Connections between permutation clusters and generalized Stirling permutations

Justin Troyka
Davidson College

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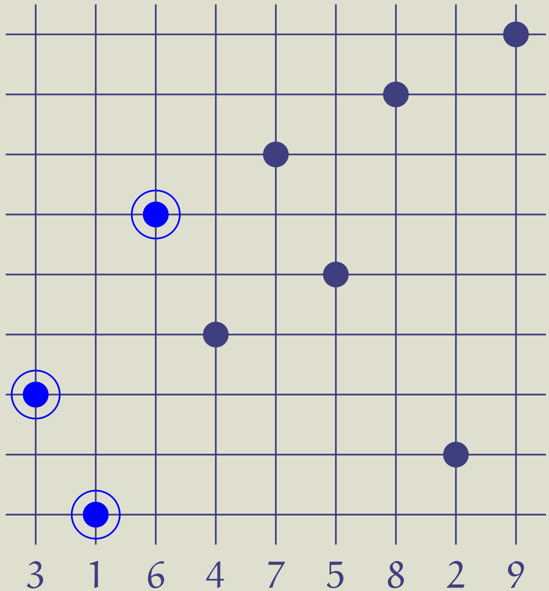
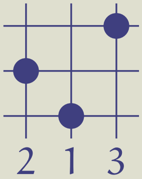


- The highlighted dots are a consecutive occurrence of



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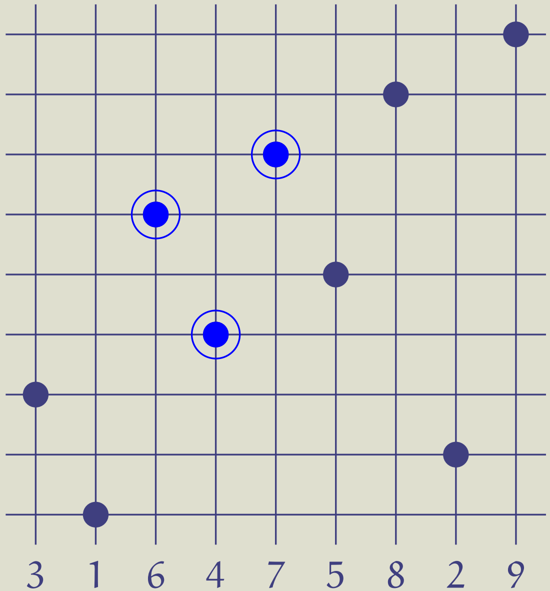

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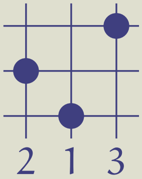

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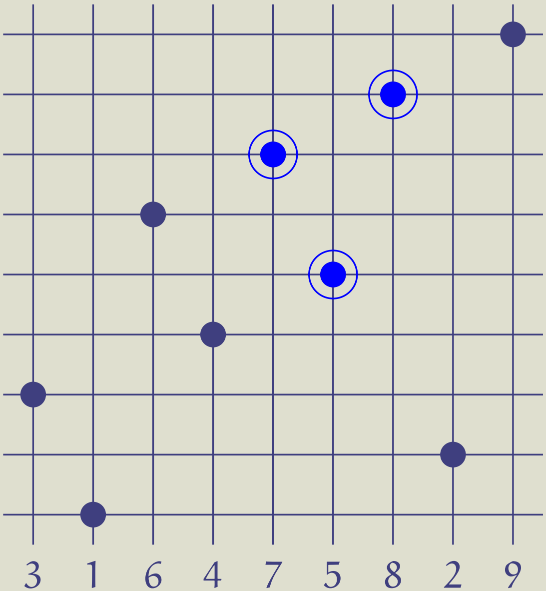

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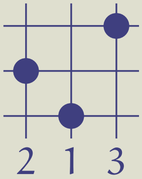

