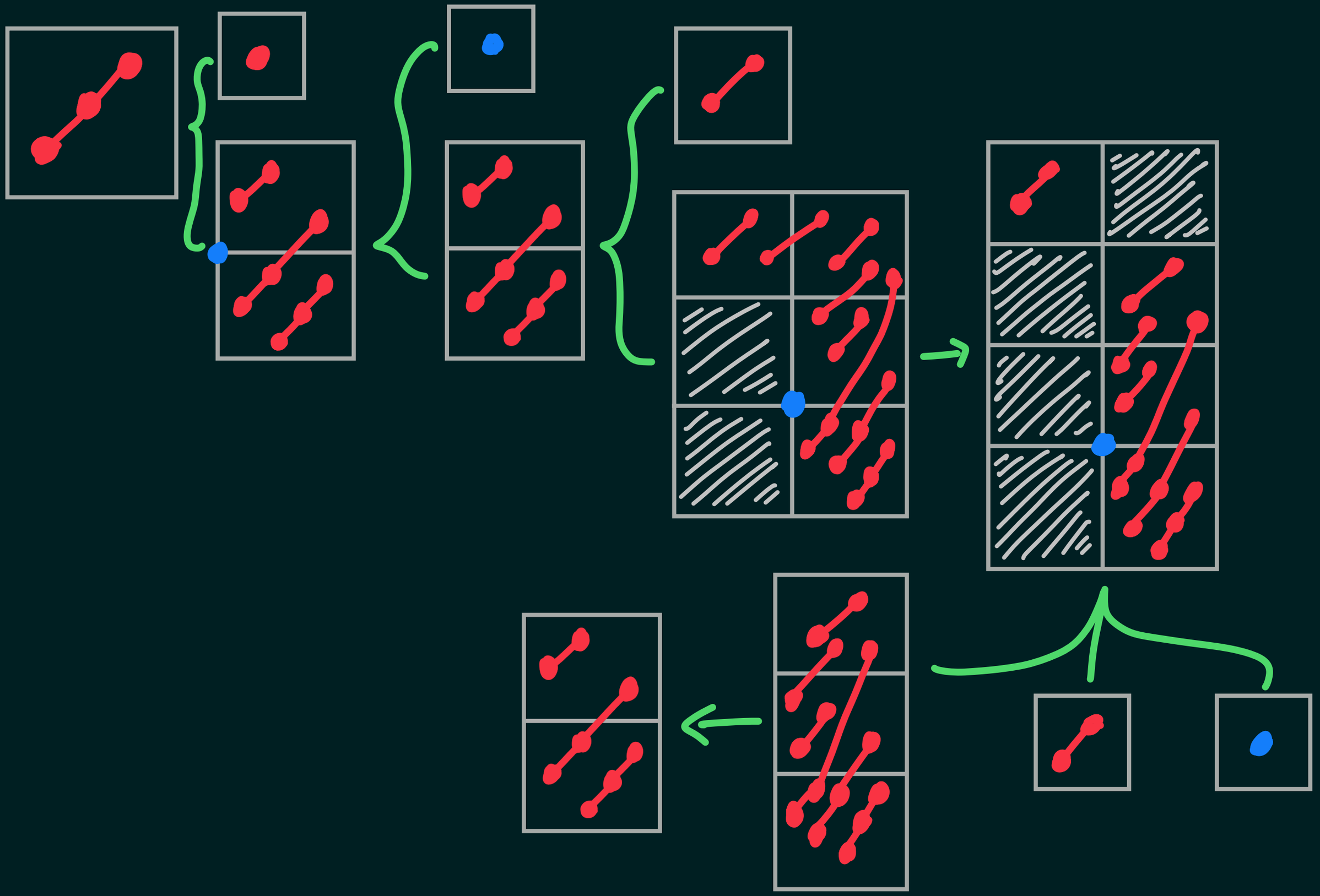


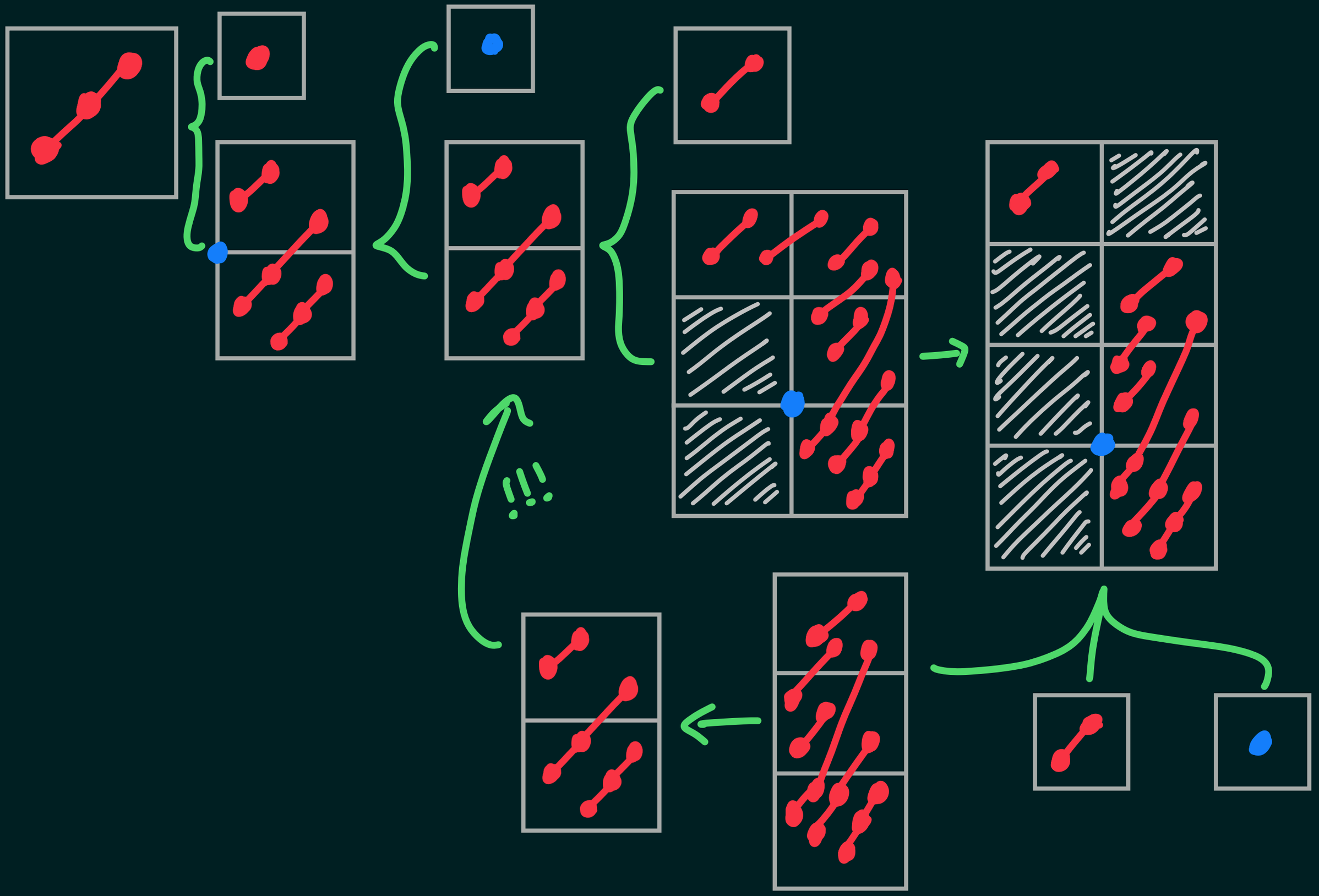


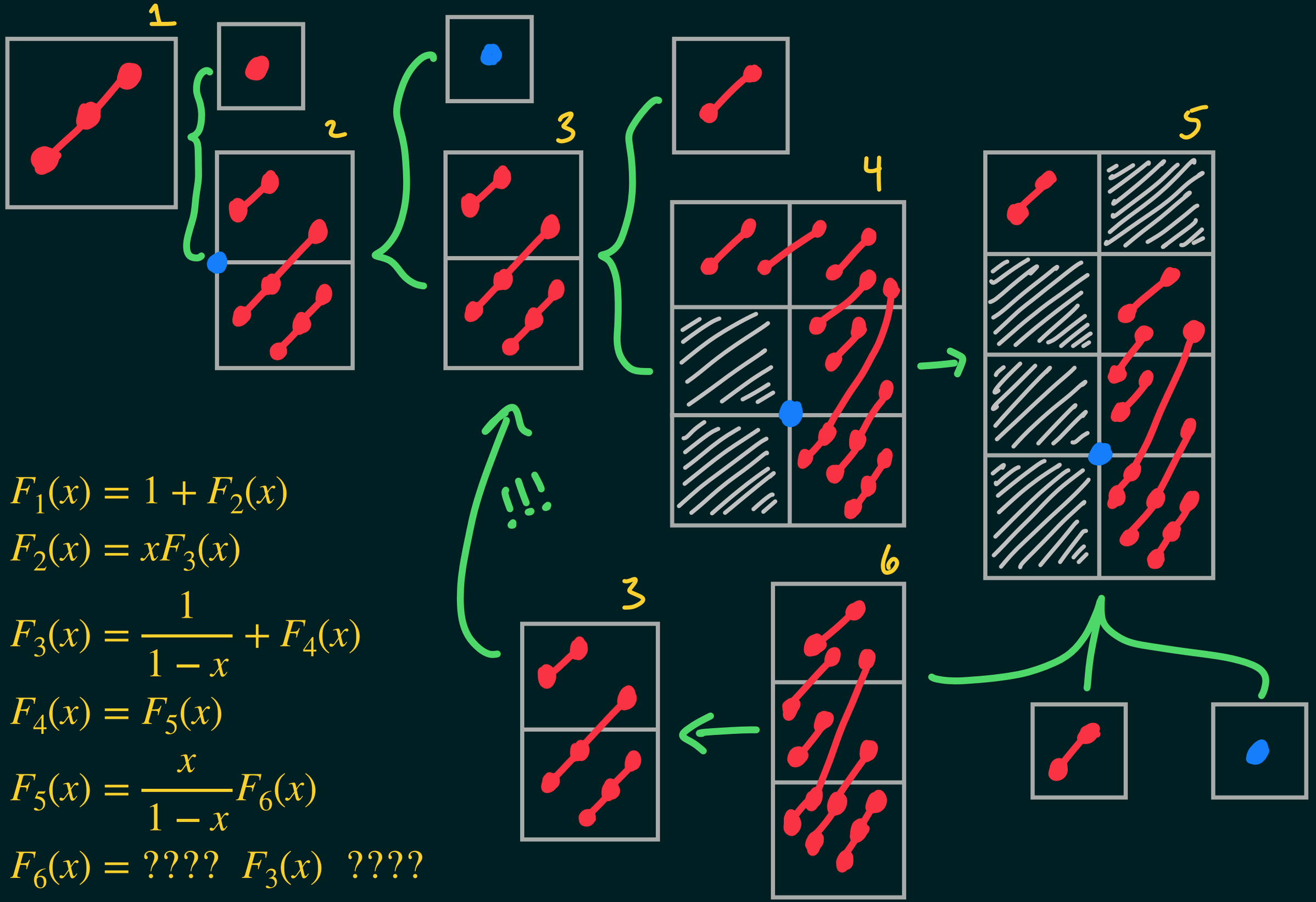












$$F_1(x) = 1 + F_2(x)$$

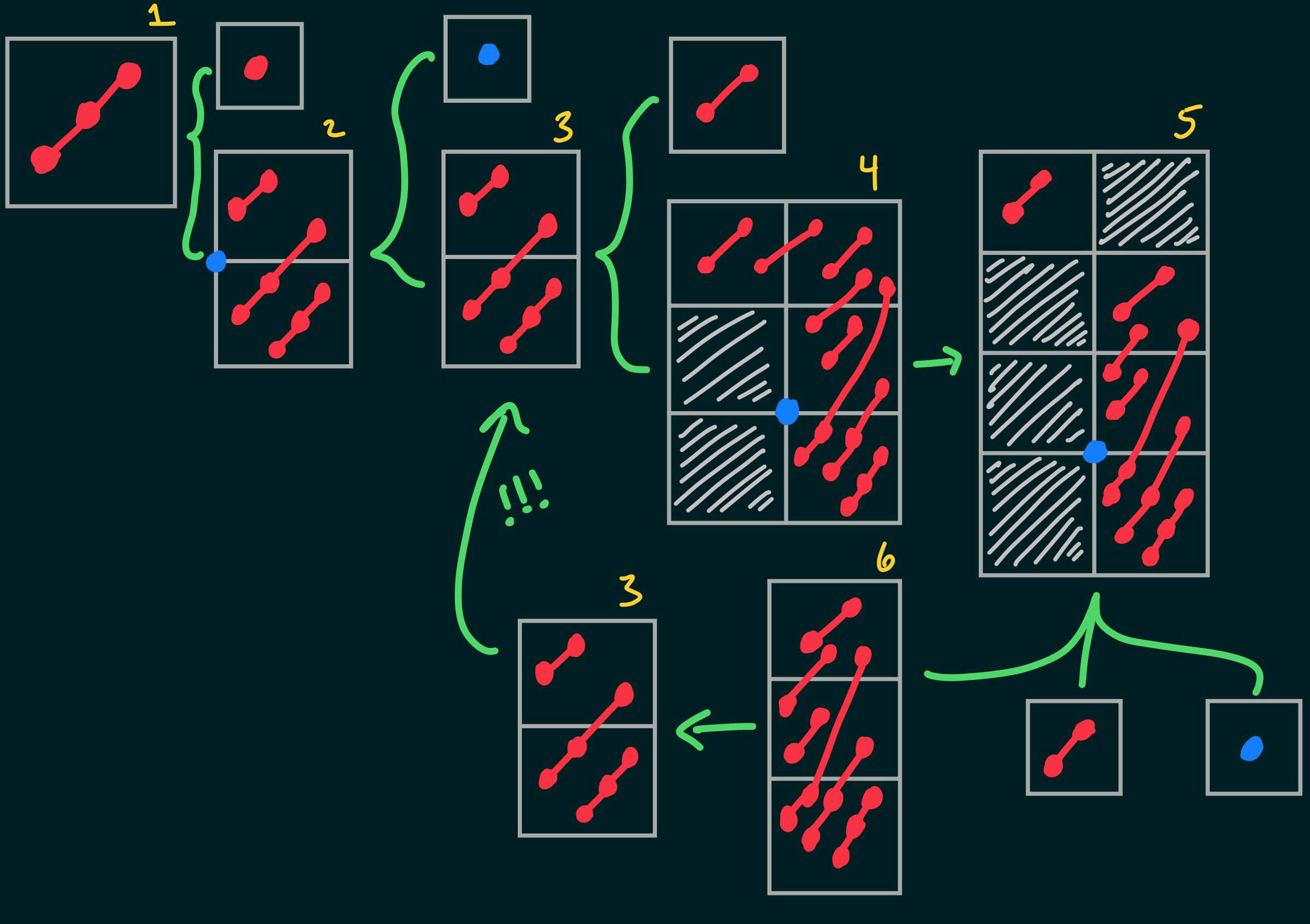
$$F_2(x) = xF_3(x)$$

$$F_3(x) = \frac{1}{1-x} + F_4(x)$$

$$F_4(x) = F_5(x)$$

$$F_5(x) = \frac{x}{1-x}F_6(x)$$

$$F_6(x) = ????\ \ F_3(x)\ \ ????$$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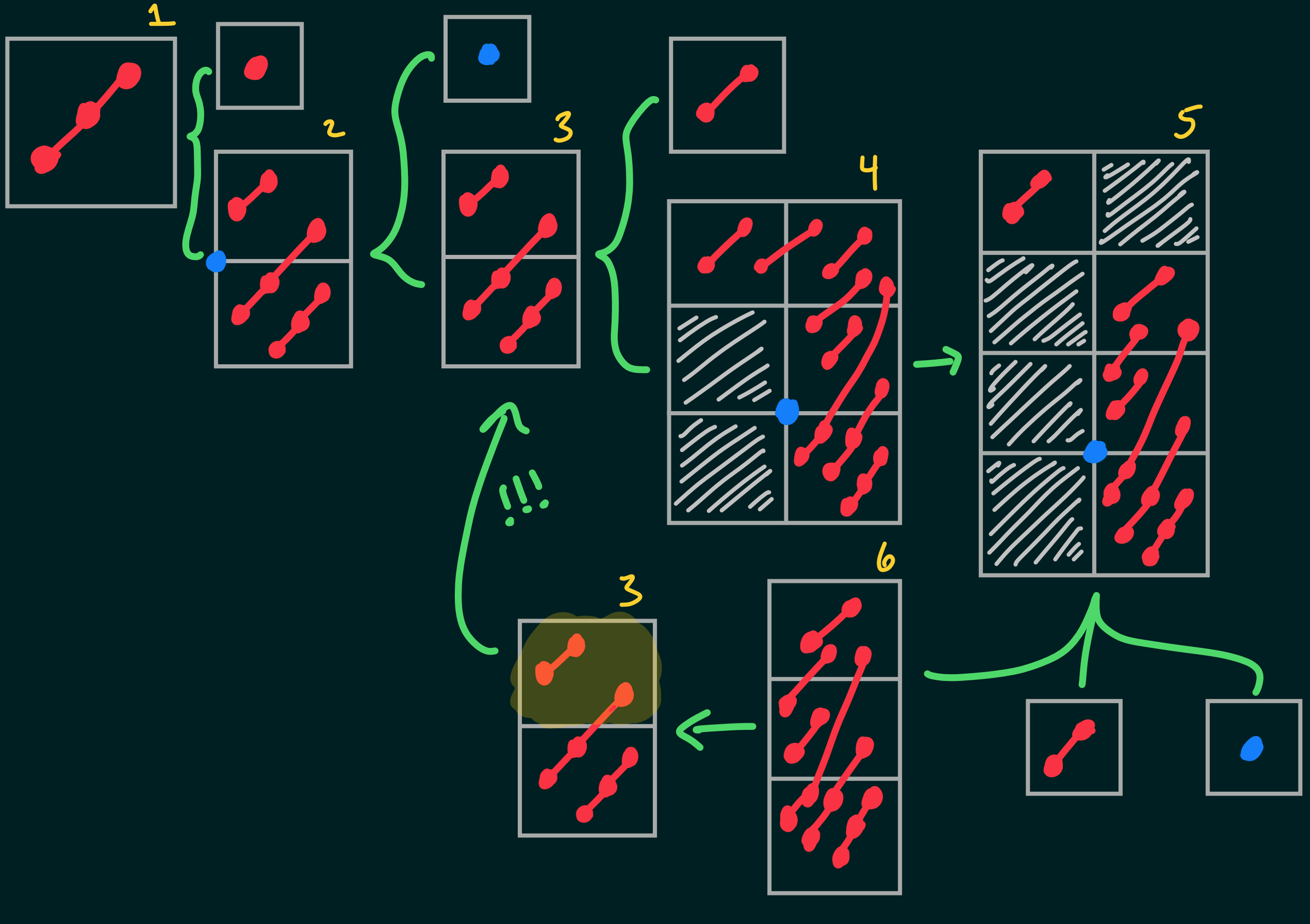
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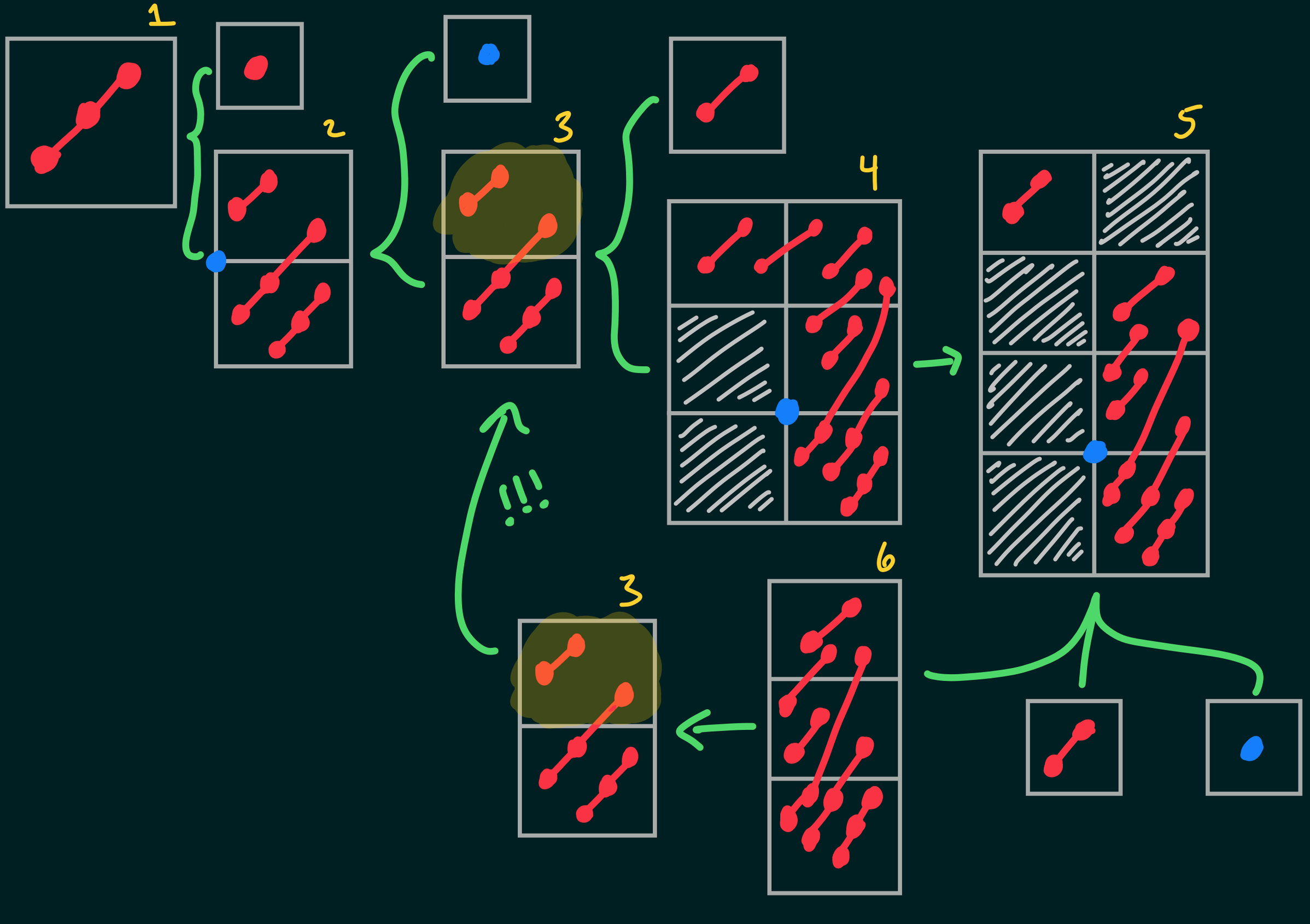
