Preimages under the Bubblesort operator

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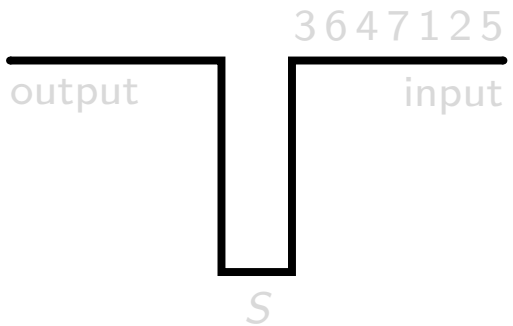
Joint work with Mathilde Bouvel and Lapo Cioni

Three sorting algorithms

There are several sorting algorithms operating by sorting in phases.

Stacksort



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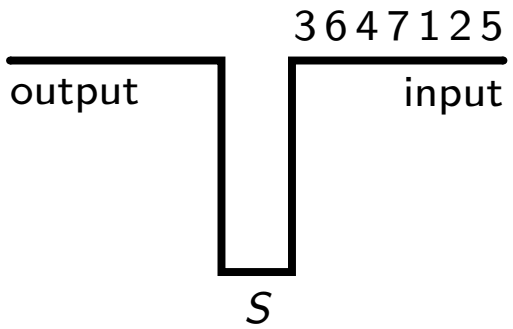

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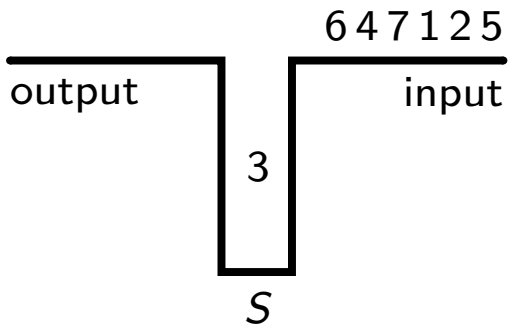

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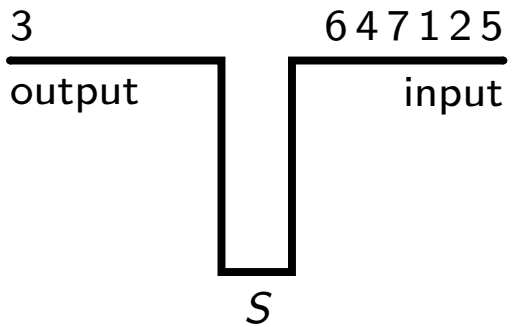

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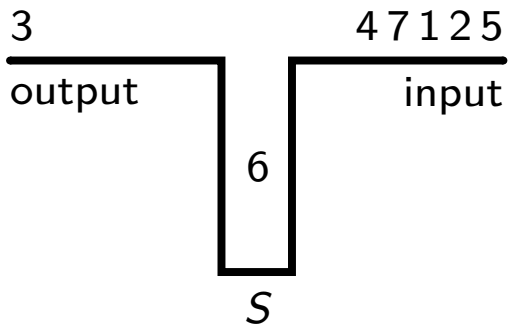

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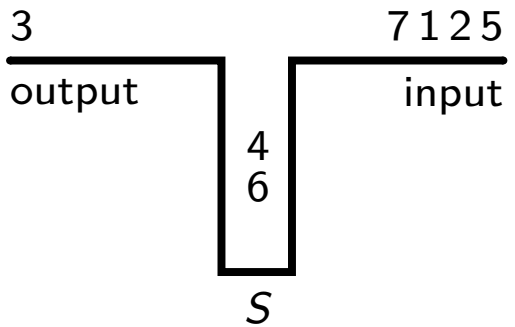

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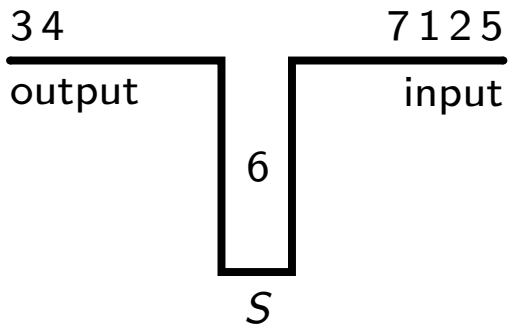

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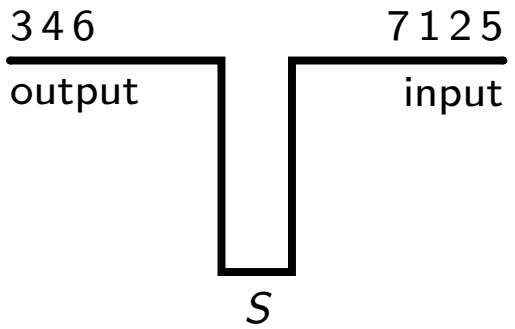

