Pattern-Avoiding Involutions & Brownian Bridge

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w/ Chris Coscia

321-avoiding Involutions

- $I_n(321)$

321-avoiding Involutions

- $\mathcal{I}_n(321)$
- $|\mathcal{I}_n(321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ Simion & Schmidt '85

321-avoiding Involutions

- $Iv_n(321)$
- $|Iv_n(321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ Simion & Schmidt '85

also counts Simple Random Walk Bridges

Other Related Results

- Connections to Motzkin Paths (2011 Barnabei Bonetti Silimbani)
  - & $\mathrm{Inv}_n(4321, 3412)$
- Exact & Asymptotic Enumeration (2013 Bona Homberger Pantone Vatter)
- Fixed-pt distribution (Miner Rizzolo S. 2017)

and many more!

Brownian Bridge

Thm (Donsker)

- Let W_n be a simple random walk bridge from $(0,0)$ to $(n, 0)$.

$$\boxed{\frac{1}{n^{1/2}} W_n(\lfloor nt \rfloor) \longrightarrow_d b_t}$$

where b_t denotes Brownian bridge & the convergence is in distribution

Main Result

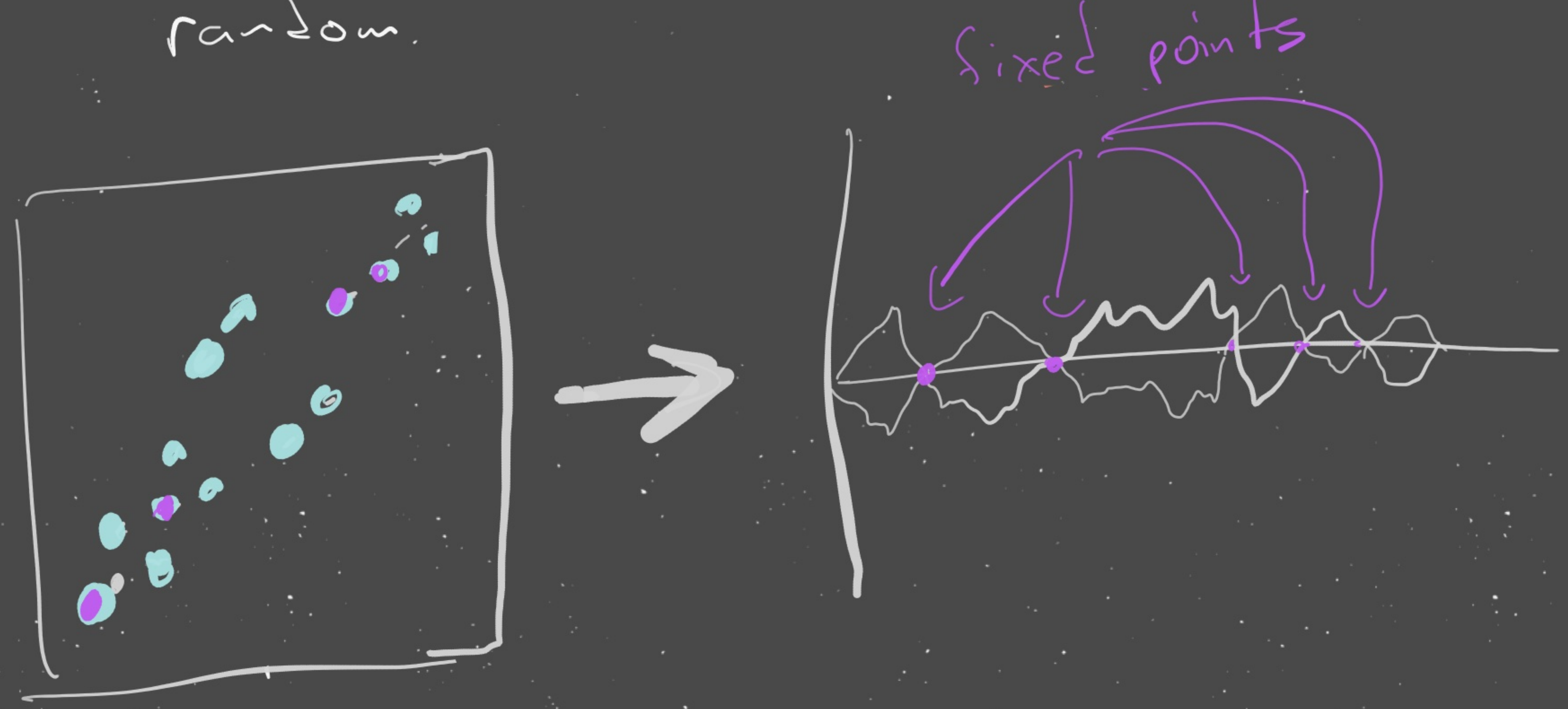
Let b_t denote Brownian bridge, and $\sigma_n \in \mathrm{Inv}(321)$ chosen uniformly at random. Then