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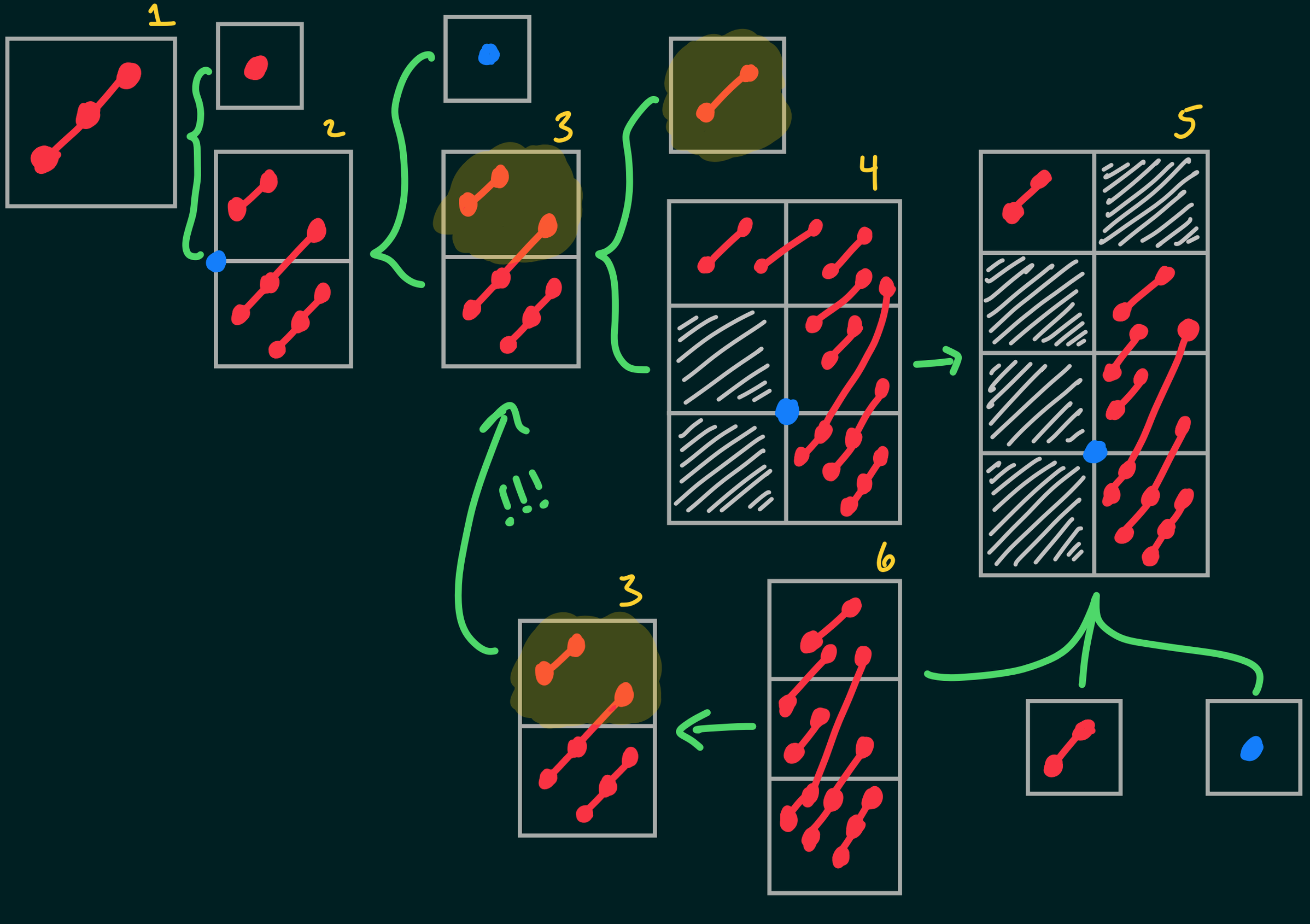
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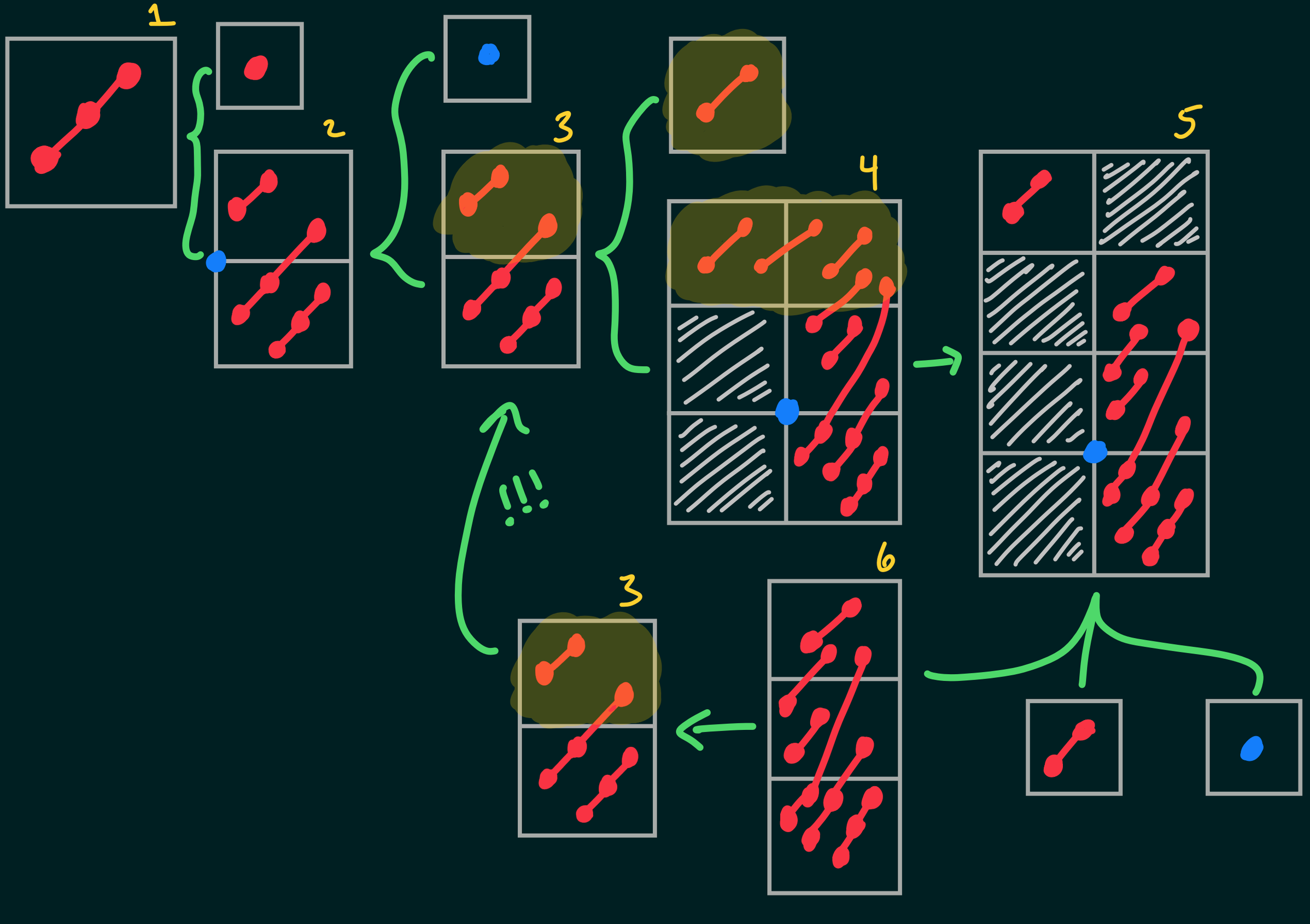
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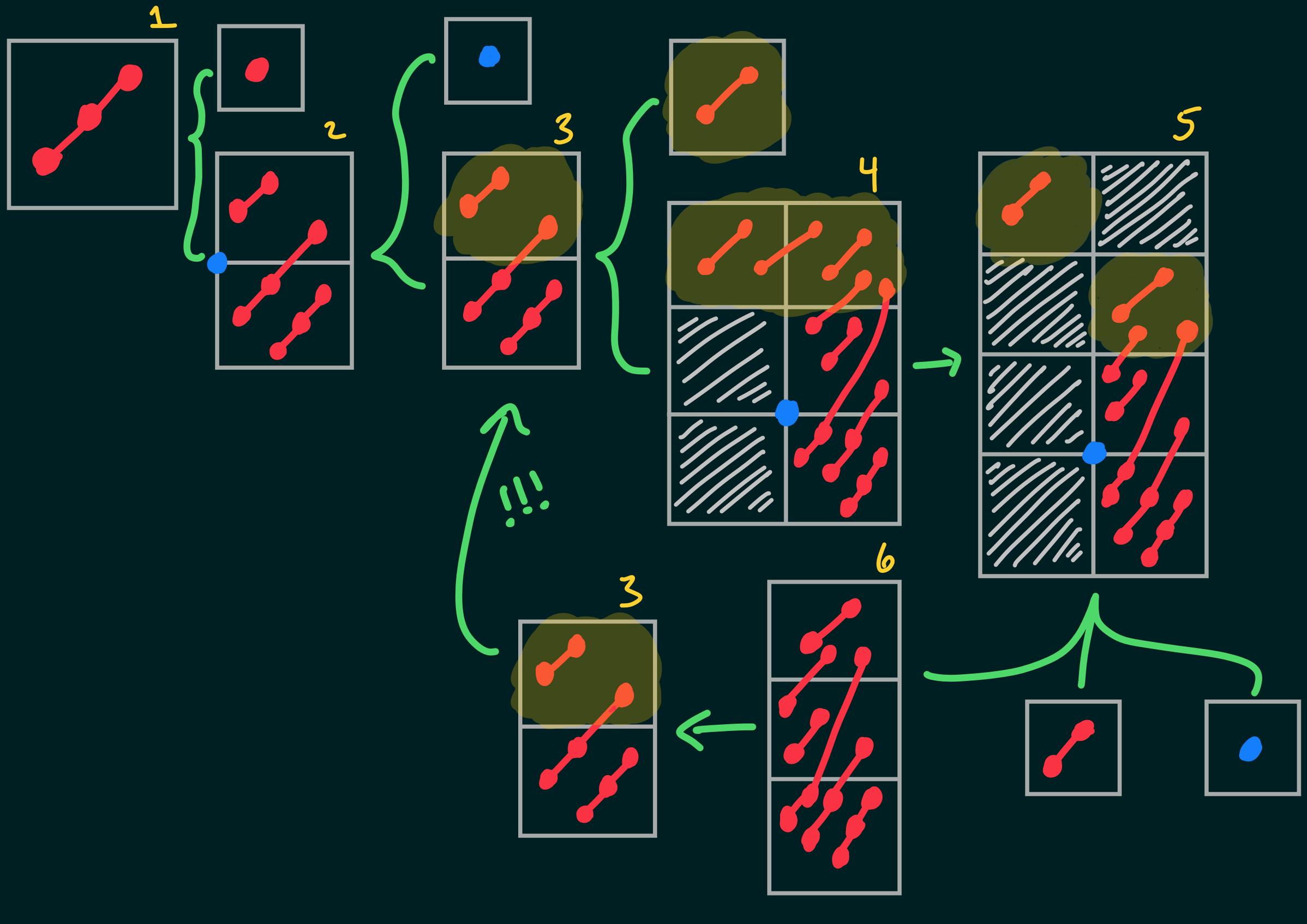
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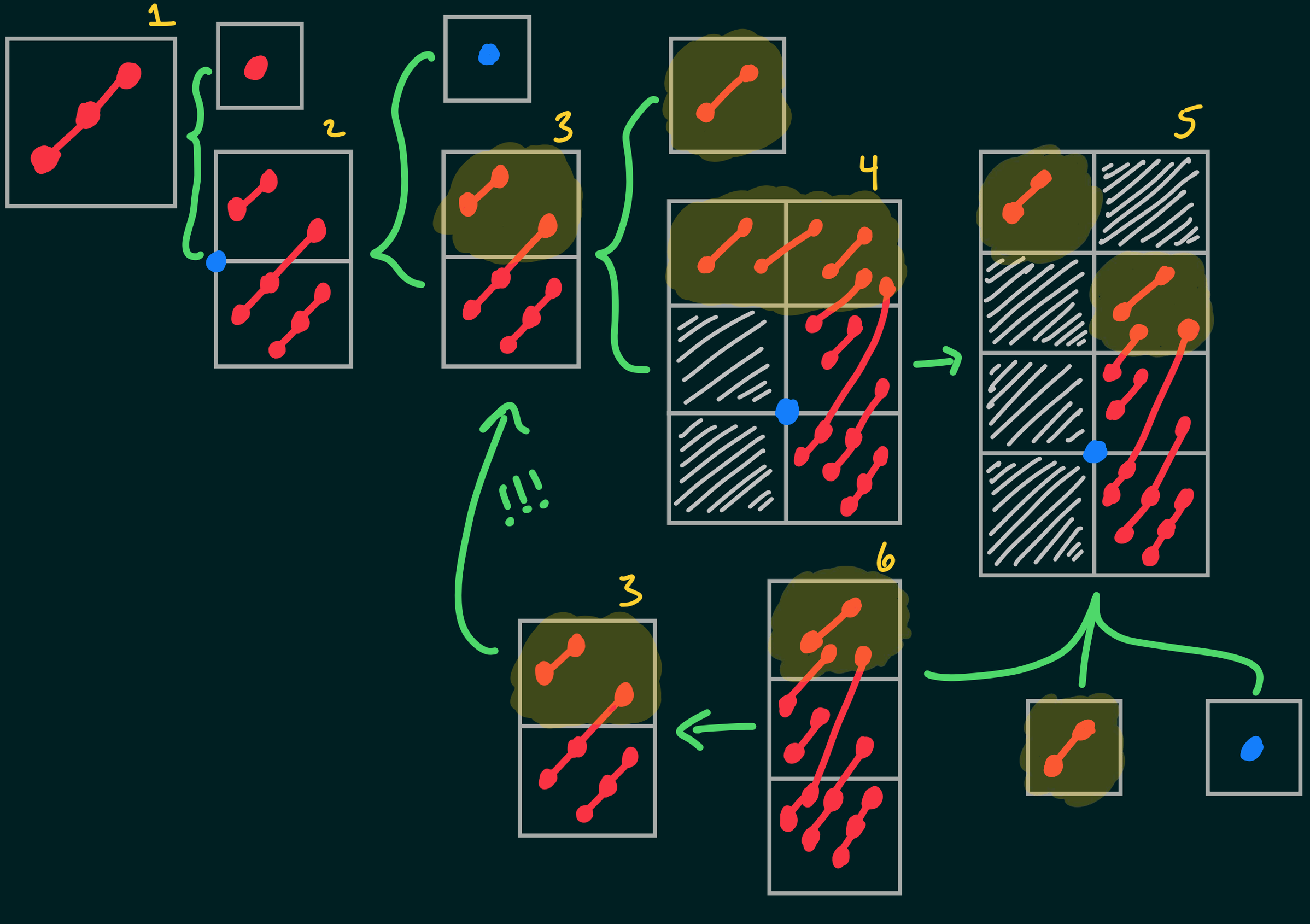
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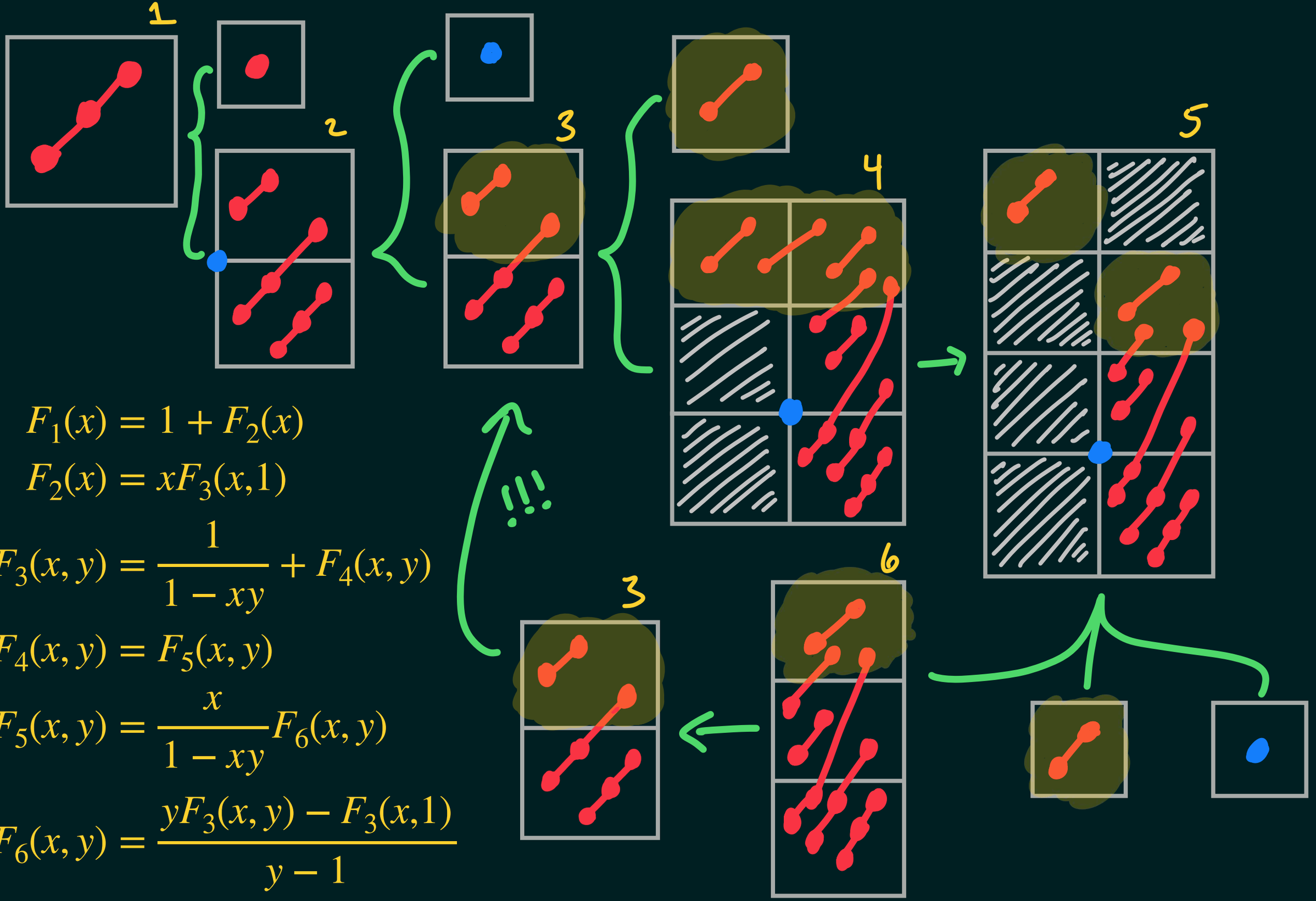


