The shallow permutations are the unlinked permutations

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Statistics motivation

It is 400 BC, you have reports from various footraces among the same runners in preparation for the Olympics, and the only information you have is the order in which the runners finished.

You want to do statistics to get the "true ranking" of the runners from fastest to slowest.

To do statistics, you want some measurement of the "noise" in your data and how it is distributed.

You might have thought there was something not quite right about your elementary school teacher marking matching or ranking problems by the number of pairs incorrectly matched or items incorrectly ranked. (It's not possible to get exactly 1 wrong!) What are alternatives?

Some metrics on permutations

Diaconis and Graham (in the 1970s, in a statistics journal!) studied several measures of "noise":

- **Kendall's tau** (aka inversions):
  $\ell(w) = \#\{(i,j) \mid i < j, w(i) > w(j)\}$
- **Spearman's disarray** (aka total displacement)
  $d(w) = \sum_i |w(i) - i|$
- **reflection length** $t(w) = n - c(w)$, where $c(w)$ is the number of cycles.

Shallow permutations

Diaconis and Graham proved that

$$d(w) \geq \ell(w) + t(w)$$

for all permutations w and asked for characterizations of those w for which we had equality.

Following Petersen and Tenner, who defined **depth** for an arbitrary reflection group and showed that depth is always half of total displacement for permutations, we call these permutations **shallow** permutations.

Shallow permutations do not form a pattern class (but their intersection with $Av(4231)$ does!)

Examples

Let $w = 45231$. Then $\ell(w) = 8$, $d(w) = 12$, $t(w) = 4$.

Let $w = 34512$. Then $\ell(w) = 6$, $d(w) = 12$, $t(w) = 4$.

Note 45231 is shallow, but 34512 and 3412 are not.

Knots, links, and diagrams

A **knot** is an isotopy class of an embedding of a circle into $\mathbb{R}^3$. A **link** is the same for several circles.

Given a knot or link, one can draw a **link diagram**, an almost 2-dimensional picture of the knot/link. A link has many diagrams that don't look like each other.

Cycle diagrams and knots

Cornwell and McNew associated to each permutation a link as follows.

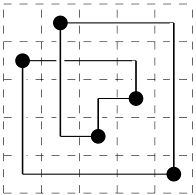
Draw the cycle diagram of the permutation:

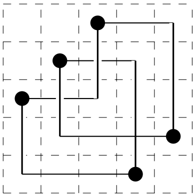
- Draw the dots $(i, w(i))$ and (i, i) for each i, and
- Connect dots with vertical and horizontal lines

To make a link diagram, have all vertical lines cross over all horizontal lines.

Unlinked permutations

A permutation is **unlinked** if its knot diagram is that of an unlink.
For example, 53412 is unlinked but 34512 is not:



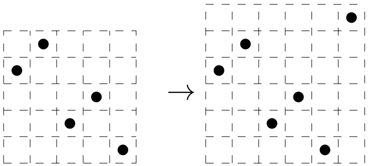


Main theorem

The shallow permutations are the unlinked permutations.

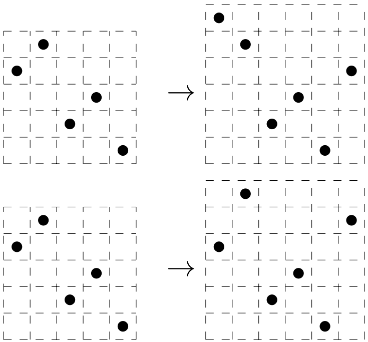
Characterization of shallow permutations, I

Hadjicostas and Monico showed that shallow permutations can all be recursively constructed as follows:
Either take a shallow permutation in S_{n-1} and add an n at the end



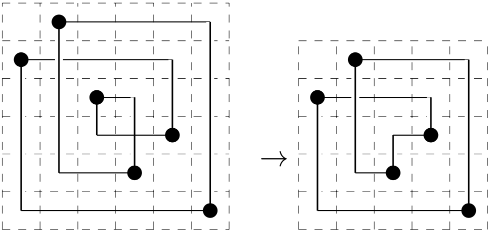
Characterization of shallow permutations, II

Take a shallow permutation in S_{n-1} and a left to right minimum or right to left maximum, and spawning two new “dots” from it above and to the right.



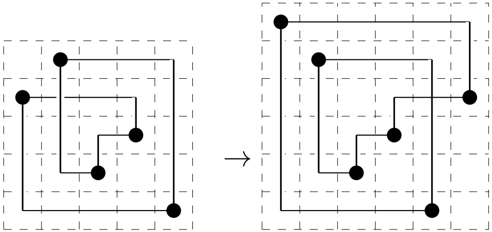
Characterization of unlinked permutations

Cornwell and McNew show that every unlinked permutation has some index i with displacement at most 1: $w(i) - i \in \{-1, 0, 1\}$, and removing that "dot" creates another unlinked permutation.



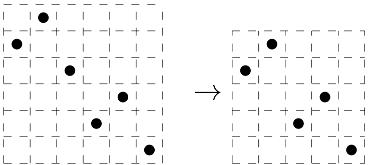
Proof shallow permutations are unknotted

To show shallow permutations are unknotted, we observe the Hadjicostas–Monico “spawning” operation doesn’t change the knot type:



Proof unknotted permutations are shallow

To show unknotted permutations are unlinked, we observe that removing a dot with $w(i) - i \in \{-1, 0, 1\}$ always preserves $d(w) - \ell(w) - t(w)$.



We have $d(564231) = 18$, $\ell(564231) = 13$, and $d(45231) = 13$, $\ell(45231) = 8$.

Consequences

Cornwell and McNew show that unknotted cycles are counted by the Schröder numbers, with a bijection to rooted signed binary plane trees, and the generating function G for unknotted permutations satisfies

$$x^2G^3 + (x^2 - 3x + 1)G^2 + (3x - 2)G + 1 = 0.$$

Hence the shallow permutations also satisfy the same recurrence.

Berman and Tenner also show shallow cycles are counted by the Schröder numbers, with a bijection to separable permutations.

Thank you

Thank you for your attention!