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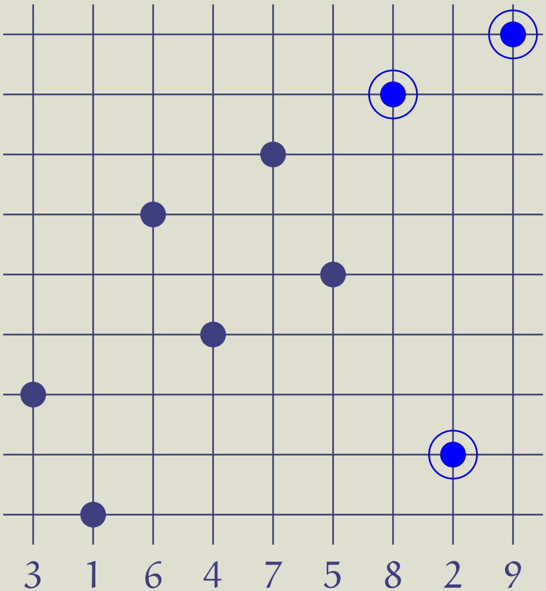

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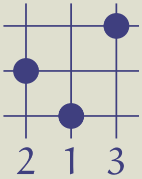

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- Thus we say 316475829 is a 213 cluster.

Permutation clusters and permutation statistics

- $\pi \in S_{2k+1}$ is a 213 cluster if for every i the consecutive pattern $\pi(2i+1)\,\pi(2i+2)\,\pi(2i+3)$ has the same relative order as 213.
- More generally, for $m \geqslant 2$, $\pi \in S_{mk+1}$ is a $2134\ldots(m+1)$ cluster if for every i the consecutive pattern $\pi(mi+1)\,\pi(mi+2)\,\ldots\,\pi(mi+m+1)$ has the same relative order as $2134\ldots(m+1)$.
- Given a permutation π:
  - the descent number of π is $\mathrm{des}(\pi) = \#\{j\colon \pi(j) > \pi(j+1)\}$;
  - the peak number is $\mathrm{pk}(\pi) = \#\{j\colon \pi(j-1) < \pi(j) > \pi(j+1)\}$;
  - the inverse descent number of π is $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1})$;
  - the inverse peak number of π is $\mathrm{ipk}(\pi) = \mathrm{pk}(\pi^{-1})$.
- We have found the enumeration of $2134\ldots(m+1)$ clusters refined by ides and refined by ipk.
- Using Zhuang's cluster method, we can use the enumeration of $2134\ldots(m+1)$ clusters refined by ides (and ipk) to count the $2134\ldots(m+1)$ avoiders by ides (and ipk).

Characterizing 213 clusters

Proposition (T. & Zhuang 2022+): Let $\pi \in S_{2k+1}$ for $k \geqslant 1$. Then π^{-1} is a 213 cluster if and only if these two conditions are met:

(i) The odd values of π form an increasing subsequence; that is, if $\pi(j), \pi(j')$ are odd and $j < j'$, then $\pi(j_1) < \pi(j_2)$.

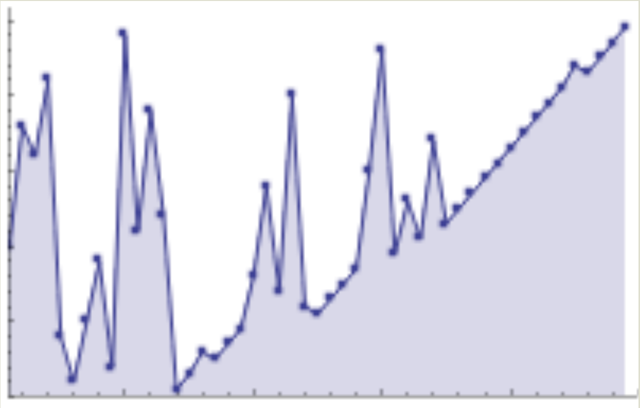
(ii) Each even value $2i$ comes before the value $2i-1$ in π; that is, $\pi^{-1}(2i) < \pi^{-1}(2i-1)$.

We write $\mathcal{P}_k$ to denote the set of permutations $\pi \in S_{2k+1}$ satisfying conditions (i) and (ii) — that is, $\mathcal{P}_k$ is the set of inverses of 213 clusters.

- The inverse of the first example is $281463579 \in \mathcal{P}_4$.

Studying ides and ipk on 213 clusters is the same as studying des and pk on $\mathcal{P}_k$.

A random element of $\mathcal{P}_{24}$



Open question: What should we call the elements of $\mathcal{P}_k$?
Conjecture: We should call them toothbrush permutations.