$$F_1(x) = 1 + F_2(x)$$

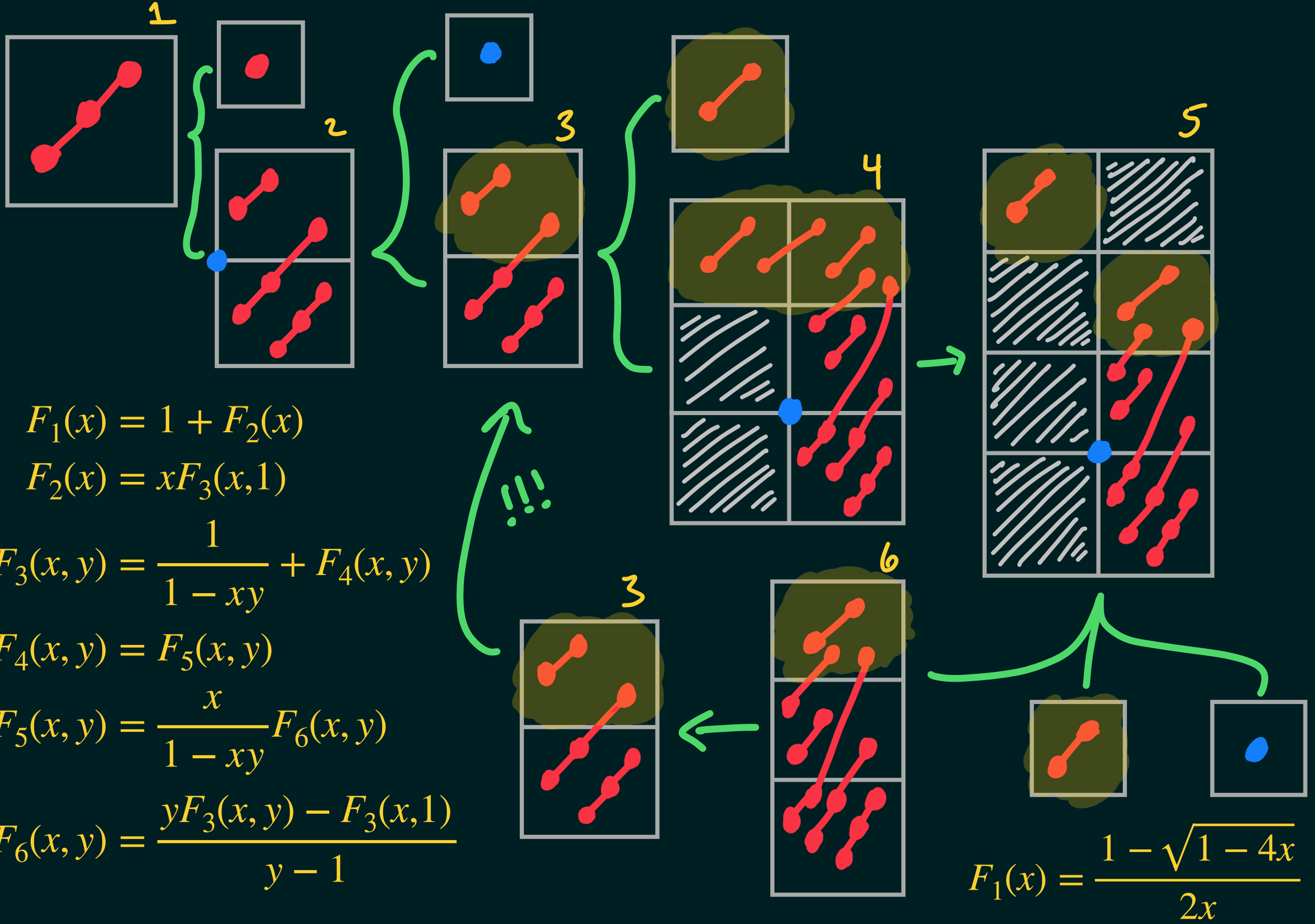
$$F_2(x) = xF_3(x,1)$$

$$F_3(x,y) = \frac{1}{1-xy} + F_4(x,y)$$

$$F_4(x,y) = F_5(x,y)$$

$$F_5(x,y) = \frac{x}{1-xy}F_6(x,y)$$

$$F_6(x,y) = \frac{yF_3(x,y) - F_3(x,1)}{y-1}$$



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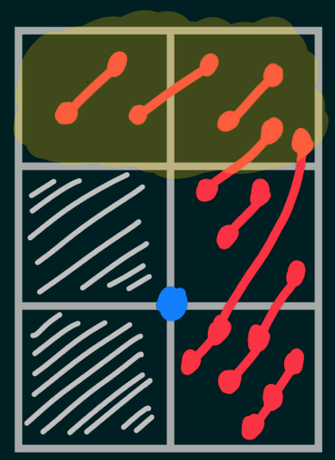
$$F_6(x,y) = \frac{yF_3(x,y) - F_3(x,1)}{y-1}$$

$$F_1(x) = \frac{1-\sqrt{1-4x}}{2x}$$

So we have:

So we have:

Tilings = sets of permutations



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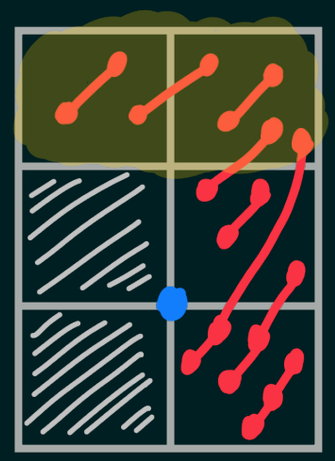
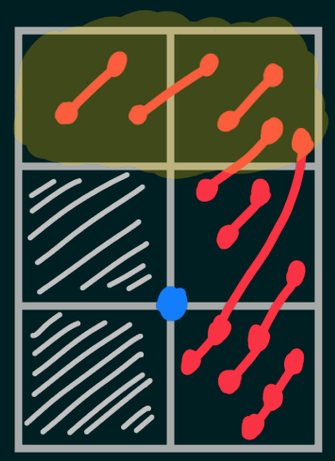


Strategies = rigorous deductions that some tilings can be decomposed in terms of others

So we have:

Tilings = sets of permutations



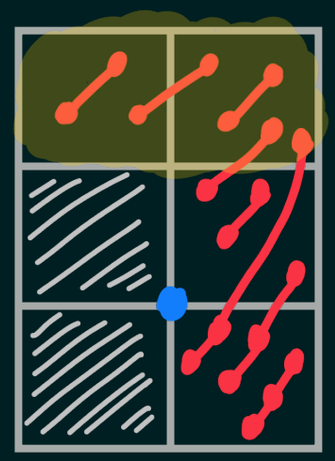
Strategies = rigorous deductions that some tilings can be decomposed in terms of others

Proof Trees = a bunch of tilings, related by strategies that are sufficient to derive enumerations, generating functions, etc

(= combinatorial specifications)

So we have:

Tilings = sets of permutations



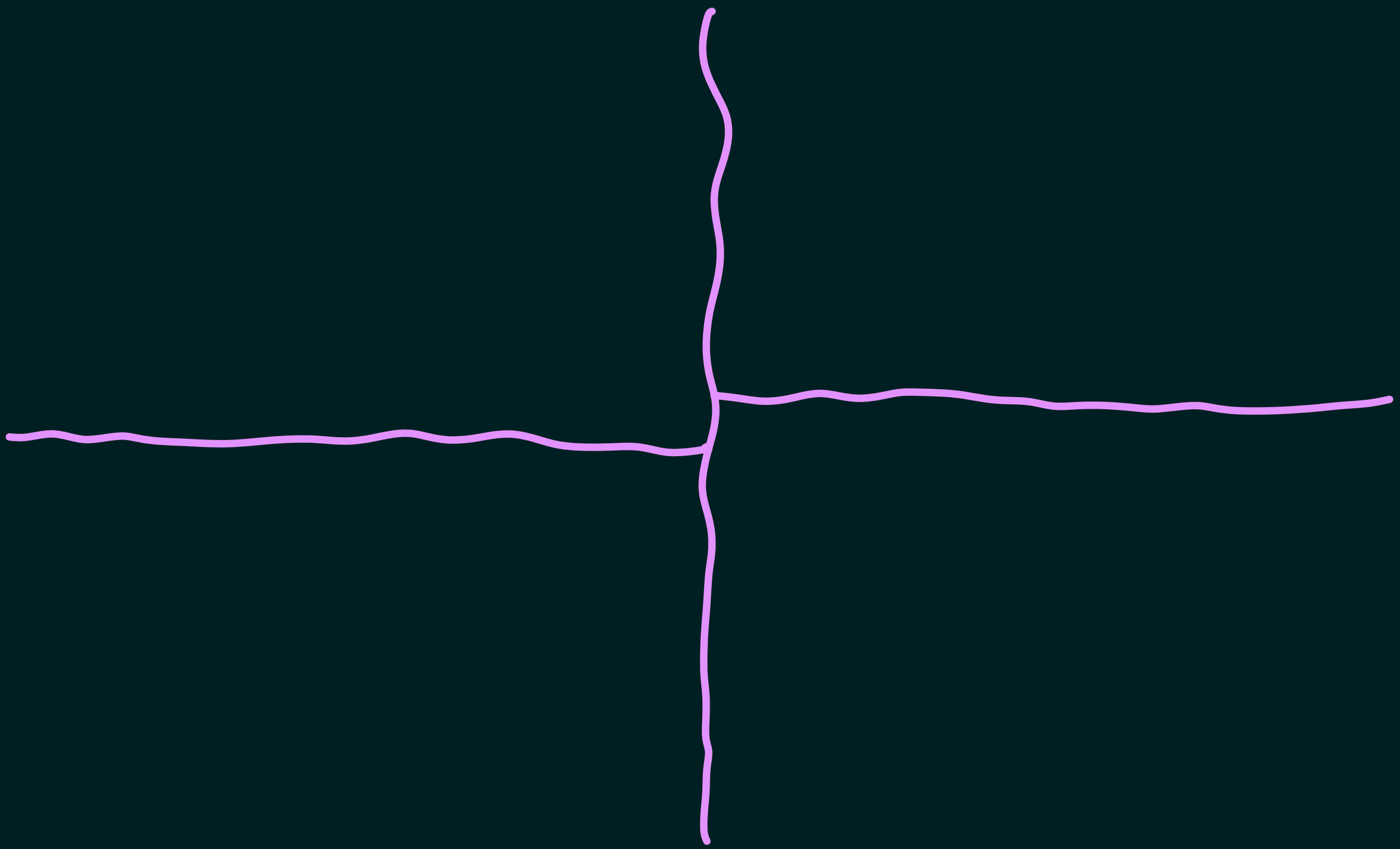
Strategies = rigorous deductions that some tilings can be decomposed in terms of others

Proof Trees = a bunch of tilings, related by strategies that are sufficient to derive enumerations, generating functions, etc

(= combinatorial specifications)

Combinatorial Exploration = repeatedly apply strategies to tilings until you find a proof tree

Point Placement



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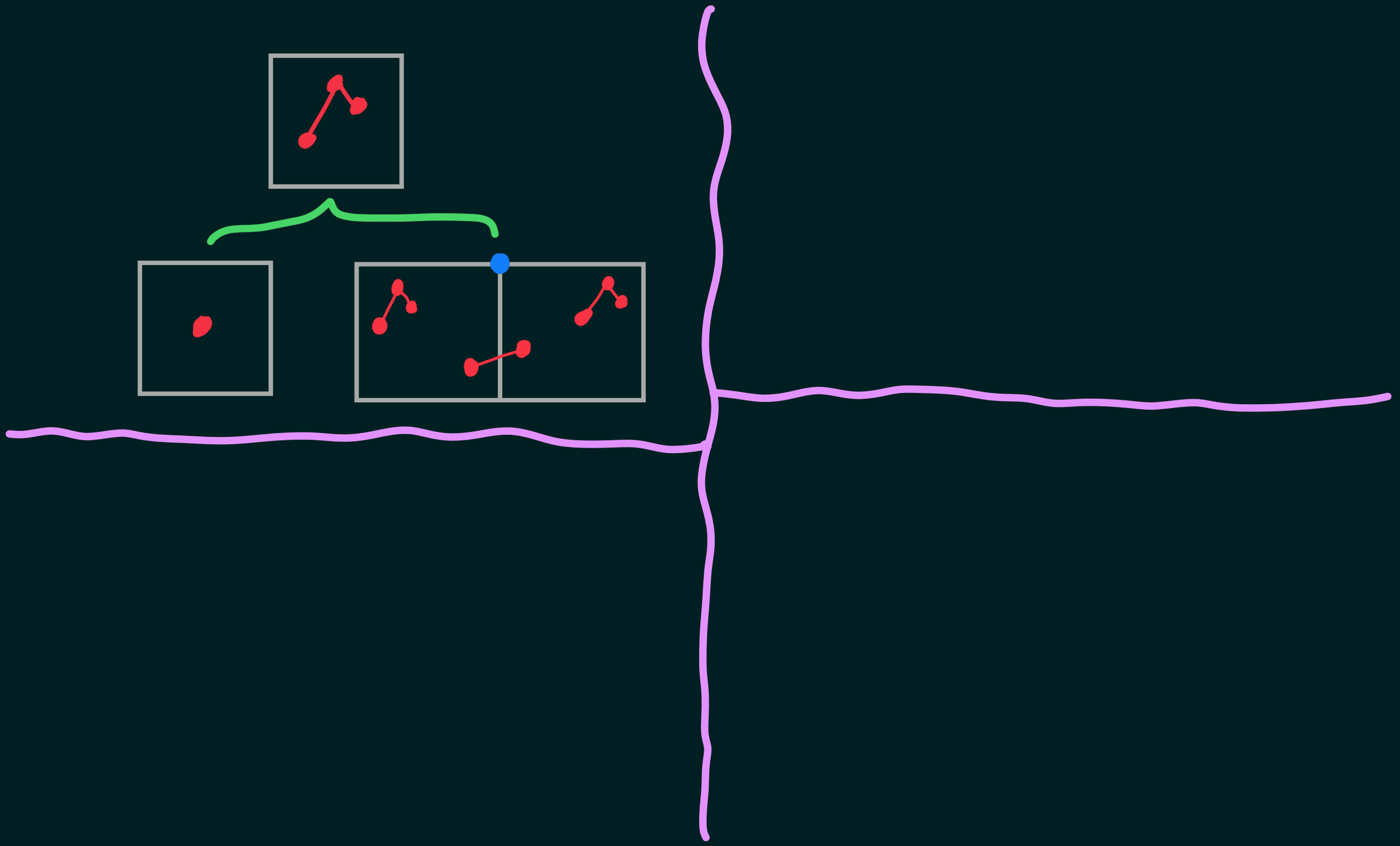

