A Lifting of the Goulden–Jackson Cluster Method to the Malvenuto–Reutenauer Algebra

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Permutation Patterns 2022

Exciting News!

Based on the paper:

Zhuang, Yan. A lifting of the Goulden–Jackson cluster method to the Malvenuto–Reutenauer algebra, `arXiv:2108.10309`.

Accepted (this morning!!!) pending minor revisions to *Algebraic Combinatorics*.

Consecutive Patterns

- Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$.
- Let $\mathfrak{S} = \bigcup_{n=0}^{\infty} \mathfrak{S}_n$.

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Let $\pi = 6351427$. Then 351 is an occurrence of 231 in π (but 352 is not).

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Example

Let $\pi = 6351427$. Then 351 is an occurrence of 231 in π (but 352 is not).

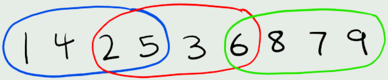
- Let $\text{occ}_\sigma(\pi)$ be the number of occurrences of σ in π.
- Given $\sigma \in \mathfrak{S}$, let $\mathfrak{S}_n(\sigma)$ denote the set of all permutations of length n which avoid σ.

Clusters

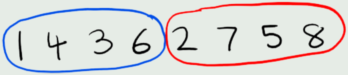
- Given $\sigma \in \mathfrak{S}$, a σ-cluster is a permutation filled with marked occurrences of σ that overlap with each other.

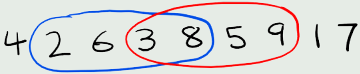
Example

An example of a 1324-cluster is



Two non-examples:



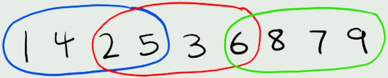


Clusters (cont.)

- Let $C_{\sigma,\pi}$ be the set of σ-clusters with underlying permutation π.
- Given a σ-cluster c, let $\mathrm{mk}_\sigma(c)$ be the number of marked occurrences of σ in c.

Example

If c is the cluster



then $c \in C_{1324,142536879}$ and $\mathrm{mk}_{1324}(c) = 3$.

The Goulden–Jackson Cluster Method for Permutations

- Let
$$F_\sigma(s,x) = \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{occ}_\sigma(\pi)} \frac{x^n}{n!} \text{ and}$$
$$R_\sigma(s,x) = \sum_{n=2}^{\infty} \sum_{\pi \in \mathfrak{S}_n} \sum_{c \in C_{\sigma,\pi}} s^{\mathrm{mk}_\sigma(c)} \frac{x^n}{n!}.$$

Theorem (Elizalde–Noy 2012)

Let $\sigma \in \mathfrak{S}$ have length at least 2. Then
$$F_\sigma(s,x) = \frac{1}{1 - x - R_\sigma(s-1,x)}.$$

- Setting $s = 0$: $\displaystyle\sum_{n=0}^{\infty} |\mathfrak{S}_n(\sigma)| \frac{x^n}{n!} = \frac{1}{1 - x - R_\sigma(-1,x)}.$

The Malvenuto–Reutenauer Algebra

- Let $\mathbb{Q}[\mathfrak{S}]$ denote the $\mathbb{Q}$-vector space with basis $\mathfrak{S}$. The Malvenuto–Reutenauer algebra is the $\mathbb{Q}$-algebra on $\mathbb{Q}[\mathfrak{S}]$ with multiplication

$$\pi \cdot \sigma = \sum_{\tau \in C(\pi,\sigma)} \tau$$

where $C(\pi, \sigma)$ is the set of all concatenations of π and σ.

Example

$12 \cdot 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421$

The Cluster Method in Malvenuto–Reutenauer

- Given $\sigma \in \mathfrak{S}$, let

$$F_\sigma(s) = \sum_{\pi \in \mathfrak{S}} \pi s^{\mathrm{occ}_\sigma(\pi)} \quad \text{and} \quad R_\sigma(s) = \sum_{\pi \in \mathfrak{S}} \sum_{c \in C_{\sigma,\pi}} \pi s^{\mathrm{mk}_\sigma(c)}.$$

Theorem (Z. 2022+)

Let $\sigma \in \mathfrak{S}$ have length at least 2. Then

$$F_\sigma(s) = \Big(\varepsilon - \iota - R_\sigma(s-1)\Big)^{-1}$$

where ε is the empty permutation and ι the permutation of length 1.

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- Define $\Phi\colon \mathbb{Q}[\mathfrak{S}] \to \mathbb{Q}[[x]]$ by $\Phi(\pi) = x^n/n!$ where n is the length of π. Applying Φ recovers Elizalde and Noy's cluster method for permutations.

Other Homomorphisms

- Let inv be the inversion number statistic.
- Define $\Phi_q \colon \mathbb{Q}[\mathfrak{S}] \to \mathbb{Q}[[q,x]]$ by $\Phi_q(\pi) = q^{\mathrm{inv}(\pi)} \dfrac{x^n}{[n]_q!}$ where n is the length of π.
- Applying Φ_q recovers a q-analogue of the cluster method for permutations (Elizalde 2016) which also keeps track of inv.