$$\frac{1}{n^{1/2}}\left[\,|\sigma_n(\lceil nt \rceil) - nt|\,\right] \longrightarrow_d |b_t|$$

w/ convergence in distribution in $D[0,1]$ equipped w/ Skorohod Topology.

Main Result

Let b_t denote Brownian bridge, and $\sigma_n \in \mathrm{Inv}(321)$ chosen uniformly at random.



Similar Results

Hoffman Rizzolo S.

2017 • $\sigma \in Av_n(321)$ e_t Brownian excursion

$$\frac{1}{(2n)^{1/2}} \left|\sigma(\lfloor nt \rfloor) - nt\right| \longrightarrow_d e_t \qquad t \in (0,1)$$

Similar Results

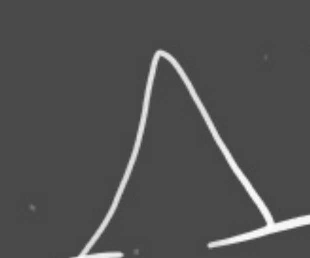
Hoffman Rizzolo S.

2017 • $\sigma \in Av_n(321)$ $\quad e_t$ Brownian excursion

$$\frac{1}{(2n)^{1/2}} \left| \sigma(\lfloor nt \rfloor) - nt \right| \to_d e_t \qquad t \in (0,1)$$

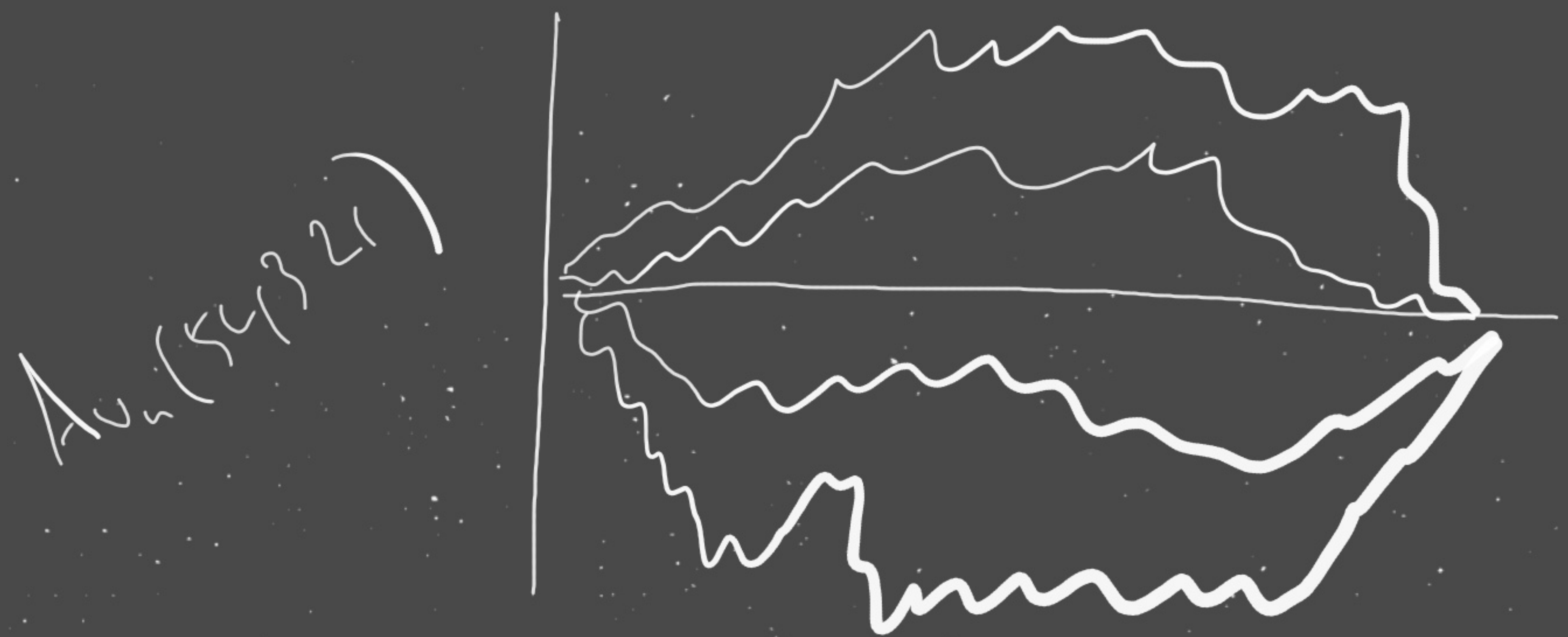
2020+ • $\rho \in Av_n(d{+}1\; d \cdots 21)$