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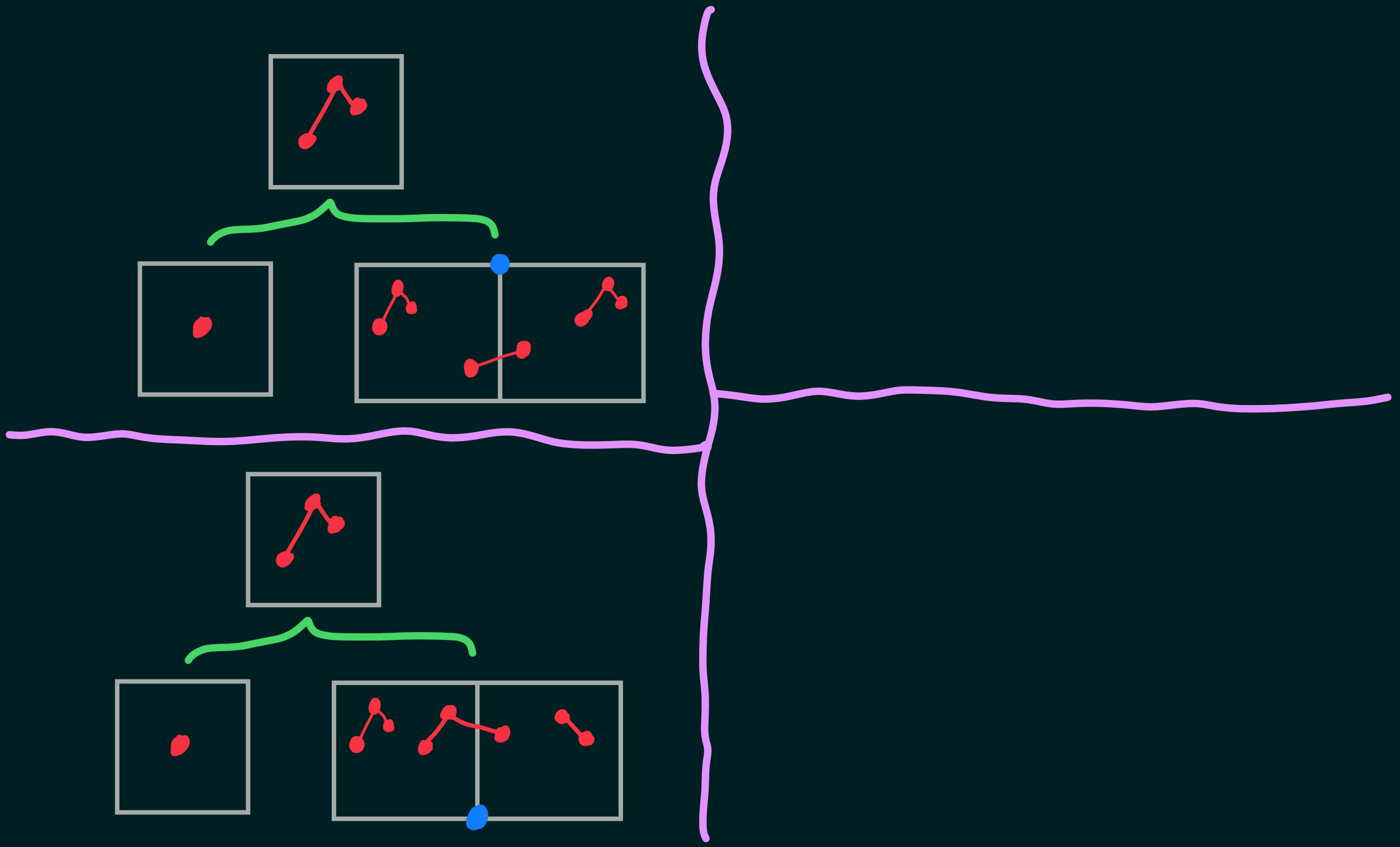

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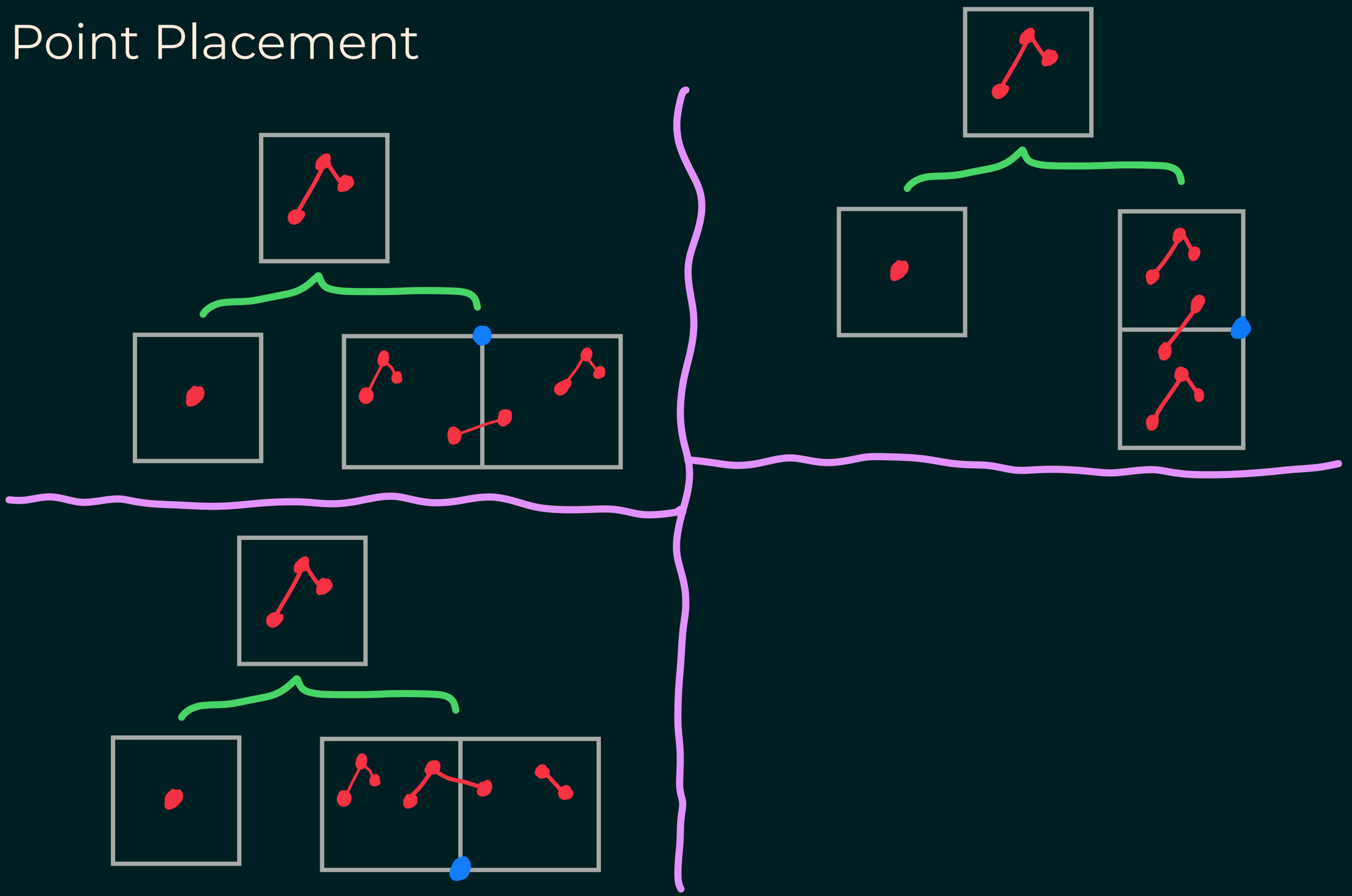

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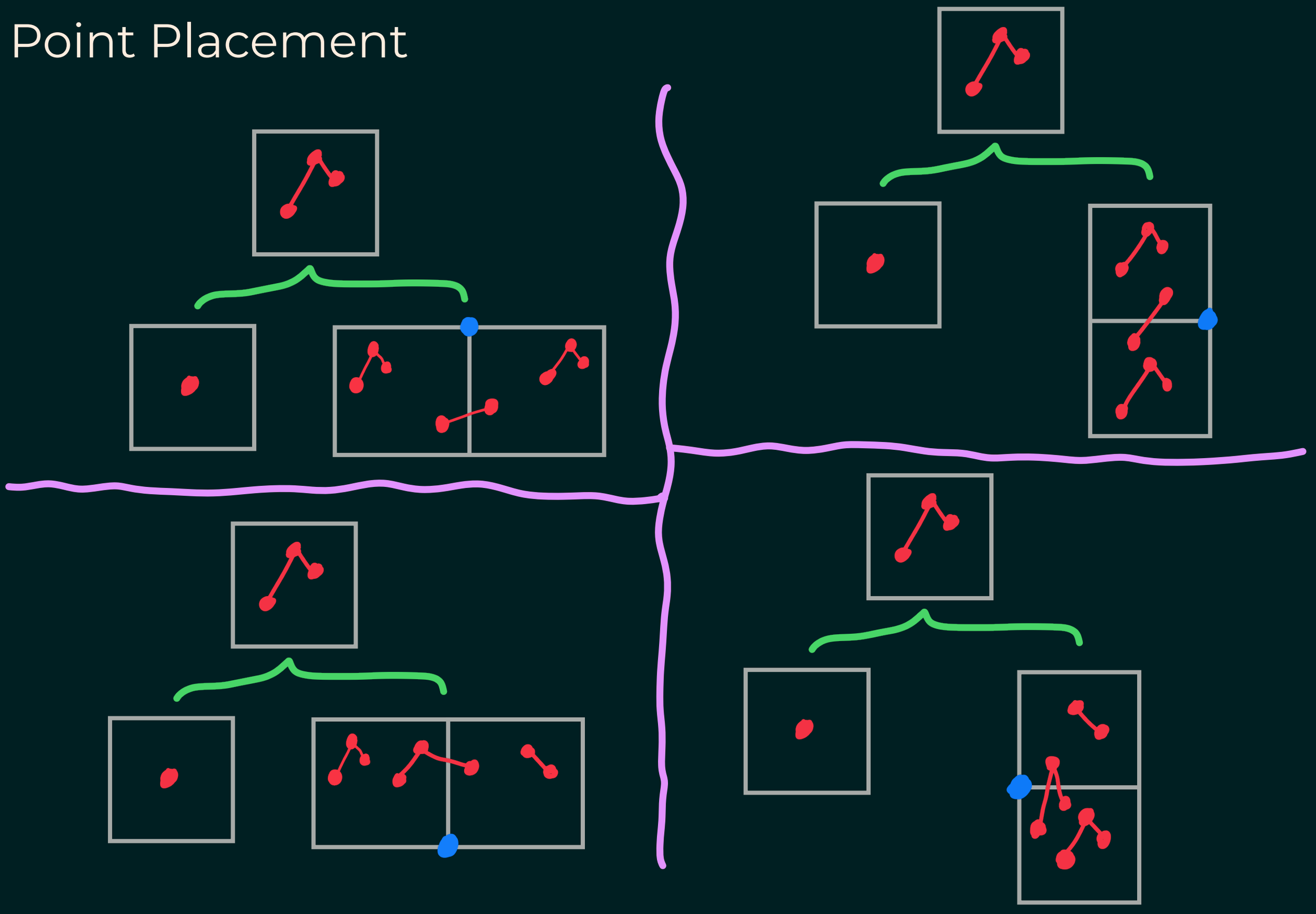
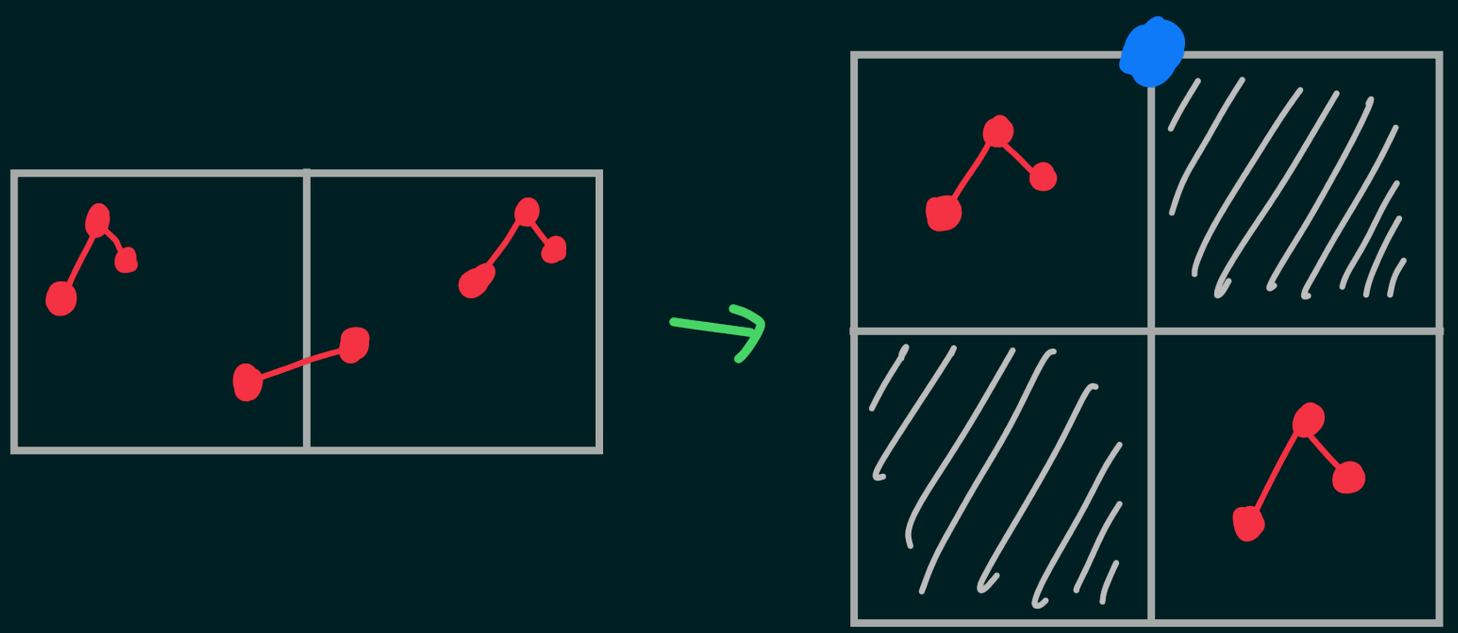
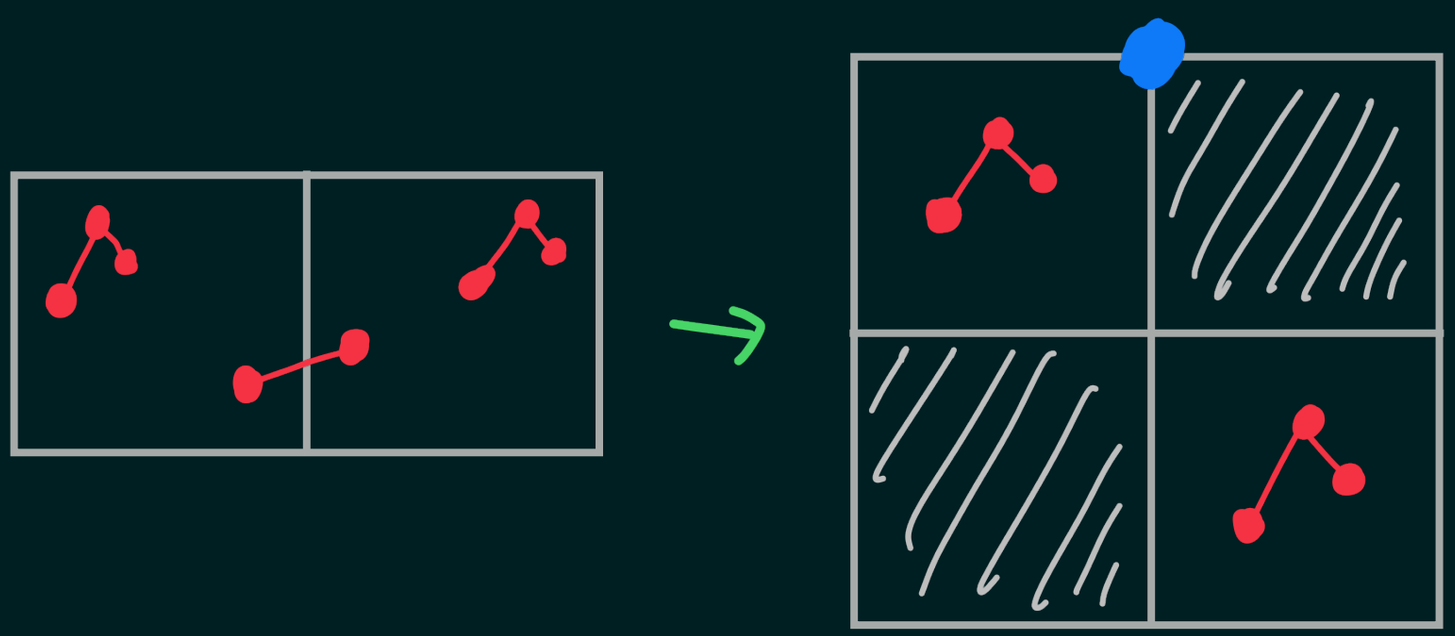
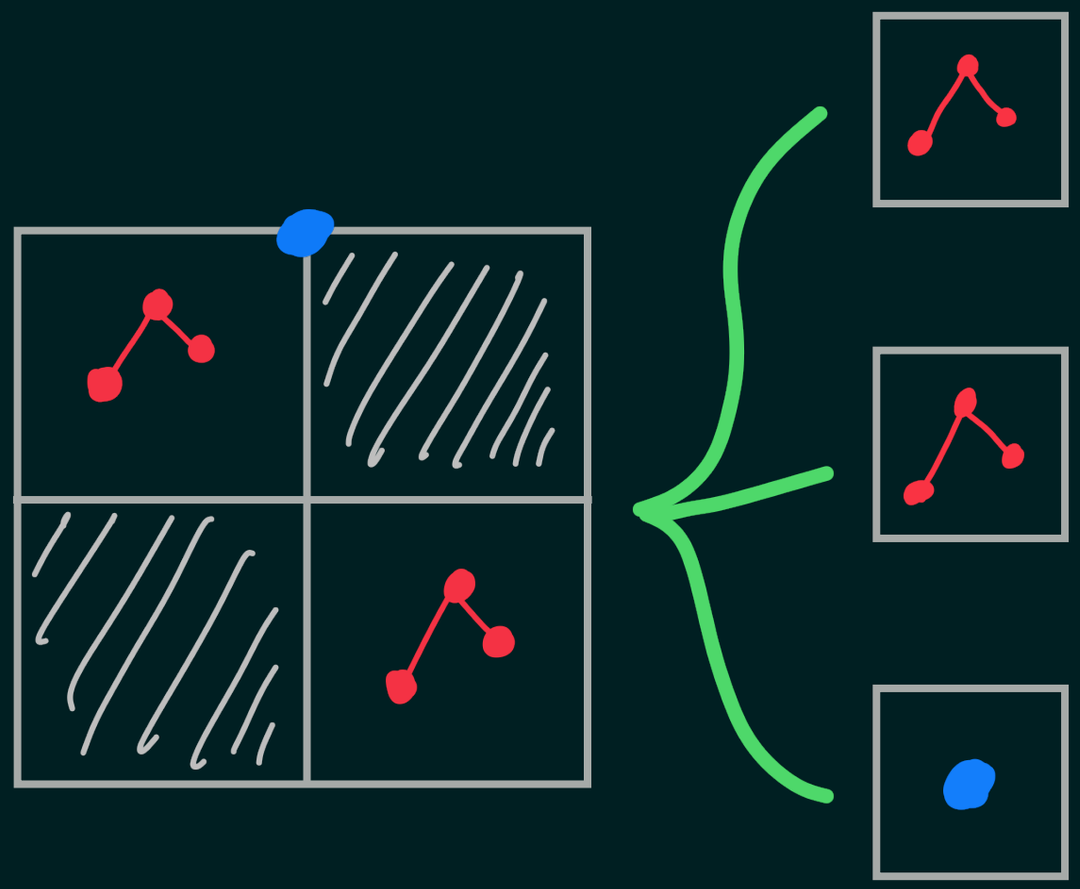


