Unimodality of q-twotorials via alternating gamma vectors

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The q-factorials

The n^{th} **q-integer** is $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$, and the n^{th} **q-factorial** is the product

$$[n]_q! := (1+q)(1+q+q^2)\ldots(1+q+q^2+\ldots+q^{n-1}).$$

For example,

$$[4]_q! = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6.$$

It is well-known that the q-factorials are both **palindromic** and **unimodal** (their coefficients satisfy $a_i = a_{d-i}$ and $a_0 \leq \cdots \leq a_{\lfloor d/2 \rfloor} \geq \cdots \geq a_d$ for their *palindromic degree* d). Each of the individual q-integers are both palindromic and unimodal, and these properties are preserved by multiplication.

The q-twotorials

We will call the polynomial $1+q^n$ the n^{th} **q-two** and define the n^{th} **q-twotorial** to be the analogous factorial-like product

$$(1+q)(1+q^2)\cdots(1+q^n).$$

For example,

$$(1+q)(1+q^2)(1+q^3) = 1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6.$$

Like the q-integers, each individual q-two is palindromic and it follows that the q-twotorials are palindromic too. Unlike the q-integers, NOT every q-two is unimodal, and proving the q-twotorials are unimodal is much more challenging. An algebraic proof was given by Stanley in 1989.

Main Question

We would like to have a combinatorial proof that the q-twotorials are unimodal. We expand this question from q-twotorials to all products of q-twos. Given a sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ whose sum is d, define

$$2_q^{[\alpha]} := (1+q^{\alpha_1})(1+q^{\alpha_2})\cdots(1+q^{\alpha_n}).$$

For α such that $2_q^{[\alpha]}$ is unimodal, is there a combinatorial proof of unimodality? For any $2_q^{[\alpha]}$, is there a cancellation-free interpretation for the *g-vector*?

The g-vectors

An important tool in proving the unimodality of a palindromic polynomial is its **g-vector**. The g-vector $g = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ of a palindromic polynomial can be computed simply by writing out its coefficients as a sum of centered vectors of all 1's. For the example q-factorial and q-twotorial we gave above, we have the following.

	1	3	5	6	5	3	1
1×	1	1	1	1	1	1	1
2×		1	1	1	1	1	
2×			1	1	1		
1×				1			

	1	1	1	2	1	1	1
1×	1	1	1	1	1	1	1
0×		1	1	1	1	1	
0×			1	1	1		
1×				1			

The g-vectors of $[4]_q!$ and $f(1,2,3;q) = (1+q)(1+q^2)(1+q^3)$ are $(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{1})$. Notice that these vectors are nonnegative, which is equivalent to the polynomial being unimodal.

The γ-vectors

The **γ-vectors** are analogous to g-vectors, but they use rows of Pascal's triangle instead of vectors of all 1's. For the examples above, we compute them as follows.

	1	3	5	6	5	3	1
1×	1	6	15	20	15	6	1
−3×		1	4	6	4	1	
2×			1	2	1		
0×				1			

	1	1	1	2	1	1	1
1×	1	6	15	20	15	6	1
−5×		1	4	6	4	1	
6×			1	2	1		
0×				1			

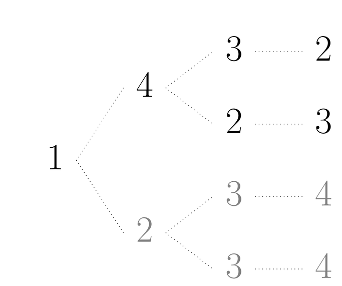
The γ-vectors of $[4]_q!$ and $f(1,2,3;q) = (1+q)(1+q^2)(1+q^3)$ are $(\mathbf{1},\mathbf{-3},\mathbf{2},\mathbf{0})$ and $(\mathbf{1},\mathbf{-5},\mathbf{6},\mathbf{0})$. Notice that these vectors are NOT nonnegative, but they do alternate in sign.