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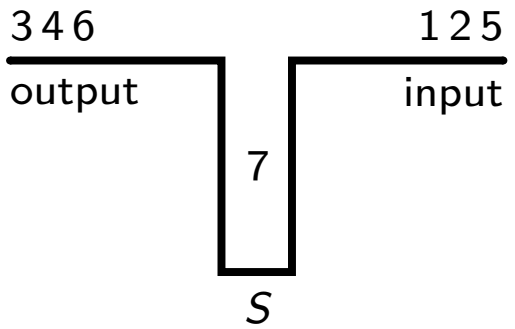

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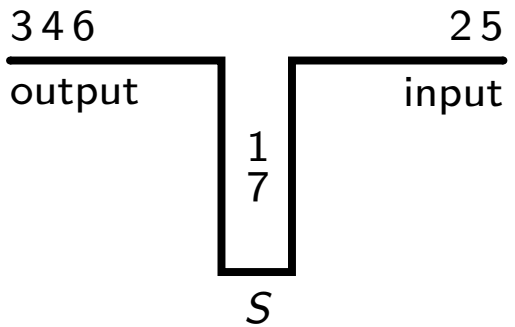

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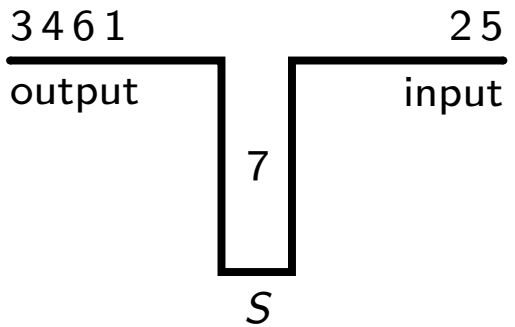

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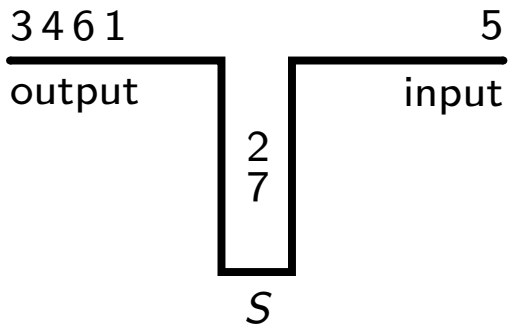

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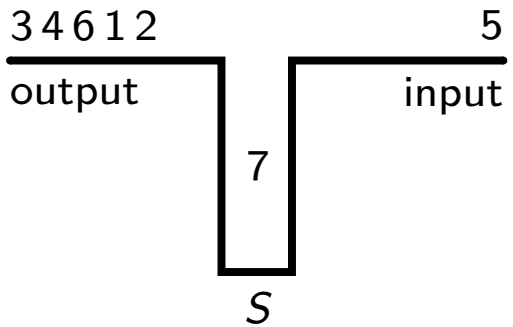

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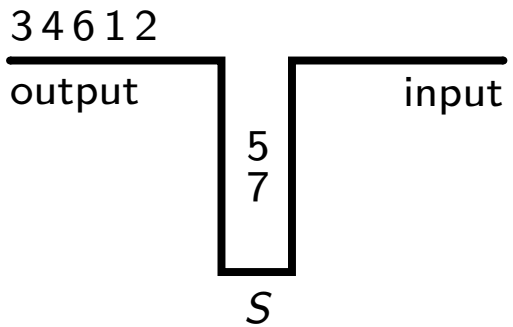

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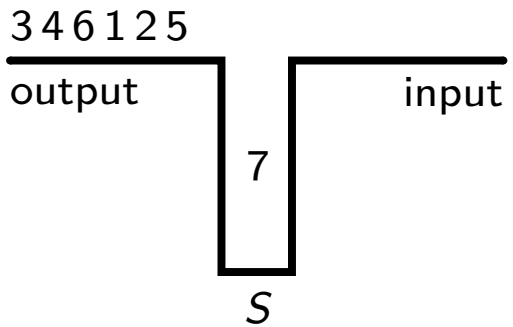

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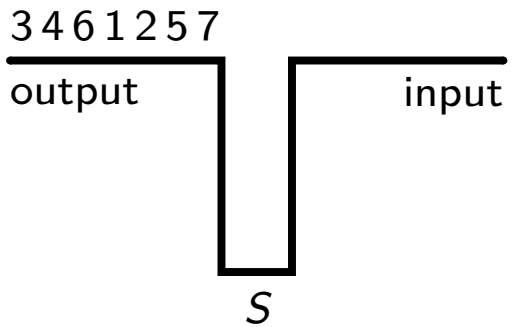