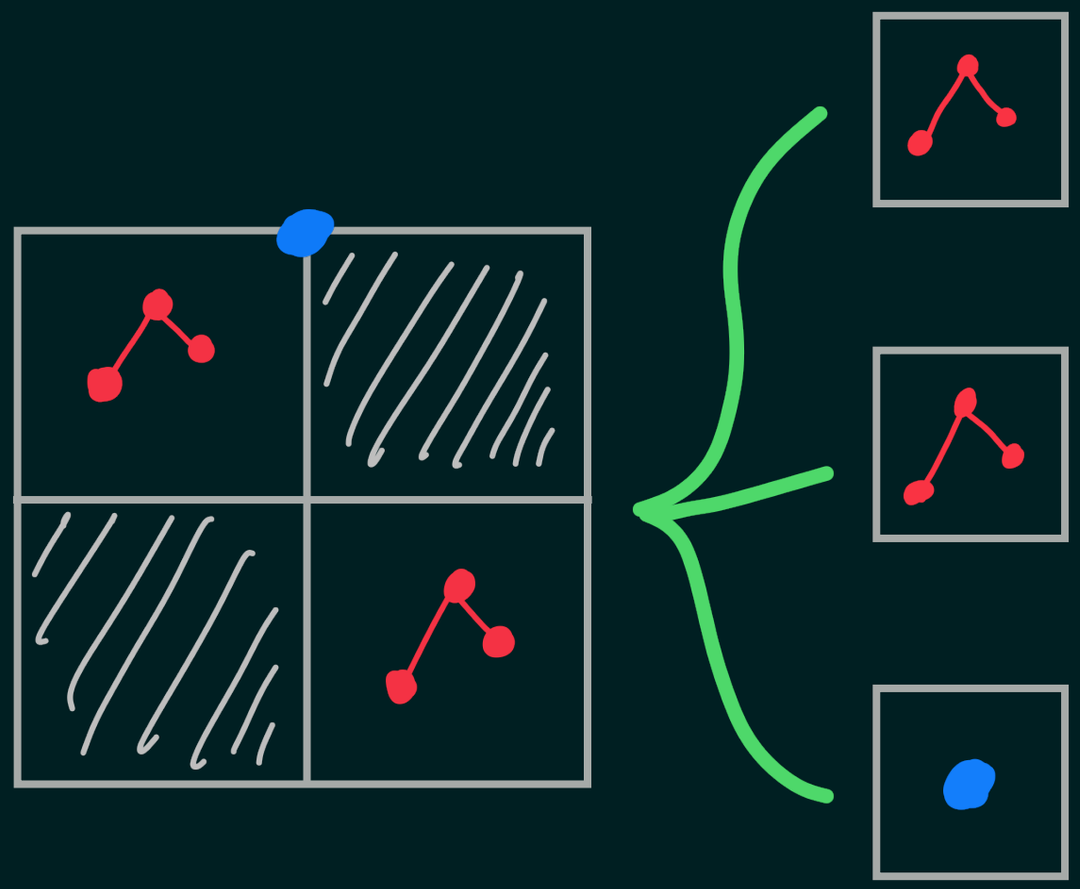


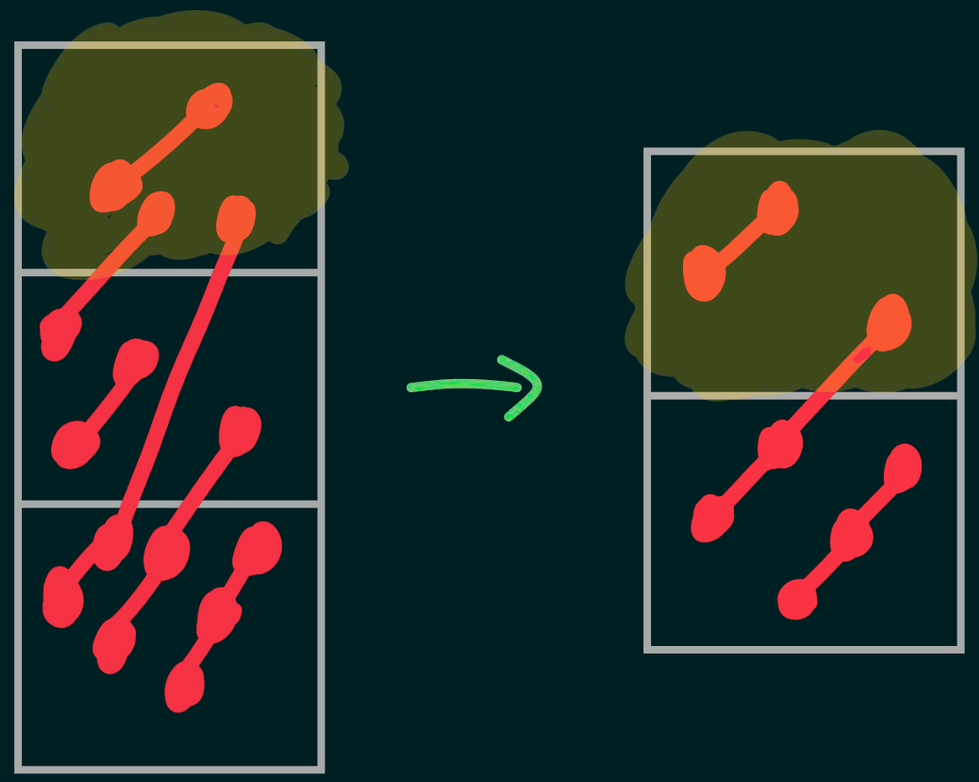
Row or Column Separation



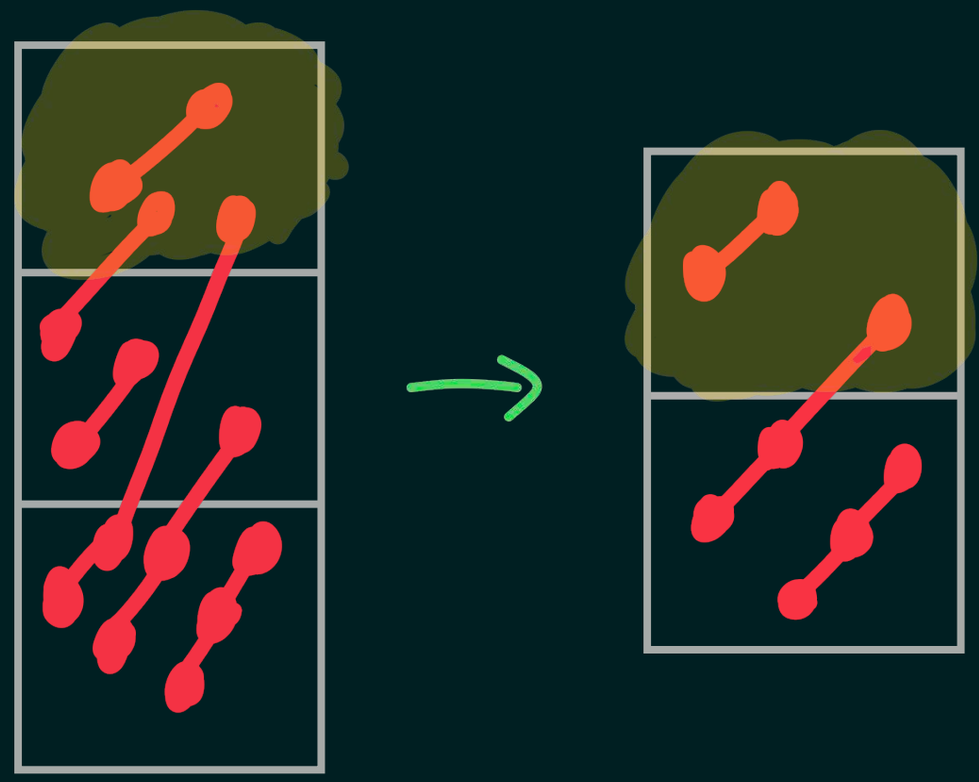


Factoring





Fusion

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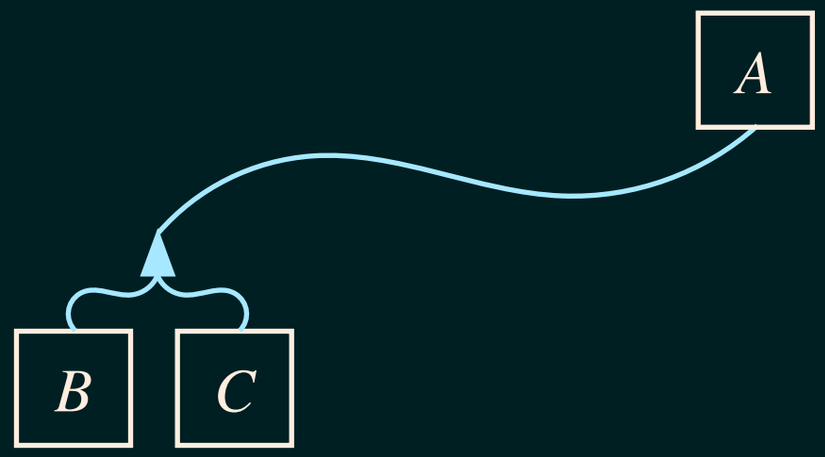
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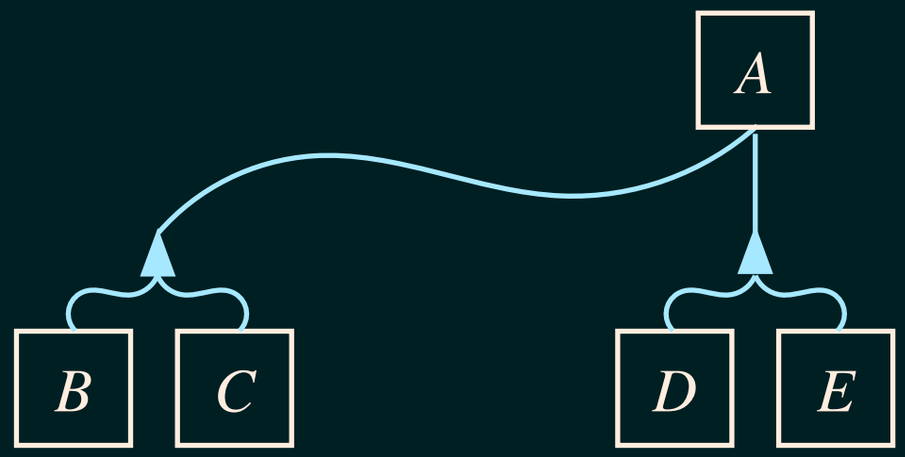
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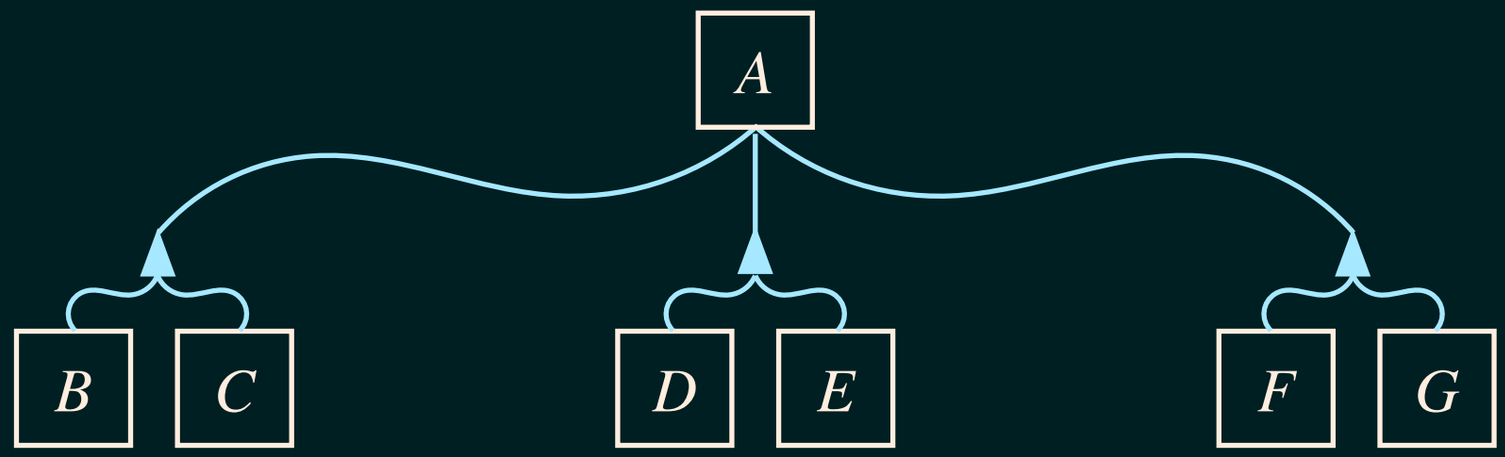
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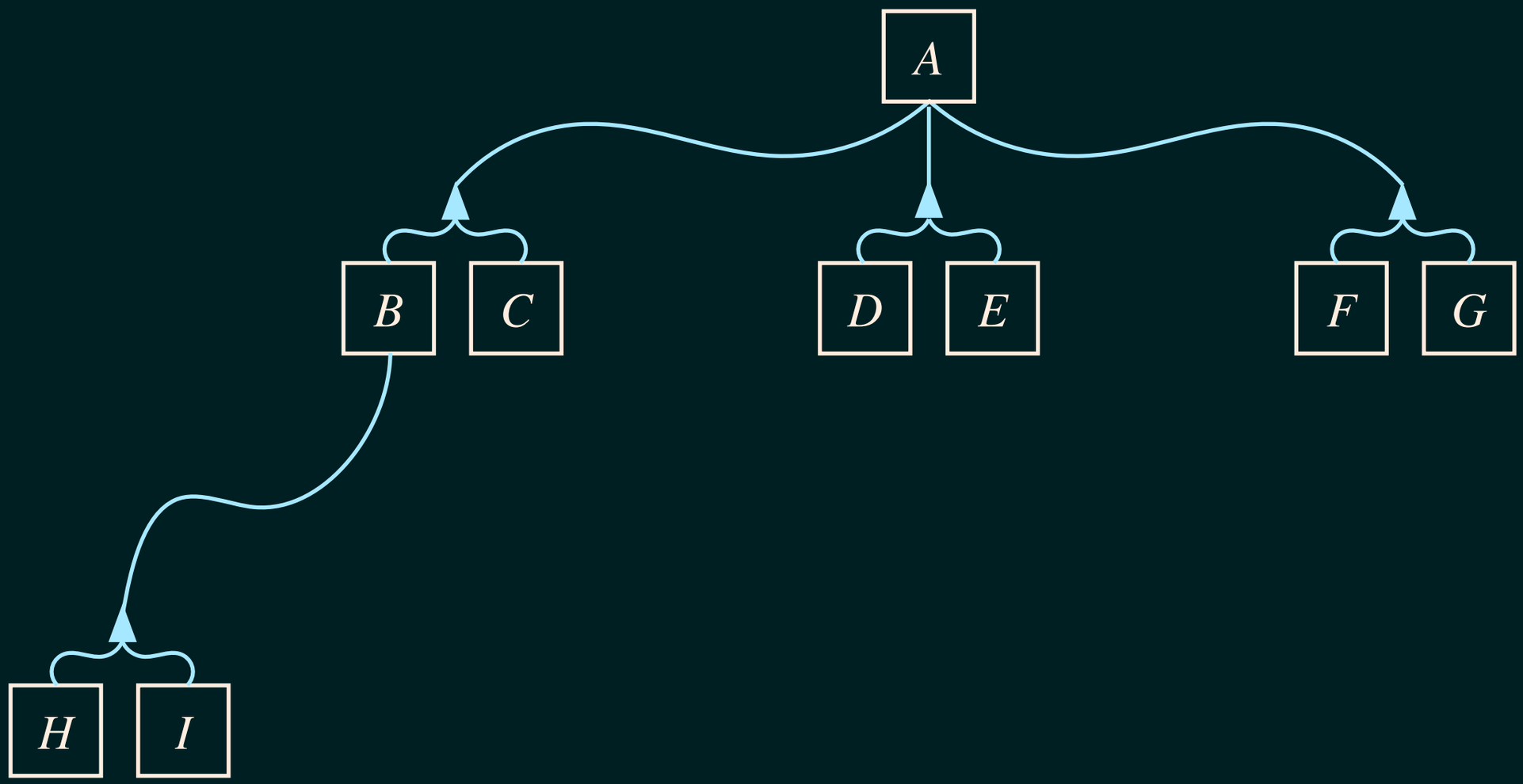
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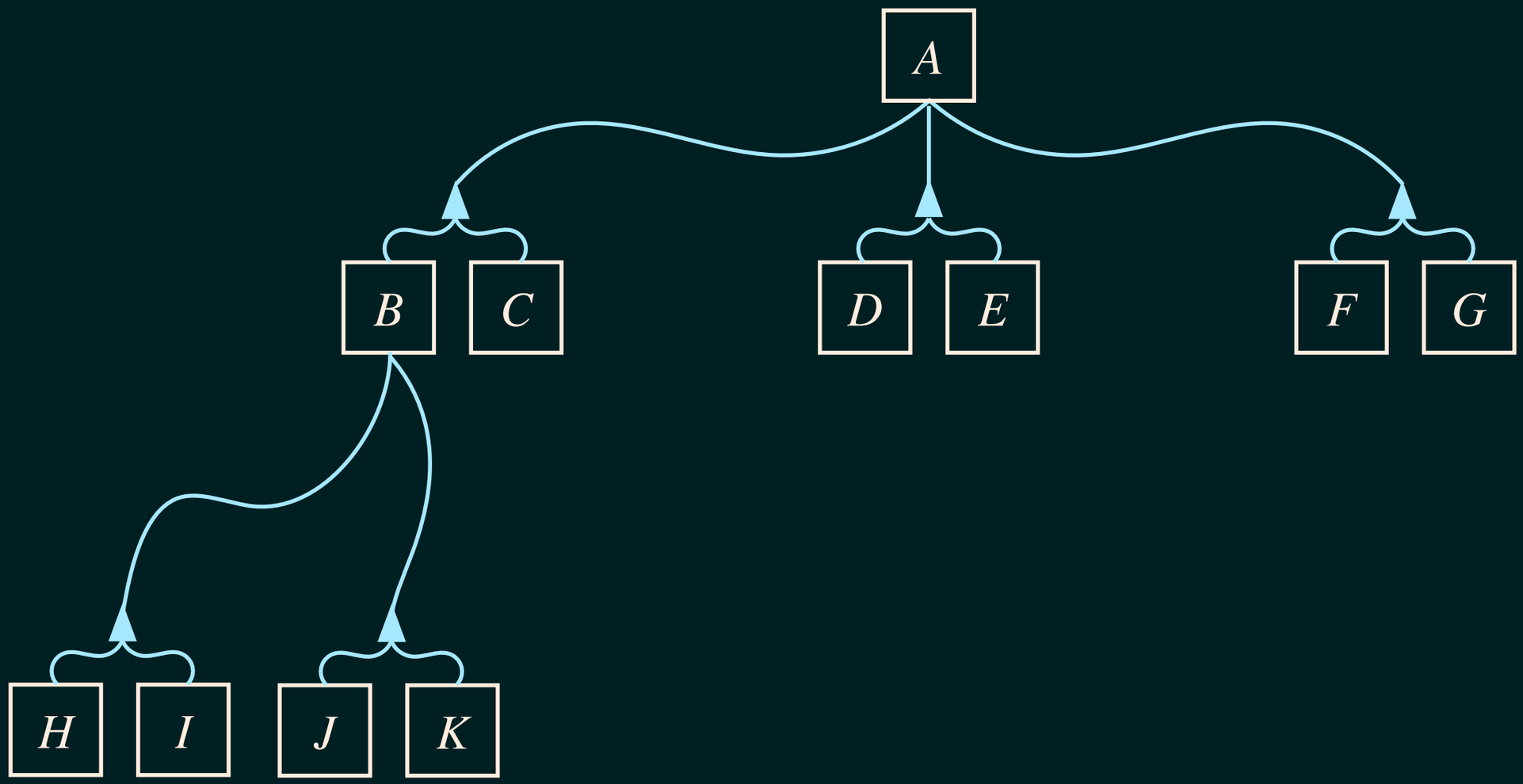