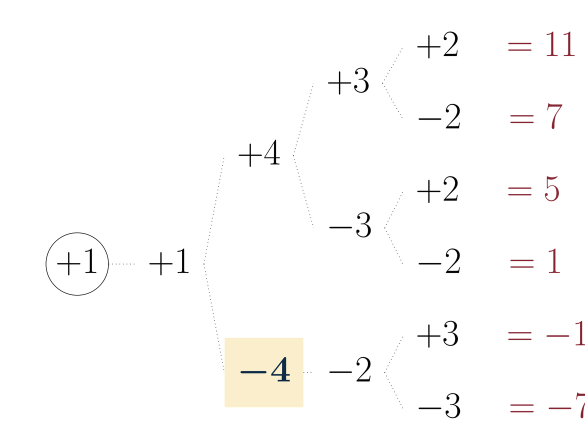
Main Strategy

We follow a suggestion by Brittenham et al. (2016) to exploit the combinatorics of the alternating γ-vectors. The alternating γ-vectors behave very nicely (their associated polynomials are multiplicative) and can be interpreted as counting *domino tilings*. The transformation from γ-vectors to g-vectors which can be interpreted using *ballot paths*. We combine these tilings and paths but we end up with a set that contains "negative" objects which should cancel with "positive" ones. We introduce a strategy to handle some of this cancellation which requires a structure we call a *g-tree*. In certain cases, we obtain a *positive* g-tree and a true proof of unimodality.

Binary trees of permutations and g-trees

A **binary tree of permutations** of $n > 1$ is a list $\bar{\pi}$ of 2^{n-2} permutations of n that can be represented as root-to-leaf paths on a vertex-labelled binary tree (see below left). We call it **synchronous** when each permutation is the same. A **g-tree** for α with respect to $\bar{\pi}$ is constructed by starting with a root $+1$ and following along in the binary tree, creating for each vertex labelled v two children labelled $+v$ and $-v$ respectively, but removing any child that creates a partial sum of zero in a root-to-leaf path. See the example for $\alpha = (1,2,3,4)$ below right, which comes from the $\bar{\pi}$ below left. In this example, we have removed paths +1-1 and +1+1-4+2. Note that the highlighted vertex -4 creates a **sign change** in the partial sum. A g-tree with no sign changes is called **positive**.





