Stirling numbers in type B

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Outline

Basic definitions

Combinatorial interpretations

Other work and open problems

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Setting $q = 1$ in this theorem recovers the fact that $\#\mathfrak{S}_n = n!$.

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Ex. If $n = 3$ then

k	1	2	3
$S([3],k)$	123	1/23, 2/13, 3/12	1/2/3
$S(3,k)$	1	3	1

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Theorem (S-Swanson)

$$S_B[n, k] = \sum_{\rho \in S_B(\langle n \rangle, k)} q^{\operatorname{inv} \rho}.$$

Outline

Basic definitions

Combinatorial interpretations

Other work and open problems

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Theorem

Π_n^B is isomorphic to the intersection lattice of Coxeter group B_n.

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Open Problem: Find $\sum_{n\geq 0} s_B[n,k]x^n/[n]!$.

Coinvariant algebras.

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Swanson and Wallach made a similar conjecture in type B. We conjecture analogues of the Artin basis in both type A and B which, if correct, would prove both conjectures.

THANKS FOR LISTENING!