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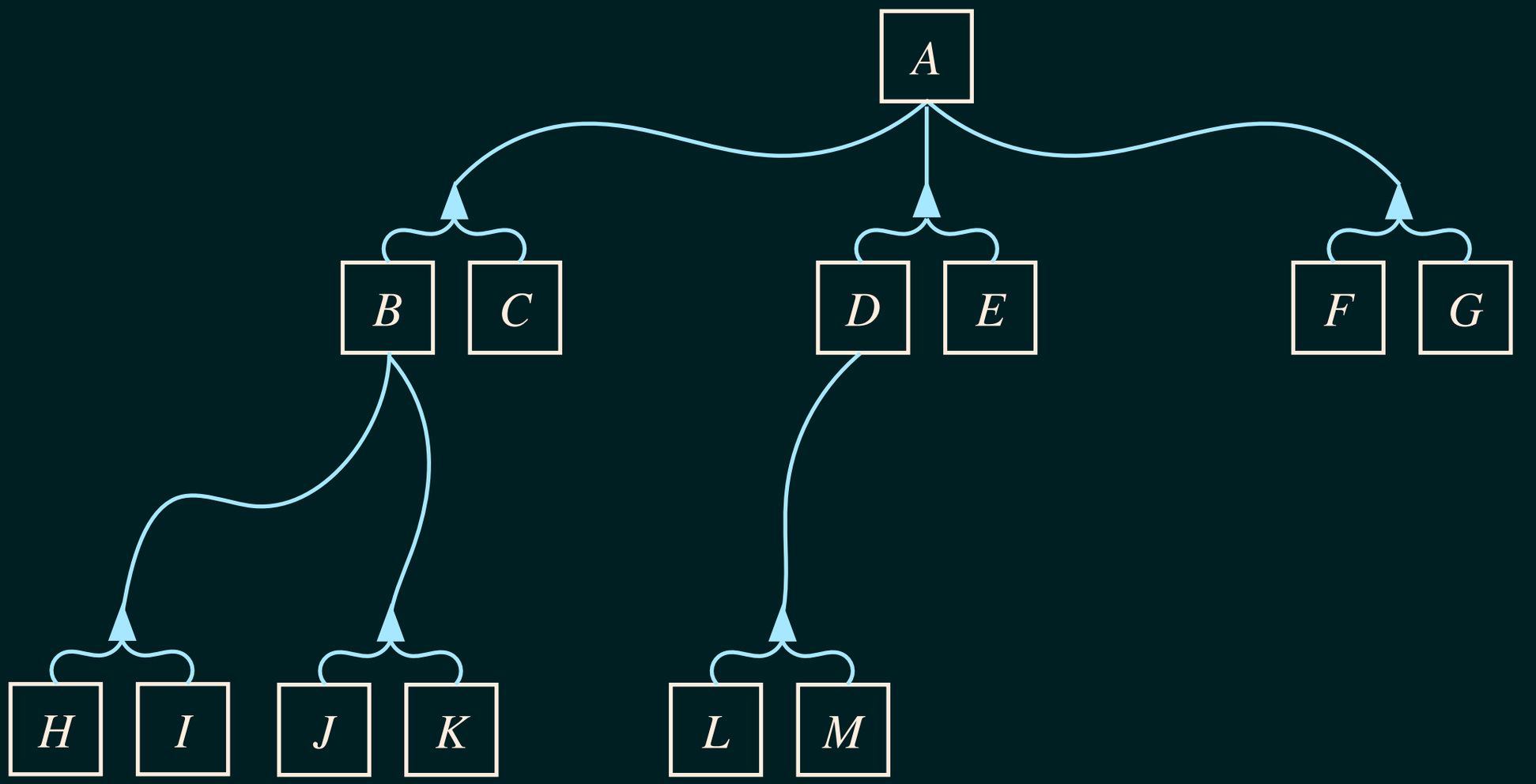
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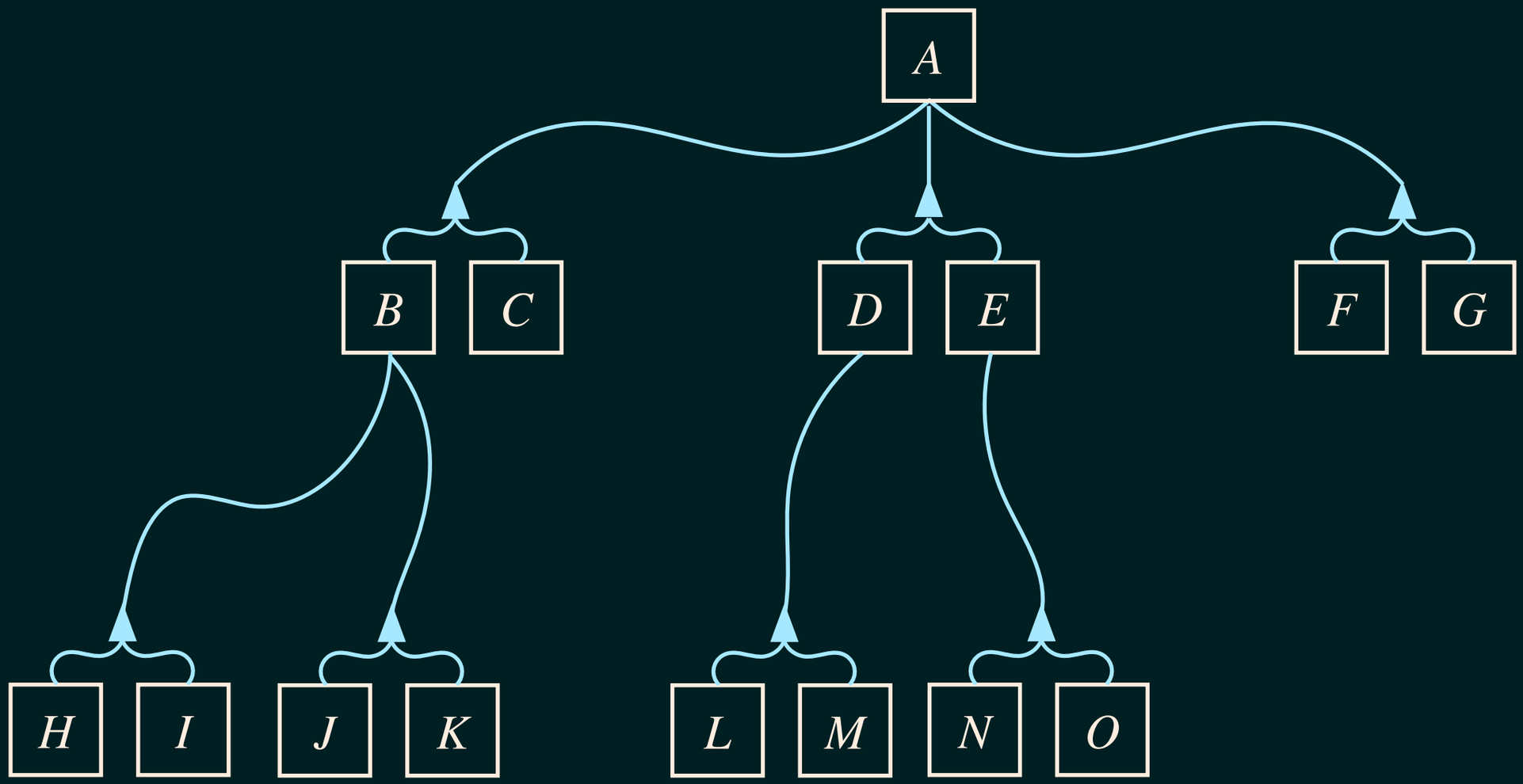
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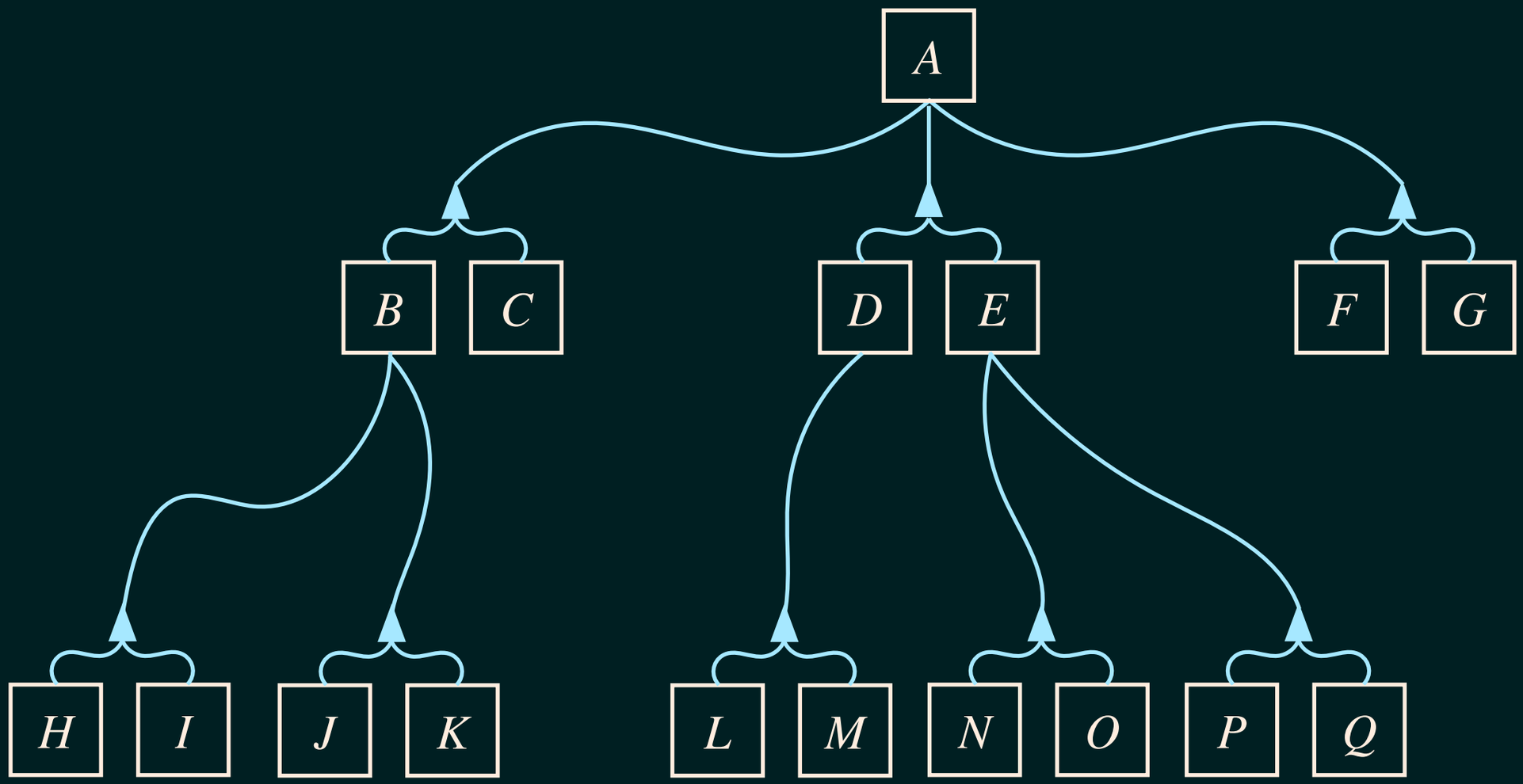
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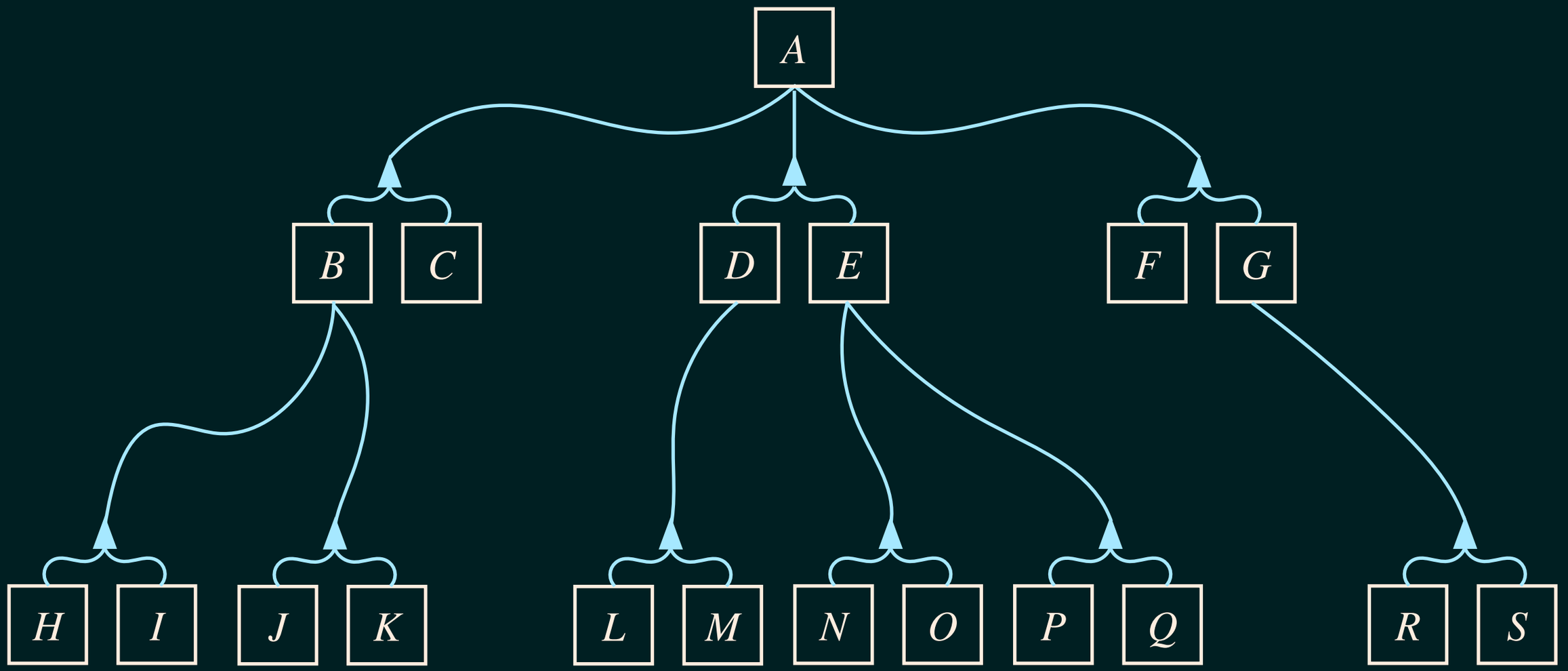
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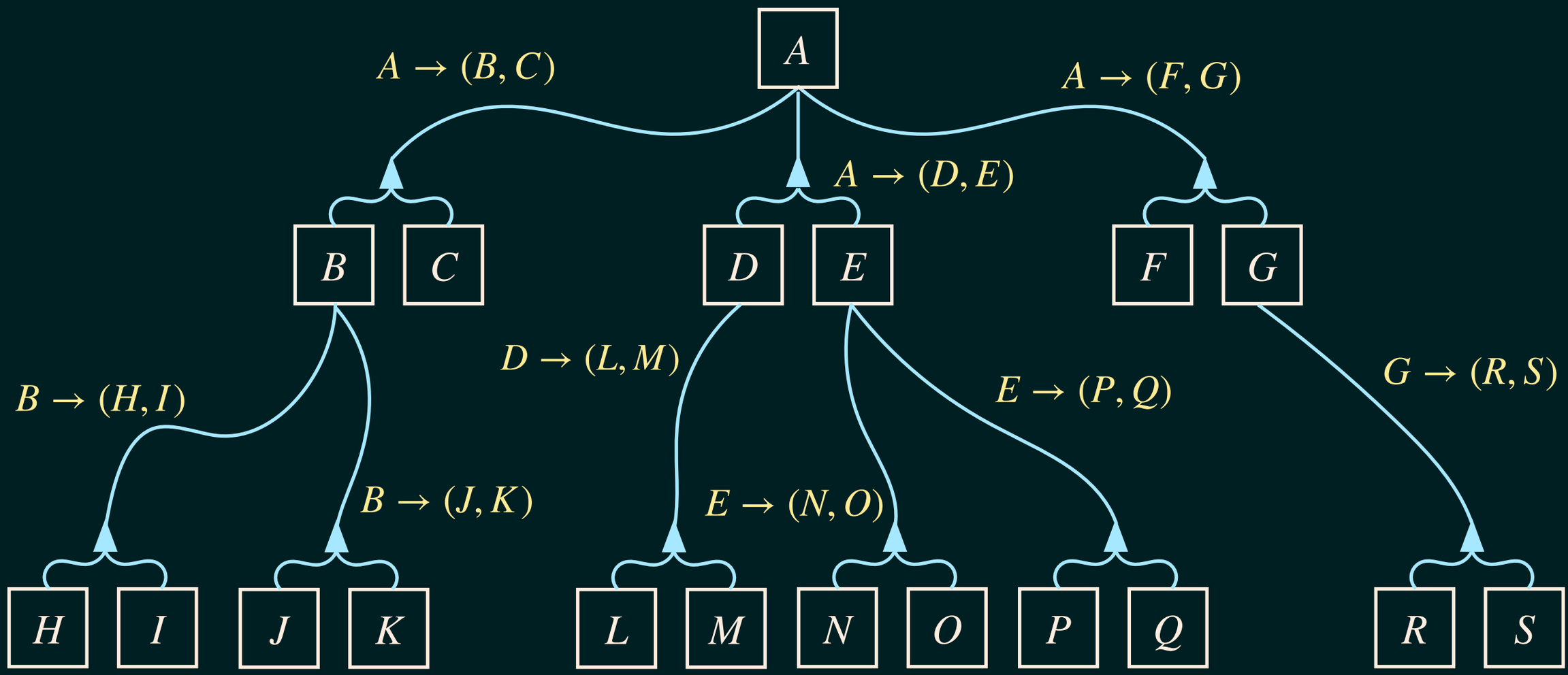
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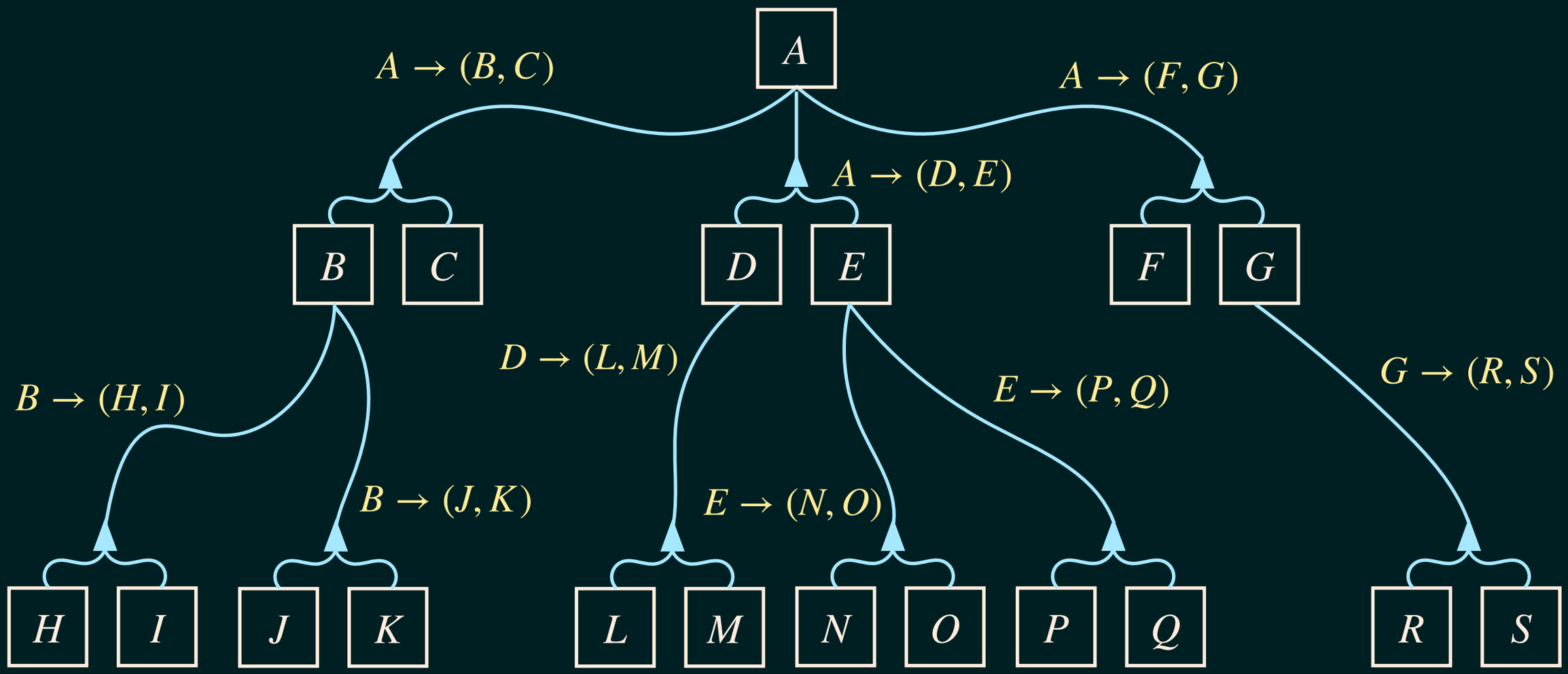