The recurrence relation for des on $\mathcal{P}_k$

- Define $p^{\mathrm{des}}(k,i) = \#\{\pi \in \mathcal{P}_k \colon \mathrm{des}(\pi) = i\}$.
- Recall the recurrence relation on the Eulerian numbers $A(n,i)$:

$$A(n,i) = i\,A(n-1,i) + (n-i+1)\,A(n-1,i-1)$$

- This is proved by looking at where the value n is inserted into a length-$(n-1)$ permutation and whether it creates a new descent.
- We use the same type of argument to prove:

Proposition (T. & Zhuang 2022+):
$p^{\mathrm{des}}(k,i) = i\,p^{\mathrm{des}}(k-1,i) + (2k-i)\,p^{\mathrm{des}}(k-1,i-1)$.

Stirling permutations

Proposition (T. & Zhuang 2022+):
$p^{\mathrm{des}}(k,i) = i\, p^{\mathrm{des}}(k-1,i) + (2k-i)\, p^{\mathrm{des}}(k-1,i-1)$.

- This is almost the same recurrence relation for the des statistic on Stirling permutations!

A Stirling permutation of size $2k$ is a permutation ρ of the multiset $\{1,1,2,2,\ldots,k,k\}$ such that the values between two t's are all at least t; that is, if $a<b<c$ and $\rho(a)=\rho(c)$, then $\rho(b) \geqslant \rho(a)$.

Let $\mathcal{Q}_k$ denote the set of Stirling permutations of size $2k$. (First studied by Gessel & Stanley 1978.)

- Example: $14412332 \in \mathcal{Q}_4$.

Connecting Stirling permutations to 213 clusters

For $\rho \in \mathcal{Q}_k$, define $\mathrm{des}(\rho) = \#\{j \colon \rho(j) > \rho(j+1)\}$, and define $q^{\mathrm{des}}(k,i) = \#\{\rho \in \mathcal{Q}_k \colon \mathrm{des}(\rho) = i\}$.

Proposition (Gessel & Stanley 1978):
$q^{\mathrm{des}}(k,i) = (i+1)\, q^{\mathrm{des}}(k-1,i) + (2k-i-1)\, q^{\mathrm{des}}(k-1,i-1)$.

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Theorem (T. & Zhuang 2022+): $p^{\mathrm{des}}(k,i+1) = q^{\mathrm{des}}(k,i)$; that is, the number of permutations in $\mathcal{P}_k$ with $i+1$ descents is equal to the number of Stirling permutations in $\mathcal{Q}_k$ with i descents.

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The statistic on $\mathcal{Q}_k$ analogous to peak number is plateau–descent number: for $\rho \in \mathcal{Q}_k$,

$$\mathrm{plde}(\rho) = \#\{j\colon \rho(j-1) = \rho(j) > \rho(j+1)\}.$$

Also define p^{pk} and q^{plde} similarly as above. Then:

Theorem (T. & Zhuang 2022+): $p^{\mathrm{pk}}(k,i) = q^{\mathrm{plde}}(k,i)$; that is, the number of permutations in $\mathcal{P}_k$ with i peaks is equal to the number of Stirling permutations in $\mathcal{Q}_k$ with i plateau–descents.

Further considerations

- Valleys can be done the same way.
- These results generalize to $2134\dots(m+1)$ clusters, for which des and pk have the same distributions as des and plde on the set of m-Stirling permutations: permutations of the multiset $\{1^m,\dots,k^m\}$ such that the values between any two t's are all at least t.
- By taking the reverse, the complement, or the reverse–complement of 213 clusters, we obtain results on 312 clusters, 231 clusters, and 132 clusters, which translate in a straightforward way from our results on 213 clusters. (Same goes for general m.)

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- By taking the reverse, the complement, or the reverse–complement of 213 clusters, we obtain results on 312 clusters, 231 clusters, and 132 clusters, which translate in a straightforward way from our results on 213 clusters. (Same goes for general m.)
- The numbers $p^{\mathrm{des}}(k, i+1) = q^{\mathrm{des}}(k, i)$ are the mth-order Eulerian numbers, introduced by Gessel 1978.
- The numbers $p^{\mathrm{pk}}(k, i) = q^{\mathrm{plde}}(k, i)$ are the $(1/2)$-Eulerian numbers, introduced by Savage and Viswanathan 2012. Sadly we do not get the $(1/m)$-Eulerian numbers for $m \geqslant 3$.