An extension of the Lindström-Gessel-Viennot Theorem

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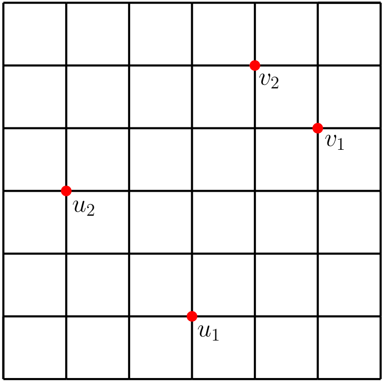
Outline

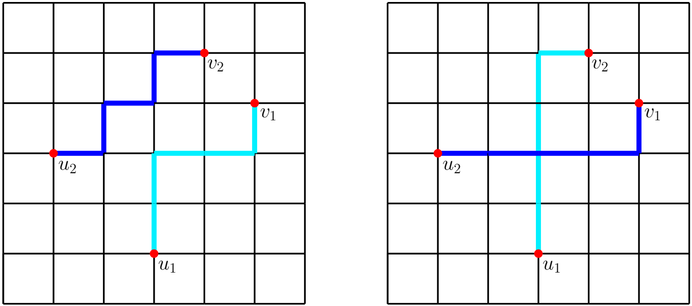
1. Two Questions
2. The Lindström-Gessel-Viennot Theorem
3. New Results
4. Application—Tiling Problems

Question 1

Consider the square lattice with the horizontal edges oriented east, vertical edges oriented north. Let $U = \{u_1, u_2\}$ be the set of starting points and $V = \{v_1, v_2\}$ be the set of ending points. How many families of non-intersecting lattice paths from U to V do we have?



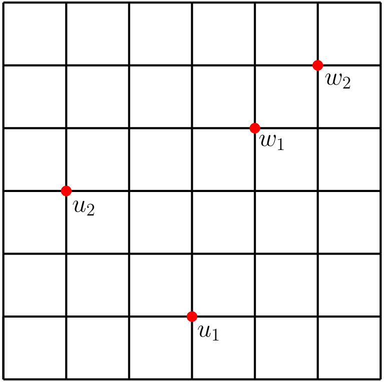
- Two paths are non-intersecting if they do not pass through the same vertex.
- A family of paths is non-intersecting if any two of the paths is non-intersecting.



(a) Non-intersecting paths (b) Intersecting paths

Question 2

Now, if we choose two different ending points $W = \{w_1, w_2\}$, then how many families of non-intersecting lattice paths from U to W do we have?



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- $\mathscr{P}^{\pi}(U, V)$ is a set of n-tuples of paths $(p_1, \ldots, p_n)$, where $p_i \in \mathscr{P}(u_i, v_{\pi(i)})$. The permutation π is called the connection type.

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- $\mathscr{P}_0^{\pi}(U, V)$ is a set of n-tuples of non-intersecting paths of the connection type π.

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We denote the weighted sum of the set of (n-tuples) of paths $\mathscr{P}$ by

$$GF(\mathscr{P}) = \sum_{p \in \mathscr{P}} \mathrm{wt}(p).$$

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Theorem [Lindström '73], [Gessel,Viennot '85], [Stembridge '90]

Let G be a directed acyclic graph. Suppose $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ are two distinct sets of vertices of G. Then

$$\sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) GF(\mathscr{P}_0^{\pi}(U, V)) = \det(M),$$

where the (i, j)-entry of the matrix M is given by $GF(\mathscr{P}(u_i, v_j))$.

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Two sets of the vertices $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ are compatible if n-tuples of non-intersecting paths only consist of paths connecting u_i to v_i for $i = 1, \dots, n$, i.e. the connection type $\pi = \mathrm{id}$.

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Corollary

If $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ are compatible, then we have

$$GF(\mathscr{P}_0^{\mathrm{id}}(U, V)) = \det(M),$$

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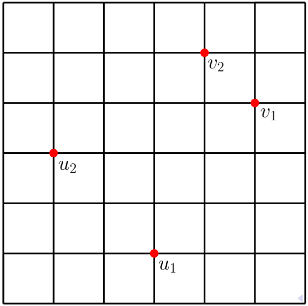
- enumeration of various types of plane partitions
- enumeration of lozenge and domino tilings
- evaluation of the Hankel determinants
- combinatorial proof of the determinent formulas (e.g. the Cauchy-Binet formula)
- combinatorial proof of the Jacobi-Trudi type identites for Schur functions.
- ...

A survey paper: Christian Krattenthaler, Lattice path enumeration, 2017. `https://arxiv.org/abs/1503.05930`

Let us go back to the first question.

- Notice that it is impossible to have a non-intersecting path connecting u_1 to v_2 and u_2 to v_1. The sets $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are compatible.