$$\frac{1}{(2dn)^{1/2}} \left(\rho(\lfloor nt \rfloor) - nt \right) \to_d$$



More complicated version of e_t

Similar Results



Connecting Bridges & 321-avoiding Involutions

Bridge W

- $W(0) = -\frac{1}{2}$
- $W(n) = \frac{(-1)^n}{2}$
- $W(i) - W(i-1) = \pm 1$

Word a

- $a(i) = A$, $W(i) > W(i-1) > 0$, $W(i) < W(i-1) < 0$
- $a(i) = B$, $W(i-1) > W(i) > 0$, $W(i-1) < W(i) < 0$
- $a(i) = C$, otherwise

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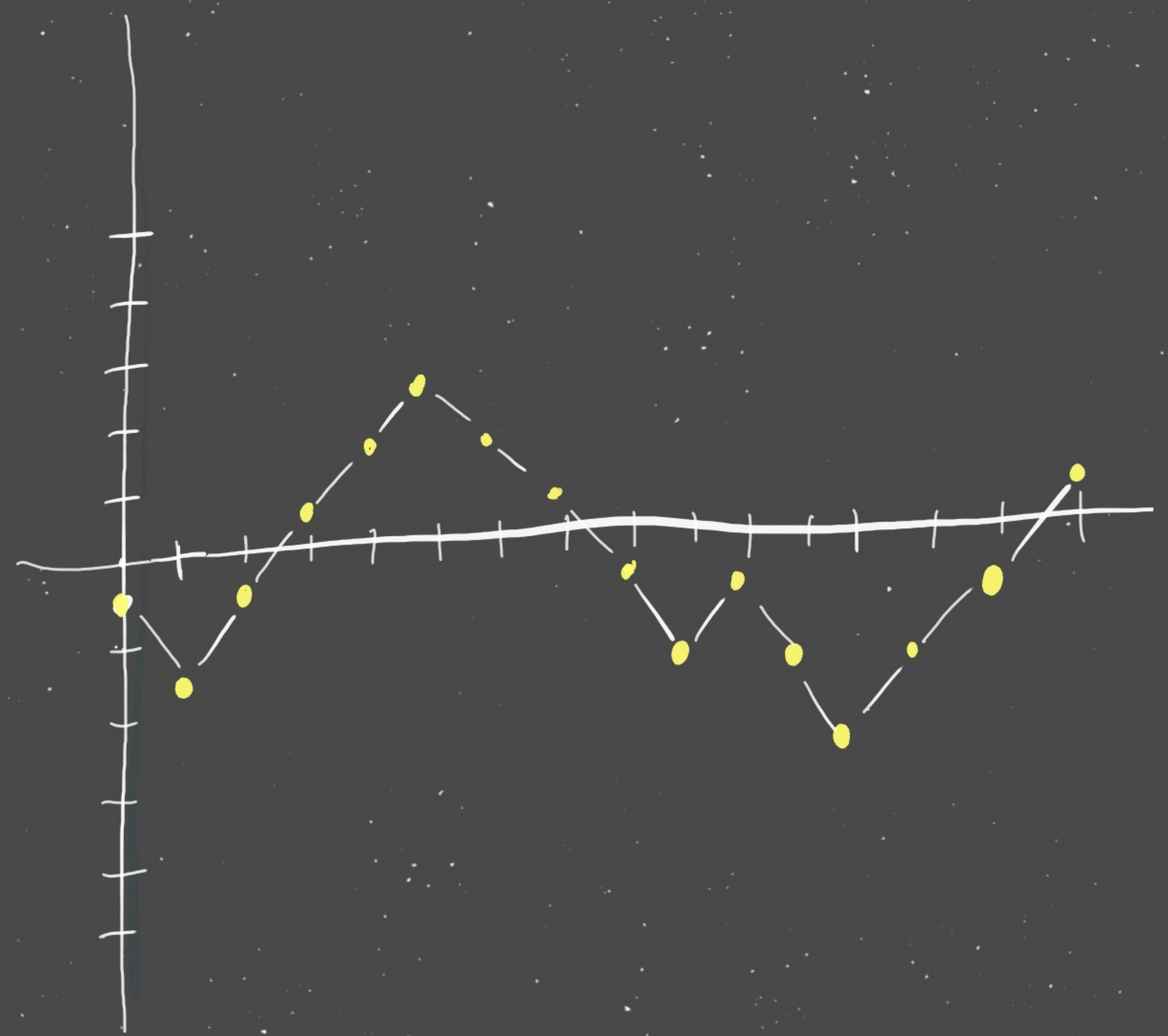

