The interval posets of permutations seen from the decomposition tree perspective

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Intervals of a permutation

The interval poset of a permutation was recently introduced and studied by Tenner. We will introduce decomposition trees into the picture and use them to answer some questions.

Definition

The interval of positive integers $[j, j+h]$ is an *interval* of the permutation $\sigma = \sigma_1 \cdots \sigma_n$ when there exists an i satisfying $\{\sigma_i, \ldots, \sigma_{i+h}\} = [j, j+h]$.

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Non-trivial and non-empty intervals are called *proper*. A permutation is *simple* if it is of size at least 4 and its only intervals are the trivial ones.
Example: $\sigma = 45683127$. Non-trivial intervals are $[4,6]$, $[4,5]$, $[5,6]$, $[1,2]$ and $[1,3]$.

Intervals of a permutation

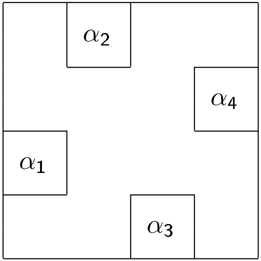
Intervals are ordered by inclusion $\rightarrow$ we have a poset!

Definition (Tenner,2022)

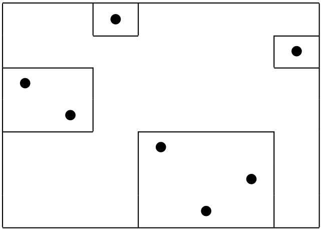
Given a permutation σ, the interval poset $P(\sigma)$ of σ is the plane embedding of the poset of the non-empty intervals of σ where the minimal elements appear in the order $\{\sigma_1\}, \{\sigma_2\}, \ldots, \{\sigma_n\}$ from left to right.
We also define $P_\bullet(\sigma)$ adding a new minimum, which is the empty interval, to $P(\sigma)$.

Substitution decomposition

$$2413[\alpha_1, \alpha_2, \alpha_3, \alpha_4] =$$



$$2413[21, 1, 312, 1] =$$



Substitution decomposition

Theorem (Albert, Atkinson, 2005)

Every permutation σ of size at least 2 can be uniquely decomposed as $\pi[\alpha_1, \ldots, \alpha_k]$ with one of the following satisfied:

- π *is simple;*
- $\pi = 12 \ldots k$ *for some* $k \geq 2$ *and all* α_i *are* $\oplus$*-indecomposable;*
- $\pi = k \ldots 21$ *for some* $k \geq 2$ *and all* α_i *are* $\ominus$*-indecomposable.*

The description of σ as $\pi[\alpha_1, \ldots, \alpha_k]$ satisfying the above is called the substitution decomposition of σ.

Applying the substitution decomposition recursively inside the α_i until we reach permutations of size 1, we can represent every permutation by a tree, called its *decomposition tree* $T(\sigma)$.

$$45683127 = 2413[123[1,1,1], 1, 21[1, 12[1,1]], 1]$$

Tree $\to$ poset

Given σ (or $T(\sigma)$), we can construct $P(\sigma)$.

- If $\sigma = 1$ (*i.e.* $T(\sigma) = \bullet$), then $P(\sigma)$ is the poset containing only one element.
- Otherwise, we consider the substitution decomposition $\sigma = \pi[\alpha_1, \dots, \alpha_k]$ of σ.
- If π is simple, we let R be the dual claw of size k. Otherwise (*i.e.* if π is $\oplus$ or $\ominus$) we denote by R the argyle poset of size k.
- We let the minimal elements of R be $p_1, \dots, p_k$, in this order from left to right.
- The poset $P(\sigma)$ is obtained by taking R and replacing, for each $i \in [1, k]$, p_i by the recursively-obtained poset $P(\alpha_i)$.

Notice that the auxiliary poset R in this procedure actually is $R = P(\pi)$.