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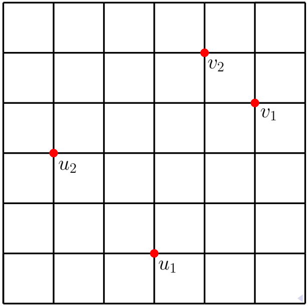
- $|\mathscr{P}(u_1, v_1)| = \binom{5}{2} = 10$, $|\mathscr{P}(u_1, v_2)| = \binom{5}{1} = 5$, $|\mathscr{P}(u_2, v_1)| = \binom{5}{4} = 5$ and $|\mathscr{P}(u_2, v_2)| = \binom{5}{3} = 10$.



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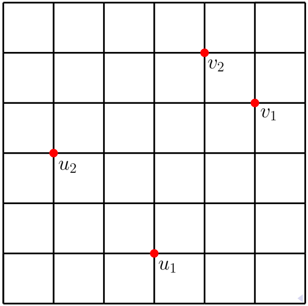
- Therefore, $|\mathscr{P}_0^{\text{id}}(U, V)| = \det \begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix} = 75.$



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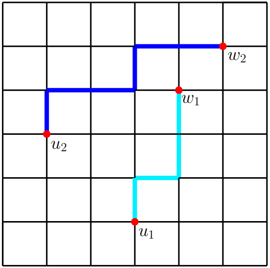
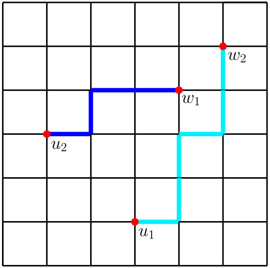
- Both connection types are possible in question 2, the sets $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$ are NOT compatible.





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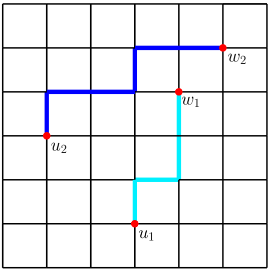
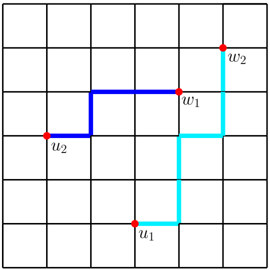
- We can still apply the LGV theorem and obtain the "signed" enumeration: $\mathrm{sgn}(\mathrm{id})GF(\mathscr{P}_0^{\mathrm{id}}(U,W)) + \mathrm{sgn}((12))GF(\mathscr{P}_0^{(12)}(U,W)) = \det M.$





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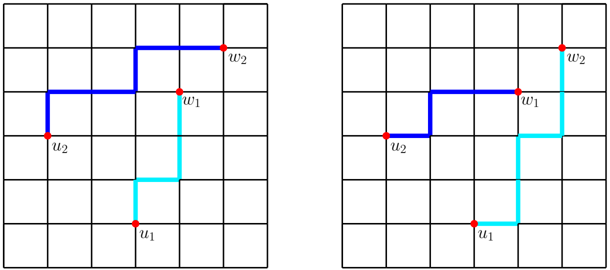
- How to find the total number of families of non-intersecting paths?



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Setting:

- Let G be a directed acyclic graph with the special property*.
- Let $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ be two sets of distinct vertices of G, not necessarily compatible.
- The new result is the "straight" enumeration

$$\sum_{\pi \in \mathfrak{S}_n} GF(\mathscr{P}_0^{\pi}(U, V)) = |\det M^*|,$$

where the (i, j)-entry of M^* depends only on the paths from u_i to v_j.

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Key ideas:

- We associate a sign (called the path sign) on each path from u_i to v_j.
- We form a new matrix M^*.
- These path signs in the det M^* cancel out the effect of the permutation signs on the LHS of the LGV theorem.

New Results—Special Property of G

Definition

An upward planar drawing of G is a drawing of G on the Euclidean plane such that

- each edge is drawn as a line segment that is either horizontal or up-pointing, and
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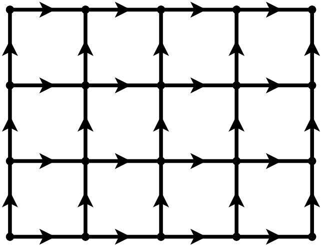
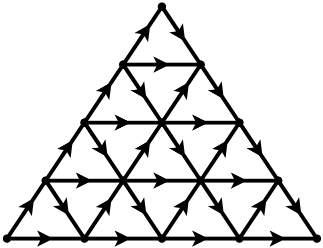
Examples:





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An *st*-planar graph is a planar, acyclic digraph with one source (a vertex with no incoming edges) and one sink (a vertex with no outgoing edges), so that these two special vertices lie on the outer face of the graph.