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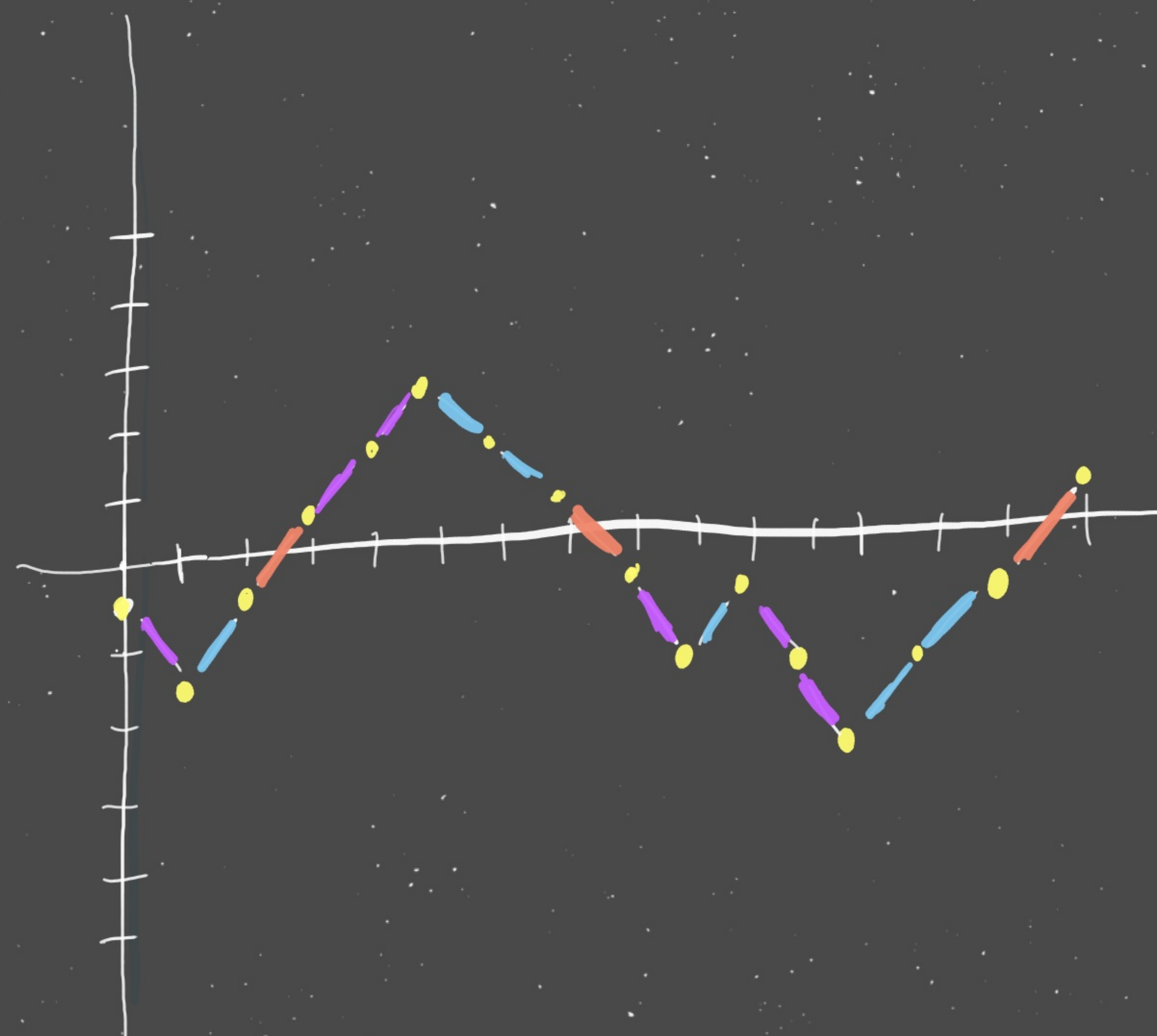