when the giant list of rules you're generating contains a subset that is a combinatorial specification, you win!

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- Random sampling routines

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- Dozens of known results and dozens of new results, and corrects several wrong results.

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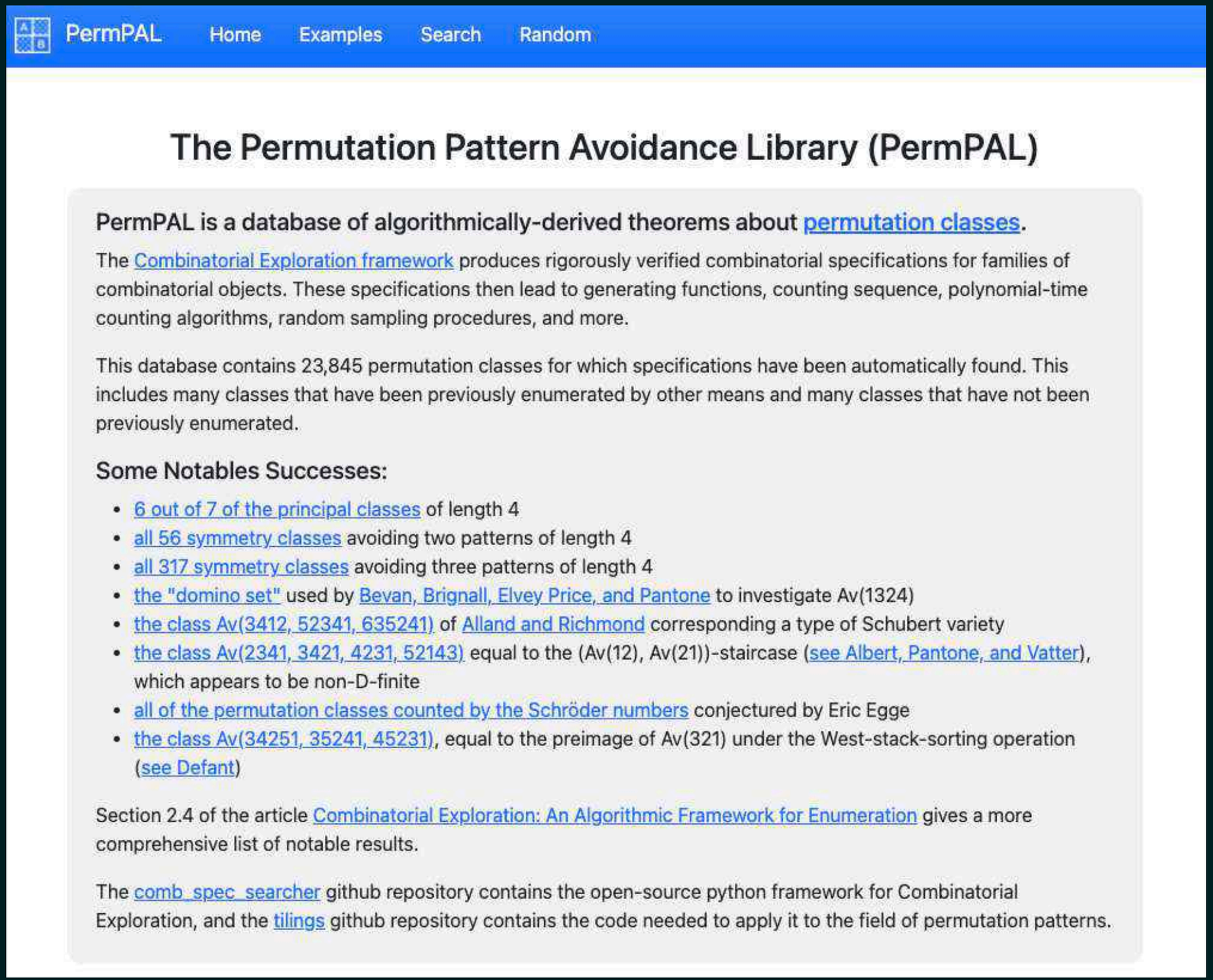
- And many more!


PermPAL
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The Permutation Pattern Avoidance Library (PermPAL)
PermPAL is a database of algorithmically-derived theorems about permutation classes.
The Combinatorial Exploration framework produces rigorously verified combinatorial specifications for families of combinatorial objects. These specifications then lead to generating functions, counting sequence, polynomial-time counting algorithms, random sampling procedures, and more.
This database contains 23,845 permutation classes for which specifications have been automatically found. This includes many classes that have been previously enumerated by other means and many classes that have not been previously enumerated.
Some Notables Successes:
6 out of 7 of the principal classes of length 4
all 56 symmetry classes avoiding two patterns of length 4
all 317 symmetry classes avoiding three patterns of length 4
the "domino set" used by Bevan, Brignall, Elvey Price, and Pantone to investigate Av(1324)
the class Av(3412, 52341, 635241) of Alland and Richmond corresponding a type of Schubert variety
the class Av(2341, 3421, 4231, 52143) equal to the (Av(12), Av(21))-staircase (see Albert, Pantone, and Vatter), which appears to be non-D-finite
all of the permutation classes counted by the Schröder numbers conjectured by Eric Egge
the class Av(34251, 35241, 45231), equal to the preimage of Av(321) under the West-stack-sorting operation (see Defant)
Section 2.4 of the article Combinatorial Exploration: An Algorithmic Framework for Enumeration gives a more comprehensive list of notable results.
The comb_spec_searcher github repository contains the open-source python framework for Combinatorial Exploration, and the tilings github repository contains the code needed to apply it to the field of permutation patterns.

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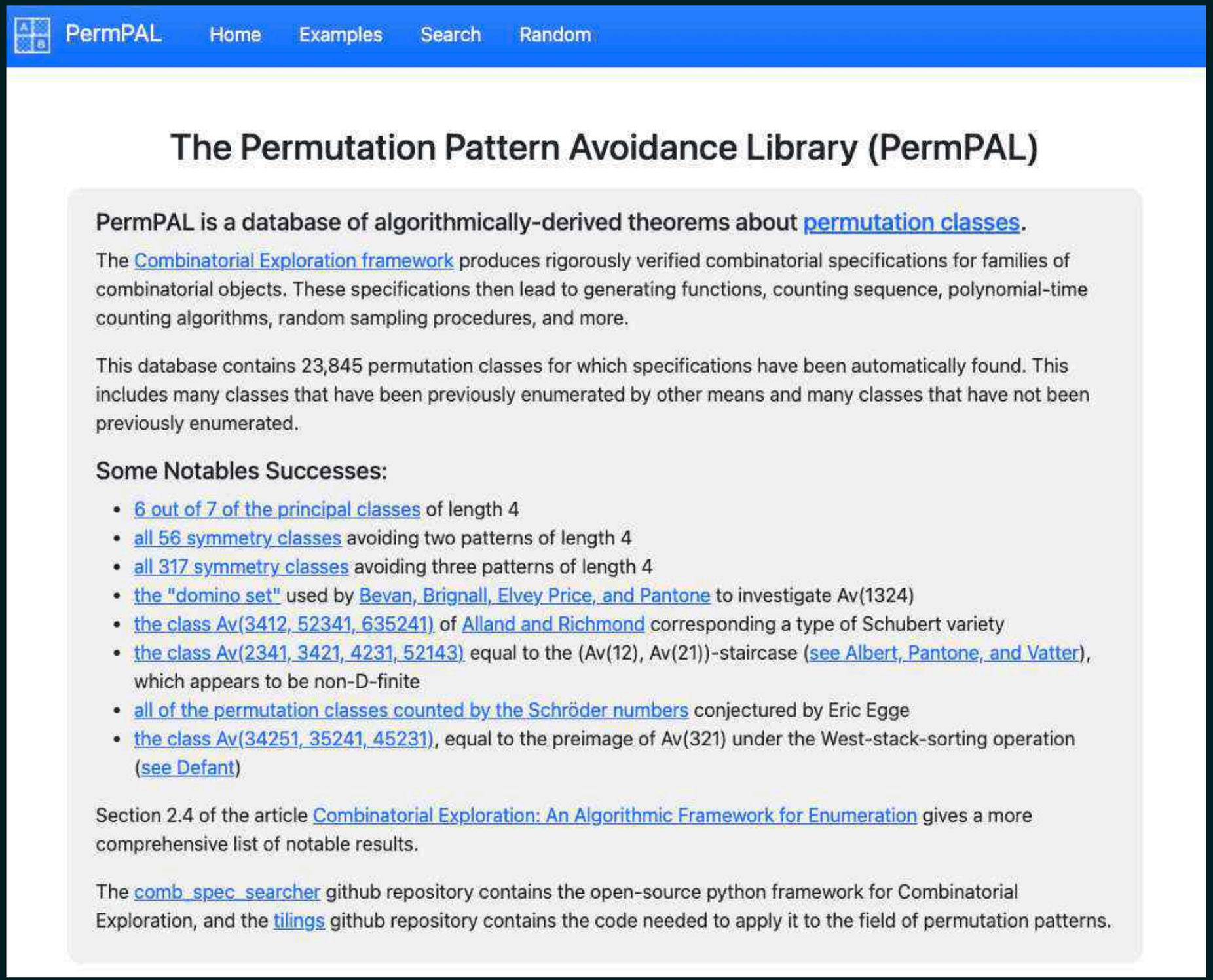

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Av(1342)

View Raw Data

Generating Function

$$\frac{-8\sqrt{-8x+1}\,x-8x^2+\sqrt{-8x+1}+20x+1}{2(x+1)^3}$$

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Counting Sequence

1, 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662, 3475090, 22214707, 144640291, 956560748, ...

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Recurrence

$$a(0)=1$$
$$a(1)=1$$
$$a(n+2)=\frac{4\,(3+2n)a(n)}{n+2}+\frac{(-8+7n)a(n+1)}{n+2},\quad n\geq 2$$

Copy to clipboard: latex Maple

Implicit Equation for the Generating Function

$$(x+1)^3F(x)^2+\left(8x^2-20x-1\right)F(x)+16x=0$$

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Specification 1 Specification 2 Specification 3 Specification 4 Specification 5

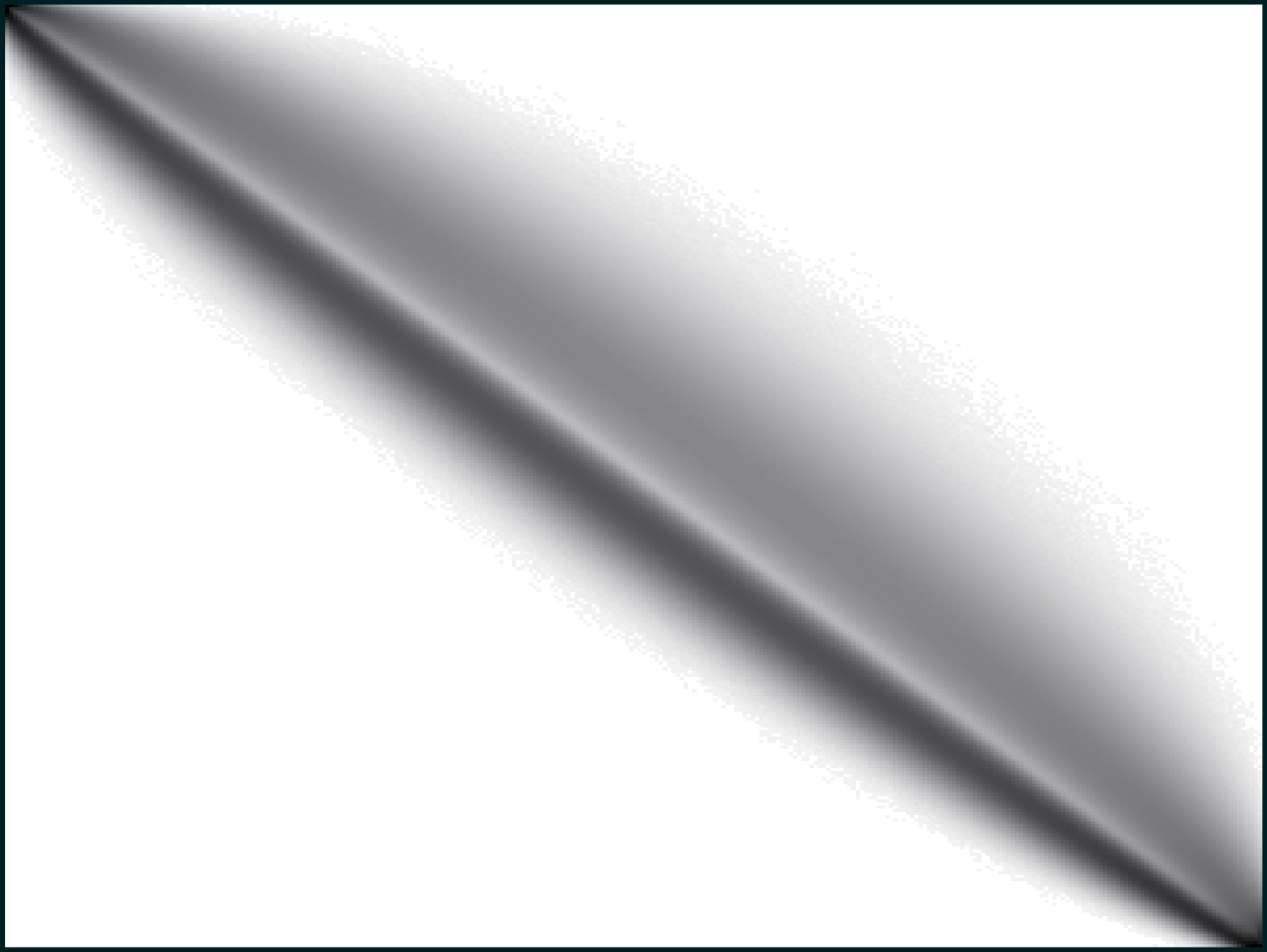
This specification was found using the strategy pack "Point And Col Placements Tracked Fusion" and has 29 rules.

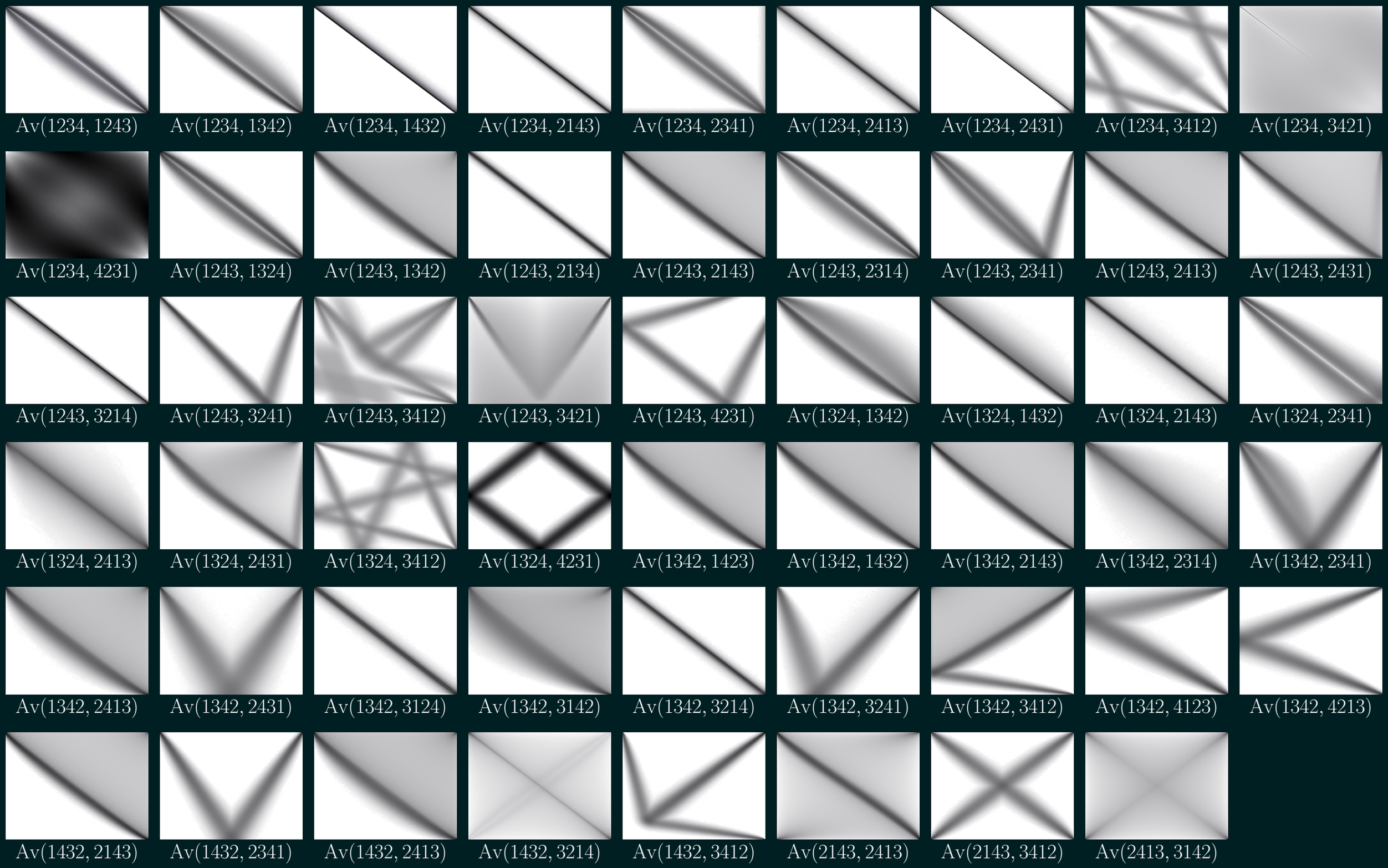
Found on May 26, 2021.
Finding the specification took 1720 seconds.