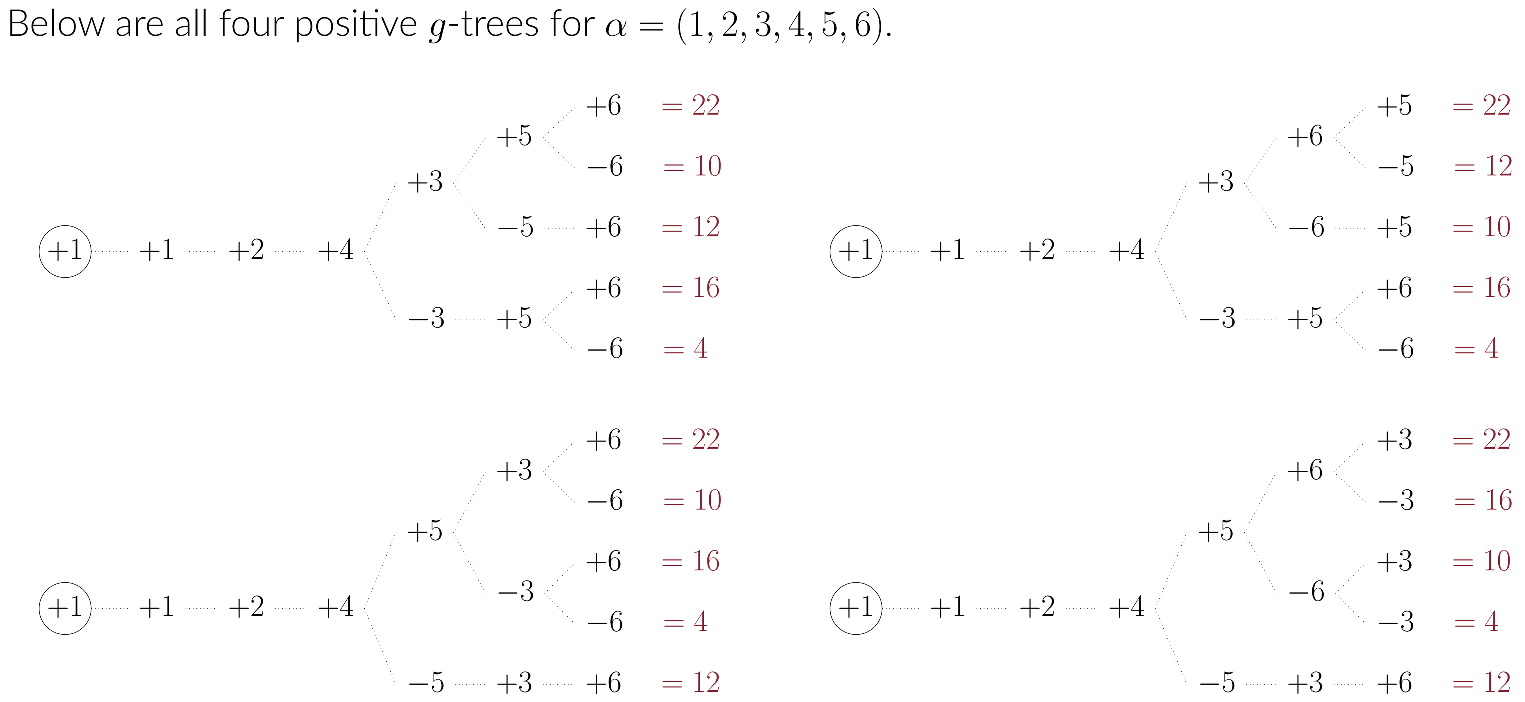
Main Theorem

Given any g-tree for α, the g-coefficient g_i is the number of maximal root-to-leaf paths whose sum is $d - 2i + 1$ minus the number whose sum is $-(d - 2i + 1)$.

In particular, if α has a positive g-tree, then its g-vector enumerates the root-to-leaf paths and we have a combinatorial proof of unimodality.

Example

Below are all four positive g-trees for $\alpha = (1,2,3,4,5,6)$.



We can see that all of our trees have branch sums of 22, 16, 12, 10, and 4. Using our main theorem, we can interpret g_i. For example, we know that $g_3 = 1$ because $i = 3$ and $d = 21$, so $d - 2i + 1 = 16$ and there is 1 branch with sum 16. The entire g-vector is (1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0).

Consequences

We were able to find positive g-trees for most of the q-twotorial cases up to $n = 24$. The table below shows for which n we can prove unimodality of the n^{th} q-twotorial with an g-tree.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	_	✓	_	✓	✓	_	✓	✓	_	✓	✓	✓	✓	✓

For the more general question about all products of q-twos, we have some sufficient conditions on α for the existence of positive g-trees. For example, given a composition of the form $\alpha = (1, 2, 4, \ldots, 2^{k-1}, \alpha_{k+1}, \ldots, \alpha_n) \vdash d$, there is a positive g-tree of α beginning $+1{+1}{+2}{+4}{+}\cdots{+2^{k-1}}$ with **maximal size** 2^{n-k} or $2^{n-k} - 1$ if and only if

$$\sum_{i=k+1}^{n} \alpha_i \leq 2^k.$$

The following table shows how many sequences α of given lengths exist with certain properties. The property in each row implies the properties in previous rows.

len α	1	2	3	4	5	6	7	8	9
$2_q^{[\alpha]}$ is unimodal	1	2	5	13	42	149	653	3369	21304
positive g-tree	1	2	5	13	41	145	626	3203	20047
synchronous positive g-tree	1	2	5	13	40	141	595	3019	18831
maximal size g-tree $+1{+1}{+2}\cdots{+2^{k-1}}\cdots$	1	2	5	13	37	121	477	2328	14328

Non-unimodal cases

We can use our main theorem to give combinatorial proofs of non-unimodality for some cases, namely when we are able to give a *cancellation-free* g-tree. As before, the following table shows how many sequences α of given lengths exist with certain properties. The property in each row implies the properties in previous rows.

len α	1	2	3	4	5	6	7	8
$2_q^{[\alpha]}$ has non-zero coefficients	1	2	6	27	192	2280	47097	1735803
cancellation-free g-tree	1	2	6	27	188	2134	40532	1313411
synchronous cancellation-free g-tree	1	2	6	25	164	1693	29414	874071

The first row above is A003513, because such α are essentially *regular sequences*, which appear in the study of subjective probability in mathematical psychology.