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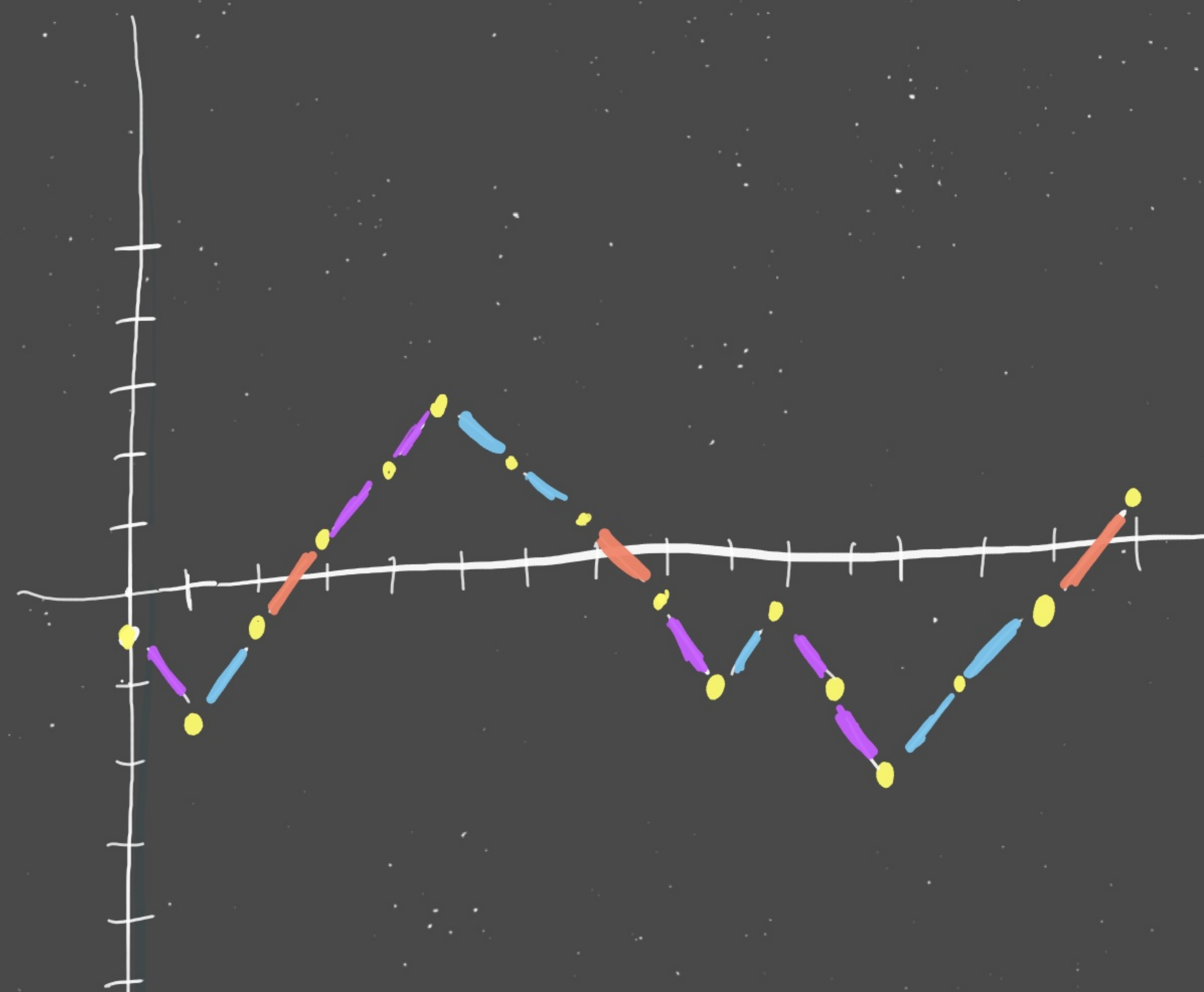