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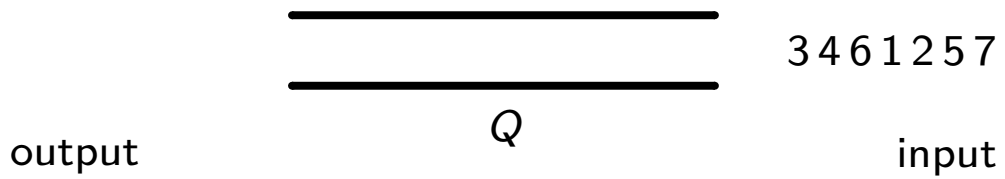

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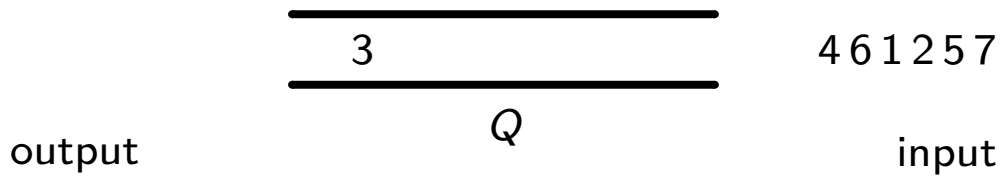

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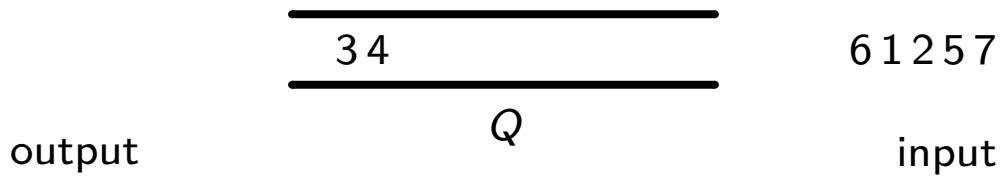

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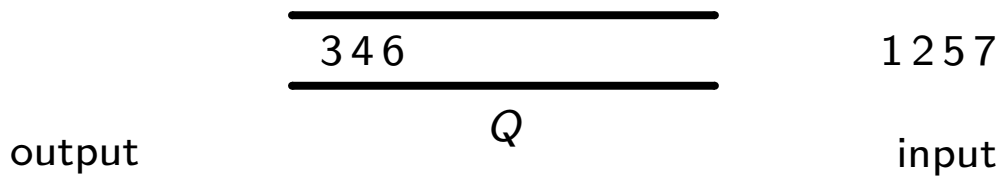

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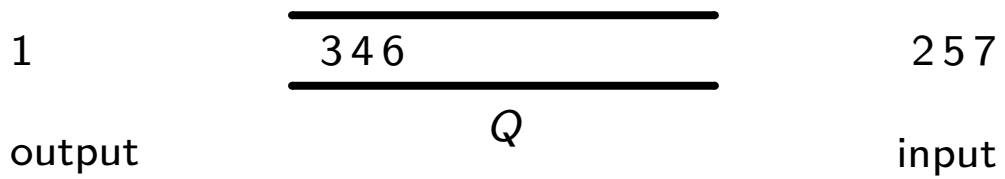

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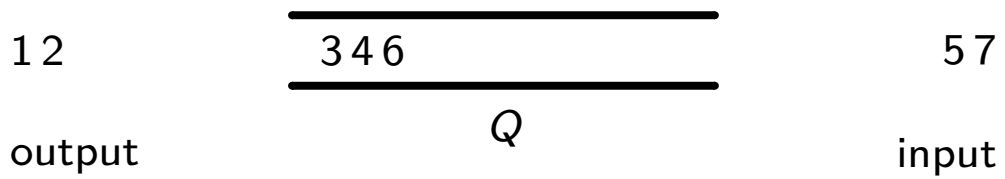

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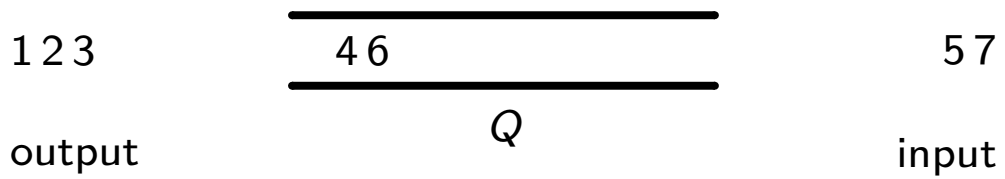