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- **Big Question:** Are there other homomorphisms on $\mathbb{Q}[\mathfrak{S}]$ for counting permutations by other statistics?

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- Applying Φ_q recovers a q-analogue of the cluster method for permutations (Elizalde 2016) which also keeps track of inv.
- **Big Question:** Are there other homomorphisms on $\mathbb{Q}[\mathfrak{S}]$ for counting permutations by other statistics?
- Given a permutation statistic st, let ist be its inverse statistic: $\mathrm{ist}(\pi) = \mathrm{st}(\pi^{-1})$.

General Principle (Z. 2022+)

For any shuffle-compatible descent statistic st, there is a homomorphism Φ_{ist} on $\mathbb{Q}[\mathfrak{S}]$ for counting permutations by ist.

Shuffle-Compatible Descent Statistics

- Let π and σ be permutations on disjoint sets of positive integers, and let $S(\pi, \sigma)$ be the set of shuffles of π and σ.

Example

Given $\pi = 13$ and $\sigma = 42$, we have

$$S(13, 42) = \{1342, 1432, 1423, 4213, 4123, 4132\}.$$

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- A permutation statistic st is shuffle-compatible if the distribution of st over $S(\pi, \sigma)$ depends only on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, and the lengths of σ and π.

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- A permutation statistic st is shuffle-compatible if the distribution of st over $S(\pi, \sigma)$ depends only on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, and the lengths of σ and π.
- Let $\mathrm{Des}(\pi)$ denote the descent set of π. Then st is a descent statistic if, for any permutations π and σ of the same length,

$$\mathrm{Des}(\pi) = \mathrm{Des}(\sigma) \implies \mathrm{st}(\pi) = \mathrm{st}(\sigma).$$

Shuffle-Compatible Descent Statistics (cont.)

- A few notable shuffle-compatible descent statistics:
  - The descent number des.
  - The peak number pk, defined by

$$\mathrm{pk}(\pi) = |\{i : 2 \leq i \leq |\pi| - 1,\ \pi_{i-1} < \pi_i > \pi_{i+1}\}|.$$

  - The left peak number lpk, defined by

$$\mathrm{lpk}(\pi) = \mathrm{pk}(\pi) + \chi(\pi_1 > \pi_2).$$

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$$\mathrm{lpk}(\pi) = \mathrm{pk}(\pi) + \chi(\pi_1 > \pi_2).$$

- Thus, we have homomorphisms Φ_{ides}, Φ_{ipk}, Φ_{ilpk} for counting permutations by the statistics ides, ipk, and ilpk.
- Applying these homomorphisms yields new specializations of our generalized cluster method.

Example: An "ides-Refined" Cluster Method

- Define the Hadamard product $*$ on formal power series in t by

$$\left(\sum_{n=0}^{\infty} a_n t^n\right) * \left(\sum_{n=0}^{\infty} b_n t^n\right) := \sum_{n=0}^{\infty} a_n b_n t^n.$$

- Let $f^{*\langle n\rangle} = \underbrace{f * \cdots * f}_{n \text{ times}}$.

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- Let $f^{*\langle n\rangle} = \underbrace{f * \cdots * f}_{n \text{ times}}$.
- Let
$$A^{\text{ides}}_{\sigma,n}(s,t) = \sum_{\pi\in\mathfrak{S}_n} s^{\text{occ}_\sigma(\pi)} t^{\text{ides}(\pi)+1}$$
$$R^{\text{ides}}_{\sigma,n}(s,t) = \sum_{\pi\in\mathfrak{S}_n} t^{\text{ides}(\pi)+1} \sum_{c\in C_{\sigma,\pi}} s^{\text{mk}_\sigma(c)}.$$

Theorem (Z. 2022+)

Let $\sigma \in \mathfrak{S}$ have length at least 2. Then
$$\sum_{n=0}^{\infty} \frac{A^{\text{ides}}_{\sigma,n}(s,t)}{(1-t)^{n+1}} x^n = \sum_{n=0}^{\infty} \left(\frac{tx}{(1-t)^2} + \sum_{k=2}^{\infty} \frac{R^{\text{ides}}_{\sigma,k}(s-1,t)x^k}{(1-x)^{k+1}} \right)^{*\langle n\rangle}.$$

Additional Results

- We also have specializations of the generalized cluster method for ipk and ilpk.

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- We also have specializations of the generalized cluster method for ipk and ilpk.
- Let
$$P^{\mathrm{ipk}}_{\sigma,n}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{occ}_\sigma(\pi)} t^{\mathrm{ipk}(\pi)+1} \qquad \text{and}$$
$$P^{\mathrm{ilpk}}_{\sigma,n}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{occ}_\sigma(\pi)} t^{\mathrm{ilpk}(\pi)}.$$
- We obtain explicit generating function formulas for $A^{\mathrm{ides}}_{\sigma,n}(s,t)$, $P^{\mathrm{ipk}}_{\sigma,n}(s,t)$, and $P^{\mathrm{ilpk}}_{\sigma,n}(s,t)$ for the following σ:
  - $12\cdots m$ and $m\cdots 21$ for $m \geq 2$;
  - $12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m$ for $m \geq 5$ and $2 \leq a \leq m-2$;
  - $2134\cdots m$ and $12\cdots(m-2)m(m-1)$ for $m \geq 3$ (in progress; ongoing work with Justin Troyka).