$a = ABCAABBCABAABBC$

- $a(i) = A$, $W(i) > W(i-1) > 0$, $W(i) < W(i-1) < 0$
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- $a(i) = C$, otherwise

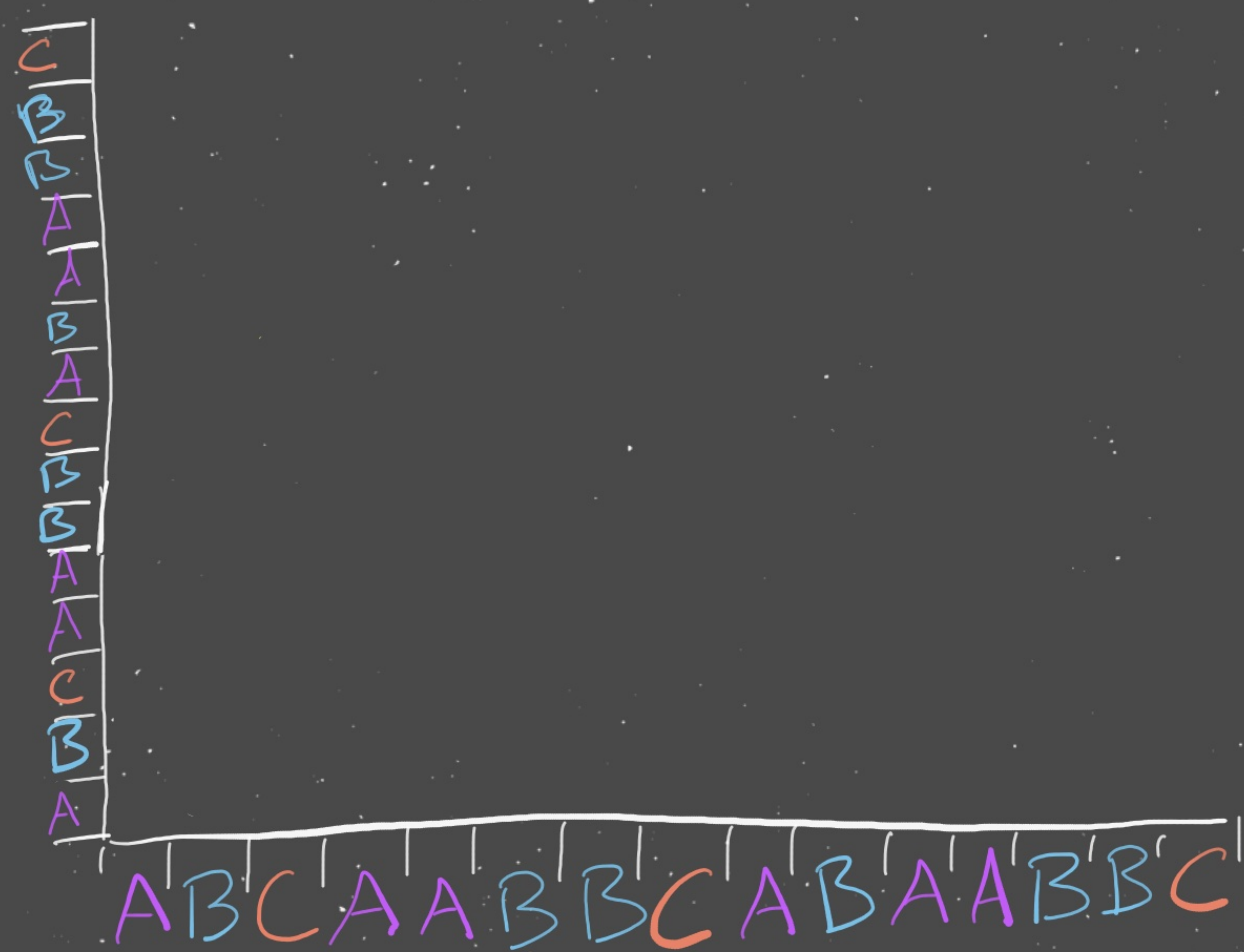
match

A^s to B^s

B to A^s

&

C^s to C^s



to create $\sigma \in \mathrm{Inv}_n(321)$

- $a(i) = A$, $W(i) > W(i-1) > 0$
  $W(i) < W(i-1) < 0$

- $a(i) = B$, $W(i-1) > W(i) > 0$
  $W(i-1) < W(i) < 0$

- $a(i) = C$, otherwise

match

A^s to B^s

B to A^s

&

C^s to C^s

to create $\sigma \in Iv_n(321)$

Main Idea

- $\frac{W_n}{n^{1/2}}$ is close to b_t

- $|\sigma(i) - i| = \begin{cases} pos_B(j) - pos_A(j) \\ pos_A(j) - pos_B(j) \\ 0 \end{cases}$ for some j

- $|W_n(i)| \approx$ (# of A^s up to i) $-$ (# of B^s up to i) ≥ 0

Main Idea

if $\quad i = pos_A(j)$ for some j

then

$$\frac{1}{n^{1/2}}\left(pos_B(j) - pos_A(j)\right) - \frac{1}{n^{1/2}}\Big[(\#A^s \text{ up to } i) - (\#B^s \text{ up to } i)\Big] \to 0$$

Main Idea

if $i = pos_A(j)$ for some j

then usually