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Theorem [Di Battista et al. '98]

A graph G has an upward planar drawing if and only if G is a subgraph of an *st*-planar graph $\widetilde{G}$ on the same vertex set.

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Now, we consider a directed acyclic graph G having an **upward planar drawing**, which is a subgraph of an *st*-planar graph $\widetilde{G}$.

New Results—Path Sign

Let s be the source and t be the sink of the st-planar graph $\widetilde{G}$, and let p be a path in the subgraph G.

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Definition

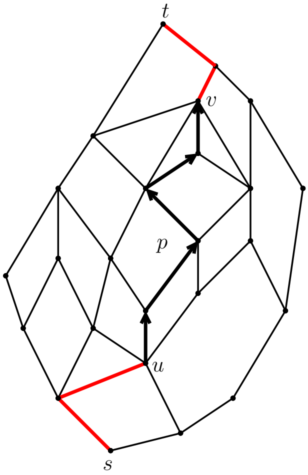
The left side of the path $p \in \mathscr{P}(u, v)$ is the closed region of the plane bounded by the following paths in $\widetilde{G}$:

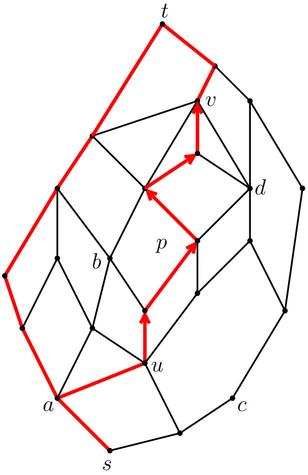
- the leftmost path from s to u,
- the path p itself,
- the leftmost path from v to t, and
- the left boundary of $\widetilde{G}$ going from s to t.

Let $L(p)$ be the collection of the starting and ending points of $U \cup V$ which are on the left side of the path p (including u and v).

New Results—Path Sign

The left side of the path p is the region bounded by red edges.





New Results—Path Sign

Definition

The path sign of a path $p \in \mathscr{P}(u, v)$ is defined to be

$$\operatorname{sgn}(p) = (-1)^{|L(p)|}.$$

In the previous graph, suppose $U \cup V = \{a, b, c, d, u, v\}$, then $L(p) = \{a, b, u, v\}$ and hence $\operatorname{sgn}(p) = (-1)^4 = 1$.

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Then the total weight of families of non-intersecting paths connecting U to V is given by

$$\sum_{\pi \in \mathfrak{S}_n} GF(\mathscr{P}_0^{\pi}(U, V)) = \left|\det M^*\right|.$$

New Results—Question 2

Apply the main theorem to question 2, our goal is to find the matrix $M^* = (m_{ij})$, where $1 \leq i, j \leq 2$.

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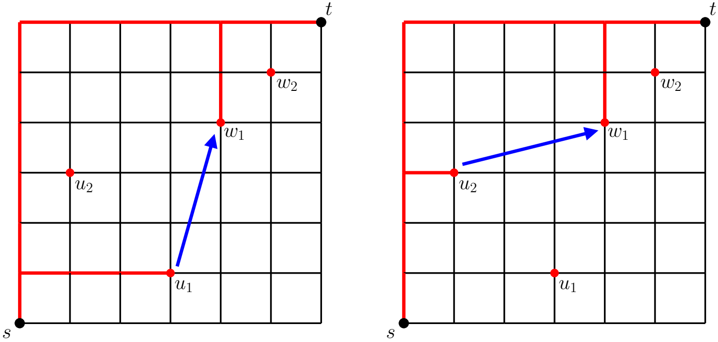
	$L(p)$	$\text{sgn}(p)$	size	matrix
$p \in \mathscr{P}(u_1, w_1)$	$\{u_1, w_1, u_2\}$	$(-1)^3$	$\binom{4}{1}$	$m_{11} = -4$
$p \in \mathscr{P}(u_2, w_1)$	$\{u_2, w_1\}$	$(-1)^2$	$\binom{4}{3}$	$m_{21} = 4$