A Real-Rootedness Conjecture

- Let

$$A_{\sigma,n}^{\mathrm{ides}}(t) = \sum_{\pi \in \mathfrak{S}_n(\sigma)} t^{\mathrm{ides}(\pi)+1},$$

$$P_{\sigma,n}^{\mathrm{ipk}}(t) = \sum_{\pi \in \mathfrak{S}_n(\sigma)} t^{\mathrm{ipk}(\pi)+1}, \quad P_{\sigma,n}^{\mathrm{ilpk}}(t) = \sum_{\pi \in \mathfrak{S}_n(\sigma)} t^{\mathrm{ilpk}(\pi)}.$$

Conjecture

Let σ be $12\cdots m$ or $m\cdots 21$ where $m \geq 3$, or $12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m$ where $m \geq 5$ and $2 \leq a \leq m-2$. Then the polynomials $A_{\sigma,n}^{\mathrm{ides}}(t)$, $P_{\sigma,n}^{\mathrm{ipk}}(t)$, and $P_{\sigma,n}^{\mathrm{ilpk}}(t)$ have real roots only for all $n \geq 2$.

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THANK YOU!

Formula for $\mathrm{occ}_{12\cdots m}$ and ides

Theorem (Z. 2022+)

Let $m \geq 2$. We have

$$\sum_{n=0}^{\infty} \frac{A^{\mathrm{ides}}_{12\cdots m,n}(s,t)}{(1-t)^{n+1}} x^n = \sum_{n=0}^{\infty} \left(\frac{tx}{(1-t)^2} + \frac{(s-1)tz^m(1-z)}{(1-t)(2-s-z+(s-1)z^m)} \right)^{*\langle n \rangle}$$

where $z = x/(1-t)$.

Formula for $\mathrm{occ}_{12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m}$ and ides

Theorem (Z. 2022+)

Let $\sigma = 12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m$ *where* $m \geq 5$ *and* $2 \leq a \leq m-2$*. Let* $i = \min(a, m-a)$*. We have*

$$\sum_{n=0}^{\infty} \frac{A_{\sigma,n}^{\mathrm{ides}}(s,t)}{(1-t)^{n+1}} x^n = \sum_{n=0}^{\infty} \left(\frac{tx}{(1-t)^2} + \frac{(s-1)t^2 z^m}{(1-t)(1-(s-1)t\sum_{l=1}^{i} z^{m-l})} \right)^{*\langle n \rangle}$$

where $z = x/(1-t)$*.*

Formula for $\mathrm{occ}_{12\ldots m}$ and ipk

Theorem (Z. 2022+)

Let $m \geq 2$. *We have*

$$\frac{1}{1-t} + \frac{1+t}{2(1-t)} \sum_{n=1}^{\infty} P^{\mathrm{ipk}}_{12\cdots m,n}(s,u) z^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{2tx}{(1-t)^2} + \frac{2t(s-1)z^m(1-z)}{(1-t^2)(2-s-z+(s-1)z^m)} \right)^{*\langle n \rangle}$$

where $u = 4t/(1+t)^2$ *and* $z = (1+t)x/(1-t)$.

Formula for $\mathrm{occ}_{12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m}$ and ipk

Theorem (Z. 2022+)

Let $\sigma = 12\cdots(a-1)(a+1)a(a+2)(a+3)\cdots m$ *where* $m \geq 5$ *and* $2 \leq a \leq m-2$. *Let* $i = \min(a, m-a)$. *We have*

$$\frac{1}{1-t} + \frac{1+t}{2(1-t)} \sum_{n=1}^{\infty} P_{\sigma,n}^{\mathrm{ipk}}(s,u) z^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{2tx}{(1-t)^2} + \frac{(1+t)(s-1)u^2 z^m}{2(1-t)(1-(s-1)u\sum_{l=1}^{i} z^{m-l})} \right)^{*\langle n\rangle}$$

where $u = 4t/(1+t)^2$ *and* $z = (1+t)x/(1-t)$.

Counting $12\cdots m$-Avoiding Permutations by ides and imaj

- Let $A_{\sigma,n}^{(\text{ides},\text{imaj})}(t,q) = \sum_{\pi\in\mathfrak{S}_n(\sigma)} t^{\text{ides}(\pi)+1} q^{\text{imaj}(\pi)}$ for $n \geq 1$.

Theorem (Z. 2022+)

Let $m \geq 2$. We have

$$\sum_{n=0}^{\infty} \frac{A_{12\cdots m,n}^{(\text{ides},\text{imaj})}(t,q)}{(1-t)(1-qt)\cdots(1-q^n t)} x^n =$$

$$1 + \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\begin{bmatrix} k+jm-1 \\ k-1 \end{bmatrix}_q x^{jm} - \begin{bmatrix} k+jm \\ k-1 \end{bmatrix}_q x^{jm+1} \right) \right]^{-1} t^k.$$