$$\frac{1}{n^{1/2}}\left(pos_B(j) - pos_A(j)\right) -$$

$$\frac{1}{n^{1/2}}\left[(\#A^s \text{ up to } i) - (\#B^s \text{ up to } i)\right] \to 0$$

good sequence

Main Idea

$$P(\text{good sequence}) > 1 - \frac{1}{c} e^{-n^c}$$

$$\frac{1}{n^{1/2}}\left[pos_B(j) - pos_A(j) \right] - \frac{1}{n^{1/2}}\left[count_A(c) - count_B(c) \right] \to 0$$

good sequence

Main Idea

Lemma

$$P(\text{good sequence}) > 1 - \frac{1}{c} e^{-n^c}$$

(Modified)

Petrov Conditions

- $\left| \text{count}_X(s) - \text{count}_X(t) - \frac{1}{2}(s-t) \right| < |s-t|^{.6}$ or $n^{.1}$
- $\left| \text{pos}_X(i) - \text{pos}_X(j) - 2(i-j) \right| < |i-j|^{.6}$ or $n^{.1}$

Main Idea

(Modified) **Petrov Conditions** $X = A$ or B (not C)

- $\left| \mathrm{count}_X(s) - \mathrm{count}_X(t) - \frac{1}{2}(s-t) \right| < |s-t|^{.6}$ or $n^{.1}$
- $\left| \mathrm{pos}_X(i) - \mathrm{pos}_X(j) - 2(i-j) \right| < |i-j|^{.6}$ or $n^{.1}$

need to show # of C^s in an interval are relatively "small"


Thank You !!