Combinatorial Models in the Representation Theory of Affine Lie Algebras

Permutation Patterns 2022

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Joint work with Carly Briggs and Cristian Lenart

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- The Tableaux model is simpler and has less structure.
- The Quantum Alcove model has extra structure which makes it easier to do several computations (energy function, combinatorial R-Matrix, ...).
- It is therefore beneficial to have an explicit isomorphism between the two models.

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Crystal graph: directed graph on B with edges colored $b \xrightarrow{i} b'$ exactly for $f_i(b) = b'$.

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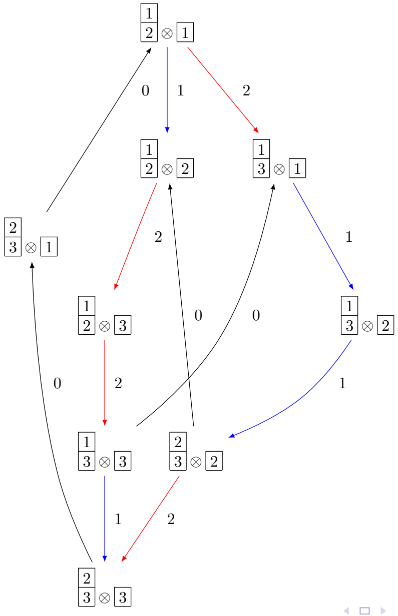
Definition. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, let

$$\mathbf{B}^{\lambda} = \mathbf{B}^{\lambda_1^{'},1} \otimes \mathbf{B}^{\lambda_m^{'},1} \otimes \ldots .$$

The crystal operators are defined on $\mathbf{B}^{\lambda}$ by a tensor product rule.

Type A KR crystal graph with $n = 3$ and shape $\lambda = (2, 1)$



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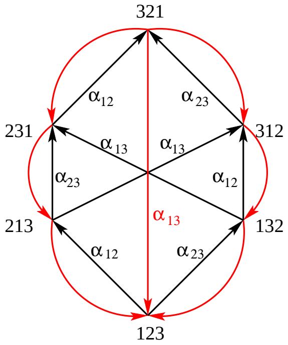

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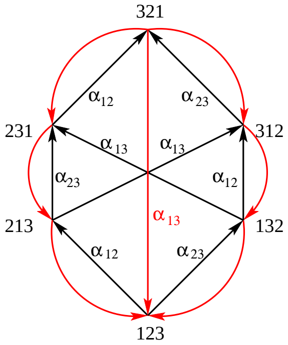

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Definition. A subset J is *admissible* if we have a path in the quantum Bruhat graph

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Theorem [Lenart , Naito , Sagaki , Schilling , Shimozono , 2017]
The collection of all admissible subsets, $\mathcal{A}(\Gamma)$,is a combinatorial model for $\mathbf{B}^\lambda$.

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Example Consider type A with $n = 4$ and $\lambda = (3, 2, 1, 0)$. Then the associated λ-chain is

$$\Gamma(\lambda) = ((3,4),(2,4),(1,4)|(2,3),(2,4),(1,3),(1,4)|(1,2),(1,3),(1,4)).$$

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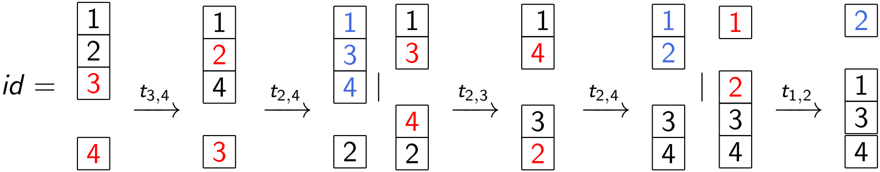

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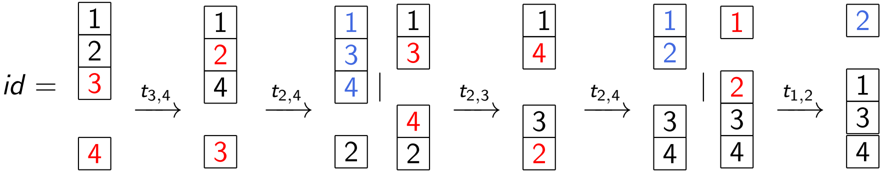

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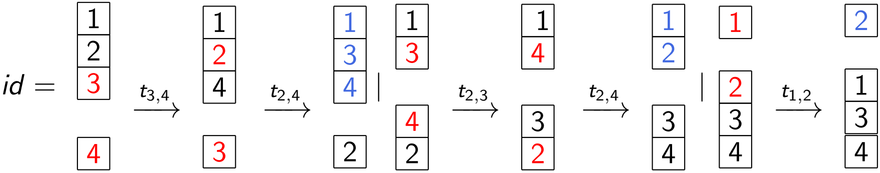

The blue columns (with entries sorted increasingly) then give us $fill(J) =$

1	1	2
3	2	
4		

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The resulting bijection is a crystal isomorphism [Lenart, Lubovsky, 2015].

The Type C_n Map

- The filling map is similar.
- The inverse map has one major change. Many KN columns have both i and $\bar{\imath}$ in them, so we use the splitting algorithm [Lecouvey] to bijectively make two columns with no $i, \bar{\imath}$ pairs in either.
- Example:

$$\begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array} \xrightarrow{\textit{split}} \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 5 & 2 \\ \hline \bar{3} & \bar{5} \\ \hline \bar{2} & \bar{4} \\ \hline \bar{1} & \bar{3} \\ \hline \end{array}$$

  The $\Gamma(k)$ in type C_n comes in two parts.
  One traverses the split columns and the other moves to the next column.
- Then similar Reorder and Path algorithms work.
- So now the reverse map is made up of a process of Split, Reorder, and Path.

The Type B_n Map

- There is a similar filling map
- For the reverse, similar to C_n, we need a splitting map.
- Recall that we now have columns of length $k - 2l$, so we need to Extend back to length k [Briggs].
- Further, the Path algorithm and Reorder algorithm no longer work.
- There is a configuration of two columns CC' that we call being blocked-off.
- Modify Path and Modify Reorder to avoid block-off pattern.

Blocked-Off Pattern

Definition: We say that columns $L = (l_1, l_2, ..., l_k), R' = (r_1, r_2, ..., r_k)$ are *blocked off at* i *by* $b = r_i$ iff $0 < b \geq |l_i|$ and

$$\{1, 2, ..., b\} \subset \{|l_1|, |l_2|, ..., |l_i|\}$$

and

$$\{1, 2, ..., b\} \subset \{|r_1|, |r_2|, ..., |r_i|\}$$

and $|\{j : 1 \leq j \leq i, l_j < 0, r_j > 0\}|$ is odd.

Example: The following columns CC' of height 5 with entries from $[\bar{8}]$ are blocked-off at 4 by 3:

1	1
4	5
$\bar{2}$	$\bar{2}$
$\bar{3}$	3
5	8

The Type D_n Map

- There is a similar filling map
- The splitting and Extend maps extend naturally.
- There is a Type D blocked-off pattern.
- Modify Path and Modify Reorder to avoid the new block-off pattern.

Recent Work

- The bijections for types B_n and D_n given here are actually crystal isomorphisms. In progress with undergraduate student.
- Efficient combinatorial computation for energy function for types B and D.
- Explicit computation of so-called *non-Dual-demazure* arrows in types B and D.

What is next?

- Explicit computation of *non-Dual-demazure* arrows in types A, C.
- Explicit embedding of the more generation rectangular shape into tensor product of column shape KR crystals.

Thank you!