Known results [B. Tenner]

Theorem (Tenner, 2022)

Let σ be a permutation, then

- $P(\sigma)$ *and* $P_{\bullet}(\sigma)$ *are planar posets;*
- $P_{\bullet}(\sigma)$ *is a lattice;*
- $P_{\bullet}(\sigma)$ *is modular if and only if* σ *is simple or* σ *has size at most* 2*;*
- $P_{\bullet}(\sigma)$ *is distributive if and only if* σ *has size at most* 2*.*

Known results [B. Tenner]

Theorem (Tenner, 2022)

Let $P = P(\sigma)$, for some σ. Denote by $rl(P)$ the number of permutations whose interval poset coincides with P.
Let T be the tree obtained from $T(\sigma)$ replacing all the labels $\oplus$ and $\ominus$ with linear and all the labels π, with π simple permutation, with prime. Then

$$rl(P) = \prod_{v \text{ non-leaf vertex of } T} rl(v)^{\varepsilon_v},$$

where the $rl(v)$ are given by

$$rl(v) = \begin{cases} 2 \text{ if } v \text{ is linear,} \\ |\{\pi \in S_k \mid \pi \text{ simple}\}| \text{ if } v \text{ is prime with } k \text{ children,} \end{cases}$$

and the exponents ε_v are given by

$$\varepsilon_v = \begin{cases} 0 \text{ if } v \text{ is linear with a linear parent,} \\ 1 \text{ otherwise.} \end{cases}$$

Counting problems: posets of size n

For every n, what is the number p_n of interval posets P with n minimal elements such that $P = P(\sigma)$ for some permutation σ?

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For every n, what is the number p_n of interval posets P with n minimal elements such that $P = P(\sigma)$ for some permutation σ?
Let:

- $\mathcal{P}$ be the family of rooted plane trees, where internal nodes carry a type which is either prime or linear, in which the size is defined as the number of leaves, and in which the number of children of any linear (resp. prime) node is at least 2 (resp. at least 4);
- $\mathcal{P}_n$ be the set of trees of size n in $\mathcal{P}$;
- $P(z) = \sum_{n \geq 0} p_n$ be p_n's generating function.

Note that $|\mathcal{P}_n| = p_n$.

Counting problems: posets of size n

Denoting a leaf with $\bullet$, the disjoint union with $\uplus$, and the sequence operator restricted to sequences of at least k components with $Seq_{\geq k}$, then

$$\mathcal{P} = \bullet \uplus Seq_{\geq 2}(\mathcal{P}) \uplus Seq_{\geq 4}(\mathcal{P}),$$

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$$\mathcal{P} = \bullet \uplus Seq_{\geq 2}(\mathcal{P}) \uplus Seq_{\geq 4}(\mathcal{P}),$$

which gives the following functional equation.

$$P(z) = z + \frac{P(z)^2}{1 - P(z)} + \frac{P(z)^4}{1 - P(z)}.$$

Rewriting

$$P(z) = z\phi(P(z)), \qquad \phi(u) = \frac{1}{1 - u\left(\frac{1+u^2}{1-u}\right)}.$$

We can apply Lagrange inversion formula to obtain a formula for p_n.

$$p_n = \begin{cases} 1 & \text{if } n = 1, \\ \frac{1}{n}\sum_{i=1}^{n-1} \sum_{k=0}^{\min\{i, \frac{n-i-1}{2}\}} \binom{n+i-1}{i}\binom{i}{k}\binom{n-2k-2}{i-1} & \text{if } n > 1. \end{cases}$$

The first terms of this sequence (starting from p_1) are
$1, 1, 3, 12, 52, 240, 1160, 5795, 29681$.
It has been added on OEIS as sequence A348479.

Counting problems: posets of size n

Theorem (Bouvel, C., Izart)

Let Λ be the function defined by $\Lambda(u) = \frac{u^2+u^4}{1-u}$. The radius of convergence ρ of the generating function $P(z)$ of interval posets is given by $\rho = \tau - \Lambda(\tau)$, where τ is the unique solution of $\Lambda'(u) = 1$ such that $\tau \in (0,1)$. The behavior of $P(z)$ near ρ is given by

$$P(z) = \tau - \sqrt{\frac{2\rho}{\Lambda''(\tau)}}\sqrt{1-\frac{z}{\rho}} + \mathcal{O}\left(1-\frac{z}{\rho}\right).$$

Numerically, we have $\tau \approx 0.2708$, $\rho \approx 0.1629$, $\sqrt{\frac{2\rho}{\Lambda''(\tau)}} \approx 0.2206$. As a consequence, the number p_n of interval posets with n minimal elements satisfies, as $n \to \infty$,

$$p_n \sim \sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}}\frac{\rho^{-n}}{n^{3/2}}.$$

Numerically, we have $\sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \approx 0.0622$, $\rho^{-1} \approx 6.1403$.

Counting problems: tree posets

For some permutations σ the interval poset $P(\sigma)$ is also a tree, i.e. every element of $P(\sigma)$ is covered by one and only one element. B. Tenner posed the problem of enumerating tree interval poset, counted with respect to the number of minimal elements.