Proof concept

We will work through the main ideas of our proof using the example $2_q^{[1,2,3]} = (1+q)(1+q^2)(1+q^3)$. First, we'll make a sequence `- | - - | - - -` which has three parts (of sizes 1, 2, and 3). Next, we are going to construct **domino tilings** on this by placing "dominos" `u d` on the parts such that the `d` immediately follows the `u` or the `u` is the end and the `d` appears back around at the start. Below are all of the possible domino tilings, and these are counted by the γ-vector of $2_q^{[1,2,3]}$.

$\gamma_0 = 1$	`-	- -	- - -`
$\gamma_1 = -5$	`-	u d	- - -`, `-	d u	- - -`, `-	- -	u d -`, `-	- -	- u d`, `-	- -	d - u`
$\gamma_2 = 6$	`-	u d	u d -`, `-	u d	- u d`, `-	u d	d - u`, `-	d u	u d -`, `-	d u	- u d`, `-	d u	d - u`
$\gamma_3 = 0$	none

Each domino creates a negation, so the tilings with one domino are "negative" objects, and the tilings with two dominos are "positive" objects (negated twice).

To get objects counted by the g-vector, we will fill in the remaining locations in our tiling with a **ballot path**, which is a sequence of `U`'s and `D`'s such that while reading from left to right there are never more `D`'s than `U`'s. Below are all of the combined domino tiling/ballot paths where the combination has a total of 3 `D`/`d`'s (this is what we need for g_3 specifically).

`U	U D	U D D`	`U	U D	D U D`	`U	D U	U D D`	`U	D U	D U D`	`U	u d	u d D`
`U	u d	U D D`	`U	u d	D U D`	`U	d u	U D D`	`U	d u	D U D`	`U	U D	u d D`
`U	u d	D u d`	`U	u d	d D u`	`U	d u	u d D`	`U	d u	D u d`	`U	d u	d D u`
`U	U D	D u d`	`U	U D	d D u`	`U	D U	u d D`	`U	D U	D u d`	`U	D U	d D u`
		`U	U U	D D D`		

Most of the objects above appear in positive–negative pairs (defined by the first swap between a `u d` and `U D`), which cancel out when we count. Only one `U | U U | D D D` remains, so we get $g_3 = 1$ for our count.

The strategy we outlined above works whenever we arrange the blocks in an order that agrees with a synchronous positive g-tree. We saw that the order 1, 2, 3 worked above because (+1) +1 +2 $^{+3}_{-3}$ is a synchronous positive g-tree. In fact, the fixed point we previously found is counted by the lower branch (1 up, 2 ups, 3 downs). However, there is is not an g-tree that corresponds to the order 1, 3, 2. It does not meet the requirements because 3 is greater than 2, which would be the sum up to that point. And indeed the strategy above fails for said order. If we try to match up positive-negative pairs as before, we encounter a problem with `U | d D u | U U` in which we cannot capitalize the domino, because a ballot path cannot begin `U D D`.

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References

[1] Charles Brittenham, Andrew T Carroll, T Kyle Petersen, and Connor Thomas. Unimodality via alternating gamma vectors. *The Electronic Journal of Combinatorics*, 23(2):P2–40, 2016.

[2] Bruce E Sagan and Jordan Tirrell. Lucas atoms. *Advances in Mathematics*, 374:107387, 2020.

[3] Richard P Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. *Ann. New York Acad. Sci*, 576(1):500–535, 1989.