System of Equations

Copy 29 equations to clipboard: latex Maple sympy

$$
\begin{aligned}
F_0(x) &= F_1(x) + F_2(x) \\
F_1(x) &= 1 \\
F_2(x) &= F_3(x) \\
F_3(x) &= F_4(x)F_5(x) \\
F_4(x) &= x \\
F_5(x) &= F_6(x, 1) \\
F_6(x, y) &= F_0(x) + F_7(x, y) \\
F_7(x, y) &= F_8(x, y) \\
F_8(x, y) &= F_{14}(x, y)F_9(x, y) \\
F_9(x, y) &= F_{10}(x, y) + F_{15}(x, y) \\
F_{10}(x, y) &= F_{11}(x, y)F_6(x, y) \\
F_{11}(x, y) &= F_1(x) + F_{12}(x, y) \\
F_{12}(x, y) &= F_{13}(x, y) \\
F_{13}(x, y) &= F_{11}(x, y)^2 F_{14}(x, y) \\
F_{14}(x, y) &= yx \\
F_{15}(x, y) &= F_{16}(x, y) \\
F_{16}(x, y) &= F_{17}(x, y)F_4(x)F_6(x, y) \\
F_{18}(x, y) &= F_0(x)F_{17}(x, y)F_4(x) \\
F_{18}(x, y) &= F_{19}(x, y) \\
F_{20}(x, y) &= F_{19}(x, y) + F_{28}(x, y) \\
F_{20}(x, y) &= F_{21}(x, y) + F_6(x, y) \\
F_{21}(x, y) &= F_{22}(x, y) \\
F_{22}(x, y) &= F_{23}(x, y)F_4(x) \\
F_{23}(x, y) &= \frac{yF_{24}(x, y) - F_{24}(x, 1)}{-1 + y} \\
F_{24}(x, y) &= F_{25}(x, y) + F_{26}(x, y) \\
F_{25}(x, y) &= F_{11}(x, y)F_5(x) \\
F_{26}(x, y) &= F_{27}(x, y) \\
F_{27}(x, y) &= F_{17}(x, y)F_4(x)F_5(x) \\
F_{28}(x, y) &= F_0(x)F_{11}(x, y)
\end{aligned}
$$




Av(1234, 1243)
Av(1234, 1342)
Av(1234, 1432)
Av(1234, 2143)
Av(1234, 2341)
Av(1234, 2413)
Av(1234, 2431)
Av(1234, 3412)
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Av(1243, 1324)
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Av(1342, 3124)
Av(1342, 3142)
Av(1342, 3214)
Av(1342, 3241)
Av(1342, 3412)
Av(1342, 4123)
Av(1342, 4213)
Av(1432, 2143)
Av(1432, 2341)
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Av(1432, 3214)
Av(1432, 3412)
Av(2143, 2413)
Av(2143, 3412)
Av(2413, 3142)

Thank you!

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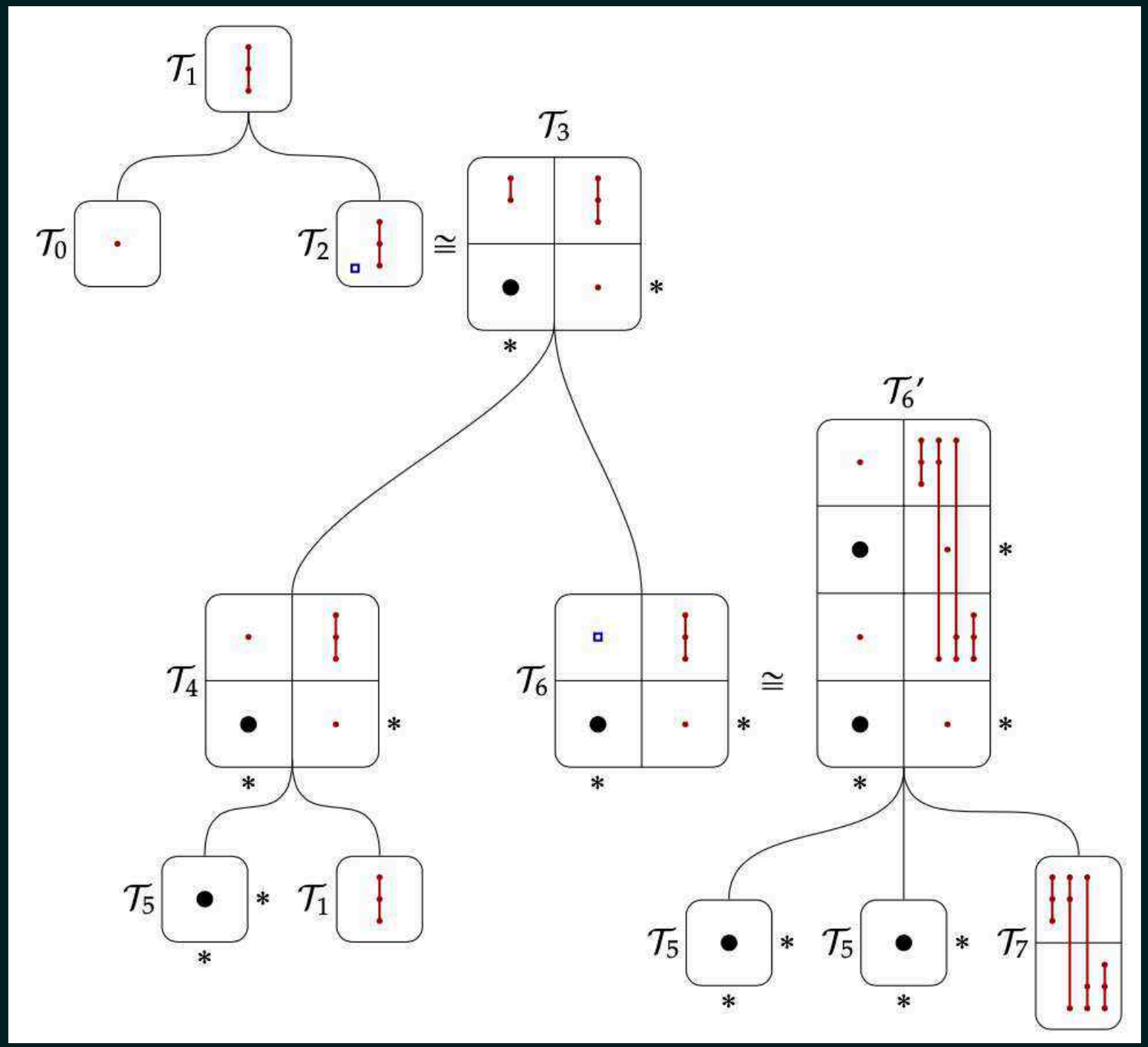
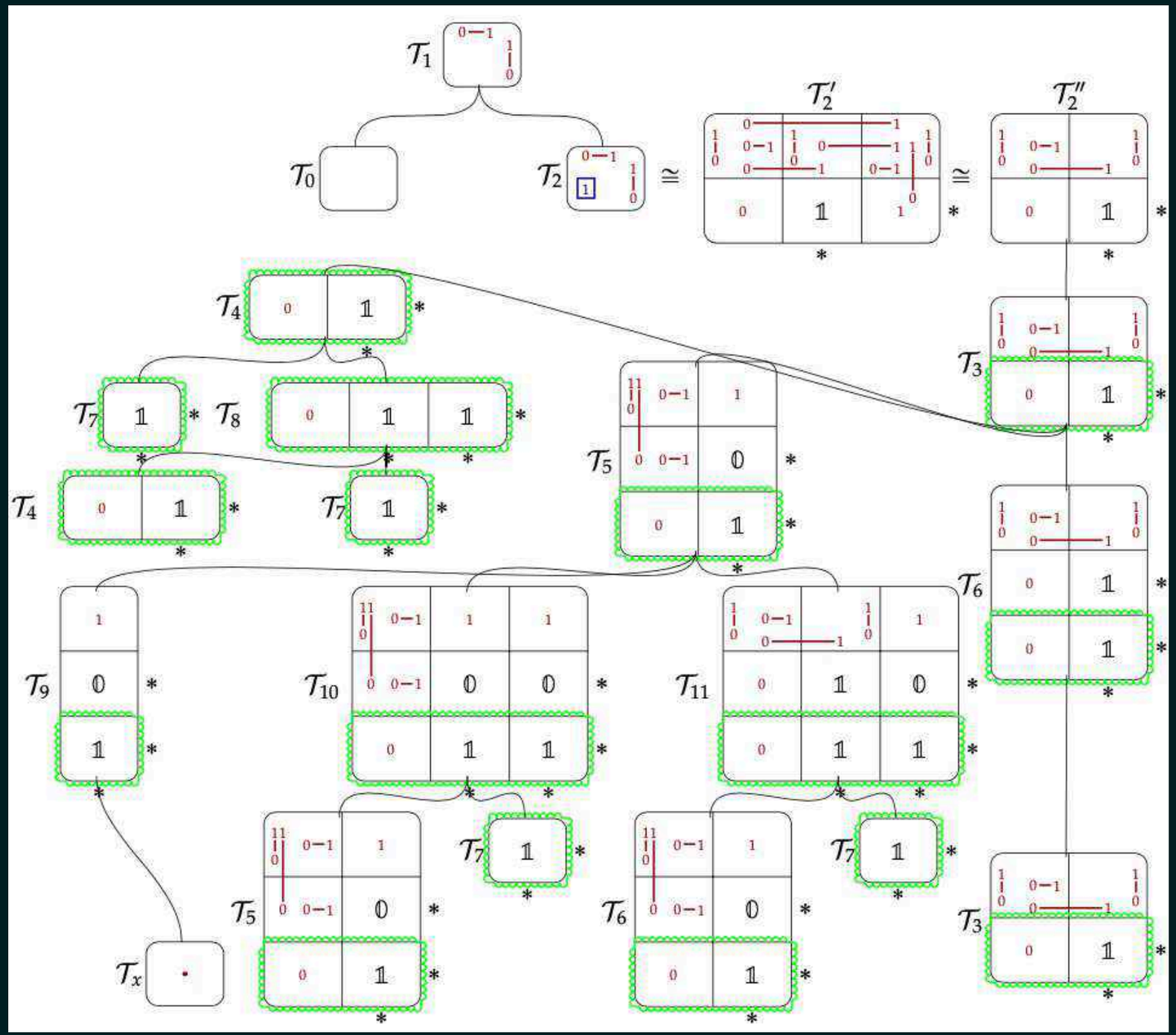


$$T_1(x) = 1 + T_2(x)$$
$$T_2(x) = T_4(x) + T_6(x)$$
$$T_4(x) = T_1(x) \cdot T_5(x)$$
$$T_5(x) = x$$
$$T_6(x) = T_5(x)^2 \cdot T_7(x)$$
$$T_7(x) = \frac{d}{dx}(x \cdot T_1(x))$$

$$T_1(x) = 1 + (x + x^2)T_1(x) + x^3\frac{d}{dx}T_1(x)$$



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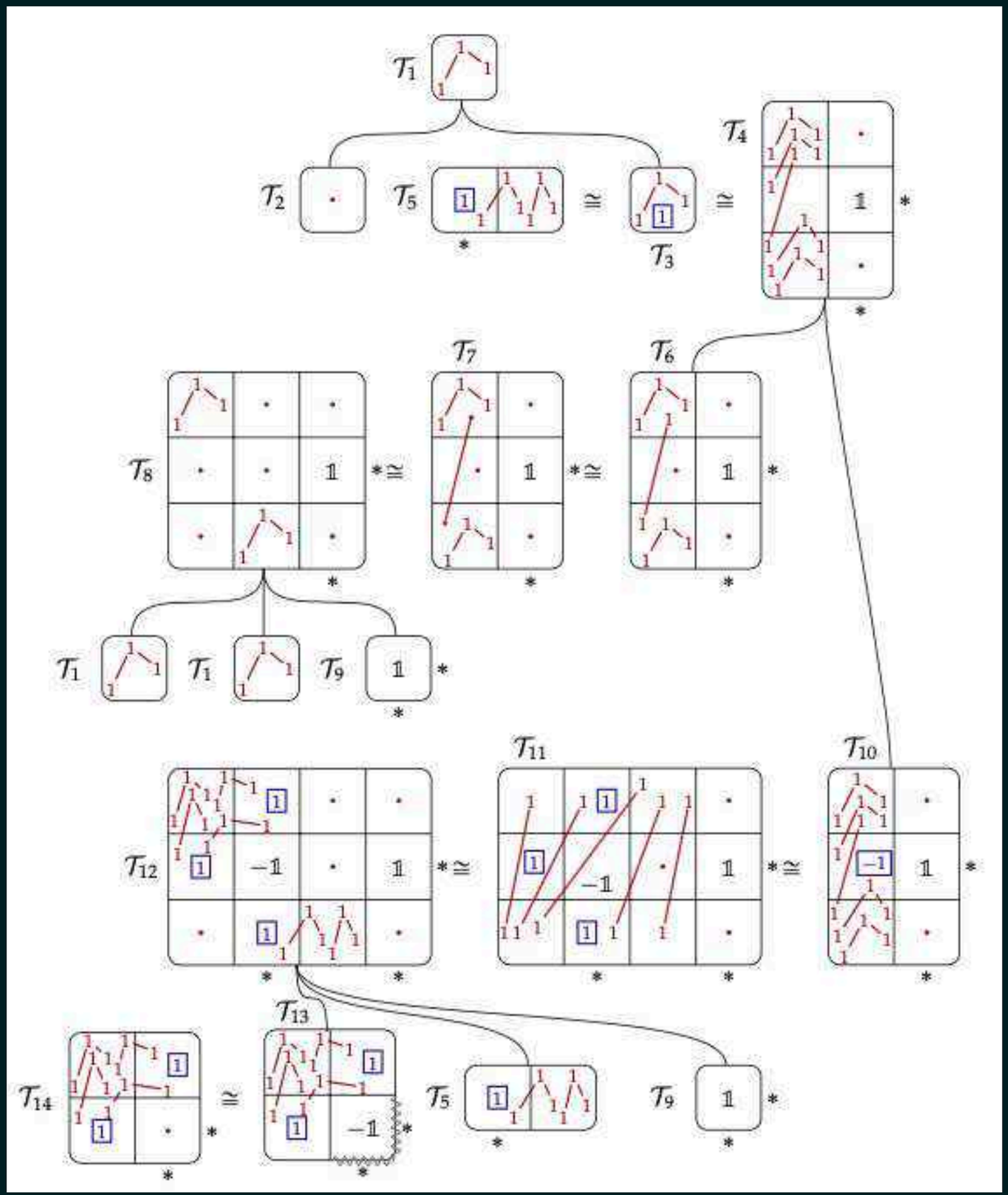
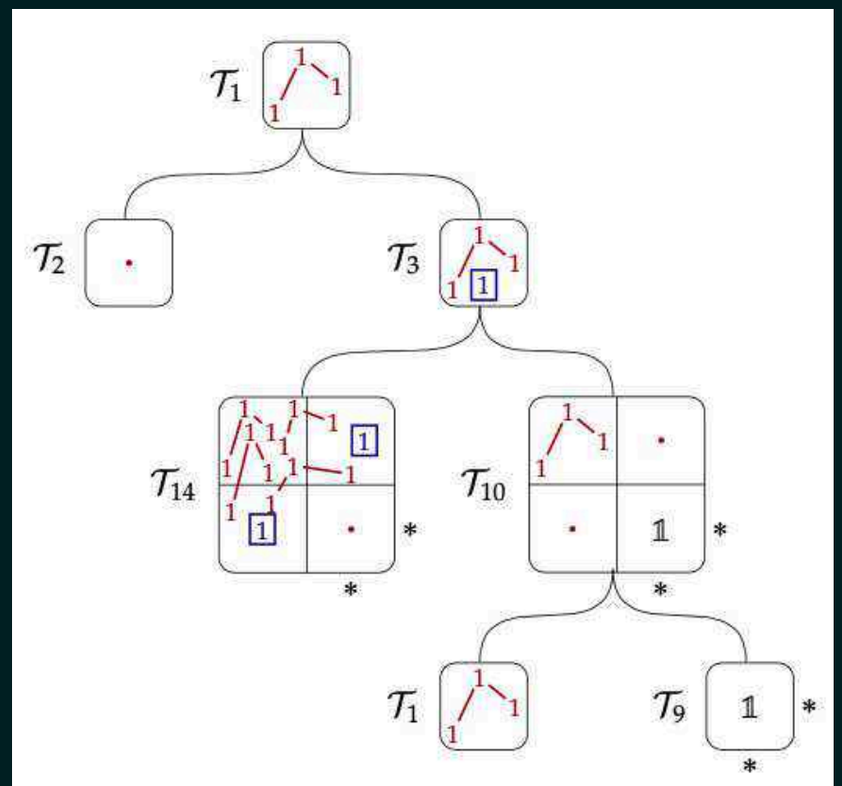
T_2(x) &= T_3(x,1) \\
T_3(x,y) &= T_4(x,y) + T_5(x,y) + T_6(x,y) \\
T_4(x,y) &= T_7(x,y) + T_8(x,y) \\
T_5(x,y) &= T_9(x,y) + T_{10}(x,y) + T_{11}(x,y) \\
T_6(x,y) &= T_3(x,xy) \\
T_7(x,y) &= xy \\
T_8(x,y) &= T_4(x,y) \cdot T_7(x,y) \\
T_9(x,y) &= 0 \\
T_{10}(x,y) &= T_5(x,y) \cdot T_7(x,y) \\
T_{11}(x,y) &= T_6(x,y) \cdot T_7(x,y)
\end{aligned}
$$

$$
T_1(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}
$$





$$T_1(x) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}$$