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Let:

- $\mathcal{T}$ be the subfamily of $\mathcal{P}$ of the trees where every linear node has exactly 2 children;
- $\mathcal{T}_n$ be the set of trees of size n in $\mathcal{T}$;
- $t_n = |\mathcal{T}_n|$;
- $T(z) = \sum_{n \geq 0} t_n$ be t_n's generating function;

Counting problems: tree posets

With the same notation as the previous case, we have

$$\mathcal{T} = \bullet \uplus (\mathcal{T} \times \mathcal{T}) \uplus Seq_{\geq 4}(\mathcal{T}),$$

which gives the following functional equation.

$$T(z) = z + T(z)^2 + \frac{T(z)^4}{1 - T(z)}.$$

Rewriting

$$T(z) = z\psi(T(z)), \qquad \psi(u) = \frac{1}{1 - u\left(1 + \frac{u^2}{1-u}\right)},$$

We can apply Lagrange inversion formula to obtain a formula for t_n.

$$t_n = \frac{1}{n}\left[\sum_{i=1}^{n-3} \sum_{k=1}^{\min\{i, \frac{n-i-1}{2}\}} \binom{n+i-1}{i}\binom{i}{k}\binom{n-i-k-2}{k-1} + \binom{2n-2}{n-1}\right].$$

The first terms of this sequence (starting from t_1) are
$1, 1, 2, 6, 21, 78, 301, 1198, 4888$, which corresponds to sequence A054515 in OEIS.

Counting problems: tree posets

In the same fashion as what we did for p_n, we find the asympotics for t_n.

Theorem (Bouvel, C., Izart)

Let Λ be the function defined by $\Lambda(u) = u^2 + \frac{u^4}{1-u}$. The radius of convergence ρ of $T(z)$ is given by $\rho = \tau - \Lambda(\tau)$, where τ is the unique solution of $\Lambda'(u) = 1$ such that $\tau \in (0,1)$. The behavior of $T(z)$ near ρ is given by

$$T(z) = \tau - \sqrt{\frac{2\rho}{\Lambda''(\tau)}}\sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\left(1 - \frac{z}{\rho}\right).$$

Numerically, we have $\tau \approx 0.3501$, $\rho \approx 0.2044$, $\sqrt{\frac{2\rho}{\Lambda''(\tau)}} \approx 0.2808$.
As a consequence, the number t_n of tree interval posets with n minimal elements satisfies, as $n \to \infty$,

$$t_n \sim \sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}}\frac{\rho^{-n}}{n^{3/2}}.$$

Numerically, we have $\sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \approx 0.0792$, $\rho^{-1} \approx 4.8920$.

The Möbius function of $P(\sigma)$

Definition

Let $(P, \leq)$ be a partially ordered set which is locally finite, and let $a, b \in P$. The *Möbius function* between a and b is recursively defined as

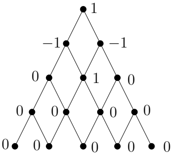
$$\mu_P(a,b) = \begin{cases} 1 & \text{if } a = b, \\ -\displaystyle\sum_{x : a < x \leq b} \mu_P(x,b) & \text{if } a < b, \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function of $P(\sigma)$

Theorem (Bouvel, C., Izart)

Let σ be a permutation of size n such that $\sigma = \pi[\alpha_1, ..., \alpha_k]$. Then for every $I \in P_\bullet(\sigma)$,

$$\mu(I, [1,n]) = \begin{cases} 1 & \text{if } I = [1,n], \\ -1 & \text{if } I \text{ is covered by } [1,n], \\ k-1 & \text{if } I = \emptyset \text{ and } \pi \text{ is either simple or } 12 \text{ or } 21, \\ 1 & \text{if } \pi \text{ is } 12\ldots k \text{ or } k\ldots 21 \text{ for some } k \geq 3 \\ & \quad \text{and } I \text{ is covered by the two coatoms of } P_\bullet(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$



Remark

Let σ be a permutation and J be an element of $P_\bullet(\sigma)$. Define $j = |J|$. Let $\mathcal{J}$ be the subposet of $P_\bullet(\sigma)$ consisting of the elements in the interval $[\emptyset, J]$ of $P_\bullet(\sigma)$. Then there exists a permutation τ (of size j) such that $P_\bullet(\tau)$ is isomorphic to $\mathcal{J}$.

We can thus compute the Möbius function of every interval of $P_\bullet(\sigma)$.

Thank you!