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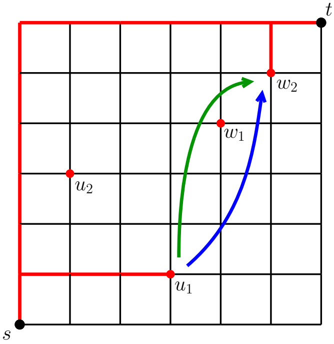
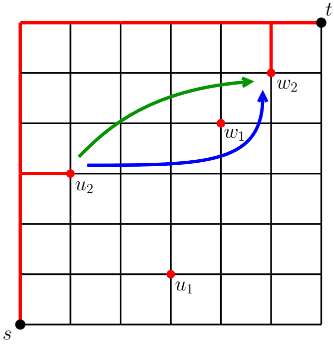
	$L(p)$	$\text{sgn}(p)$	size	matrix
$\mathbf{p} \in \mathscr{P}(u_1, w_2)$	$\{u_1, w_2, u_2\}$	$(-1)^3$	1	$m_{12} = -1 + 14$
$\mathbf{p} \in \mathscr{P}(u_1, w_2)$	$\{u_1, u_2, w_1, w_2\}$	$(-1)^4$	$\binom{6}{2} - 1$	
$\mathbf{p} \in \mathscr{P}(u_2, w_2)$	$\{u_2, w_2\}$	$(-1)^2$	$\binom{4}{2}$	$m_{22} = 6 - 9$
$\mathbf{p} \in \mathscr{P}(u_2, w_2)$	$\{u_2, w_1, w_2\}$	$(-1)^3$	$\binom{6}{2} - \binom{4}{2}$	





New Results—Question 2

We have $m_{11} = -4$, $m_{12} = 13$, $m_{21} = 4$, $m_{22} = -3$.
By the main theorem, the number of families of non-intersecting paths in question 2 is given by

$$|\det M^*| = \left|\det \begin{pmatrix} -4 & 13 \\ 4 & -3 \end{pmatrix}\right| = 40.$$

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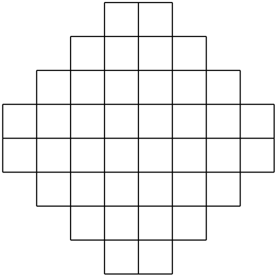
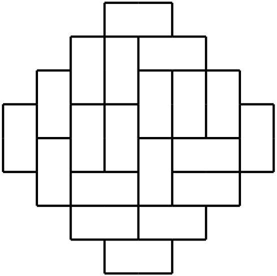
An example of a domino tiling (covered by 1×2 and 2×1 rectangles) of the Aztec diamond of order 4.

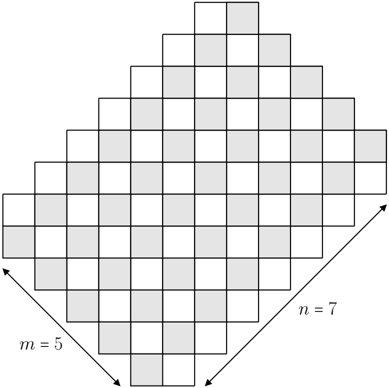


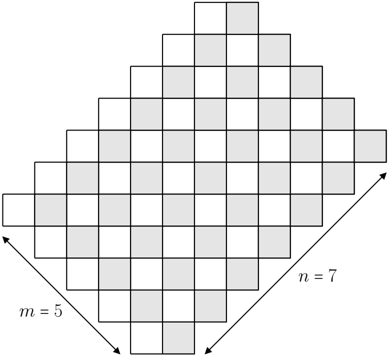


How many tilings do we have? Is there a "nice" formula?

Application—Mixed Aztec Rectangle

The Aztec rectangle (left) and the mixed Aztec rectangle (right) with the checkerboard coloring.

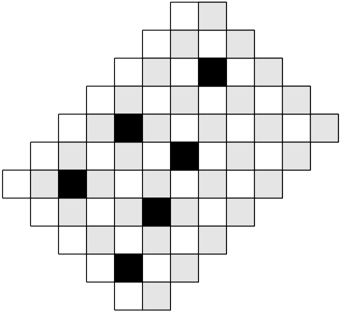


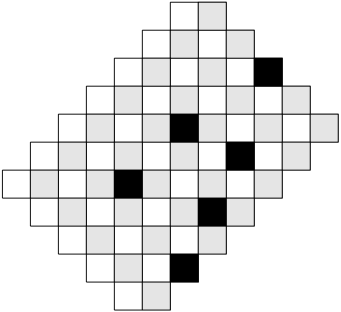


Application—Translation Invariant

Theorem [L. '22]

The number of domino tilings of the mixed Aztec rectangle with **arbitrary unit holes** is invariant under color-preserving translations of the set of holes, provided all the unit holes are still contained in the region.





Application

We consider the mixed Aztec diamond $MR_{m,n}(a, b, c)$ with four unit holes along a common horizontal line, from left to right, with colorings white-white-black-black and the spacing between them are $2a - 1$, $2b$ and $2c - 1$ units.

Theorem [L. '22]

The number of domino tilings of the region $MR_{m,n}(a, b, c)$ is given by

$$2d_{b,b}\left(\sum_{i=1}^{a}\sum_{j=1}^{c} r_{b+i+j-1}d_{a-i,a-i}d_{c-j,c-j}\right),$$

where $d_{n,k}$ is the Delannoy number and r_n is the large Schröder number. In particular, the number only depends on the separations of the four holes, and not on m, n, or the position of the left hole.

Thank You