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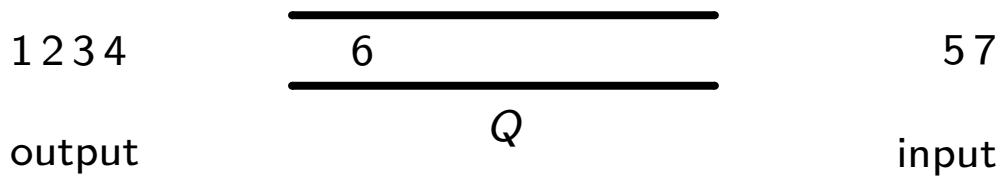

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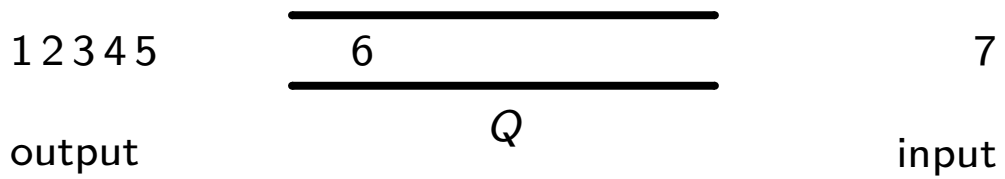

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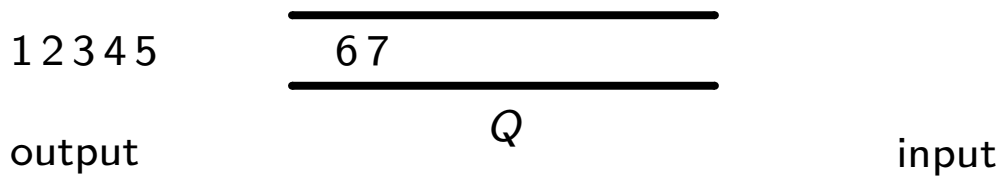

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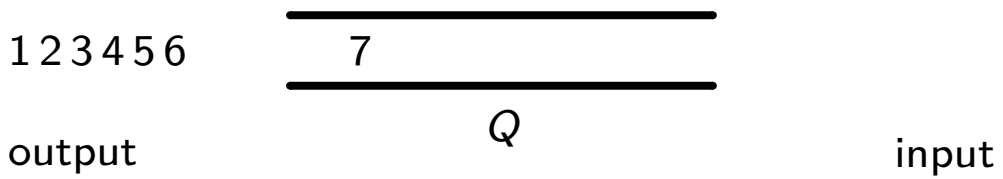

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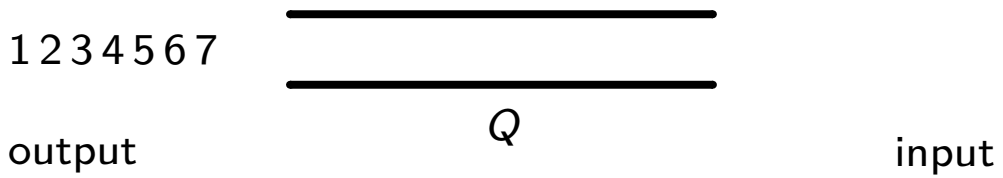

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Preimages of permutation classes

What can we say about preimages of permutation classes?

- `Stacksort`: West [1990], Claesson and Ulfarsson [2012], Defant [2020]
- `Queuesort`: Magnusson [2013]
- `Bubblesort`: Albert, Atkinson, Bouvel, Claesson, Dukes [2011]

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  - functional equation for the g.f. of permutations with at least one preimage,
- recent improvements by Defant et al. [2020-2022]
  - enumeration of permutations having exactly one preimage,
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Preimages under Bubblesort

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Bubblesort:

This talk!

An algorithm to compute preimages

$\sigma = 325146$

$$P = \{325146\}$$

for i from n down to 2:
for each $\pi \in P$:

- if π_{i-1} is not a LTR maximum, then replace π with the permutation obtained by swapping π_i and π_{i-1}
- if π_{i-1} is a LTR maximum, then add to P the permutation obtained by swapping π_i and π_{i-1}

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Theorem

Let σ be any permutation ending with its maximum. Then $P = \mathbf{B}^{-1}(\sigma)$.

Counting preimages

A preimage of $\sigma \in S_n$ is uniquely determined by a subset of its LTR maxima other than n, that can be interpreted as "guards".

$$3|25|146$$

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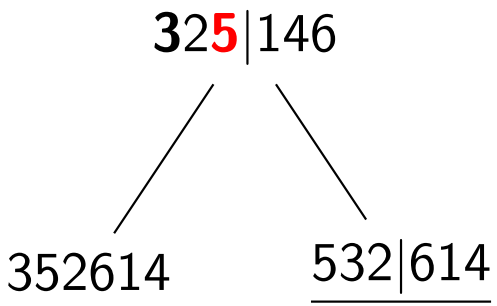

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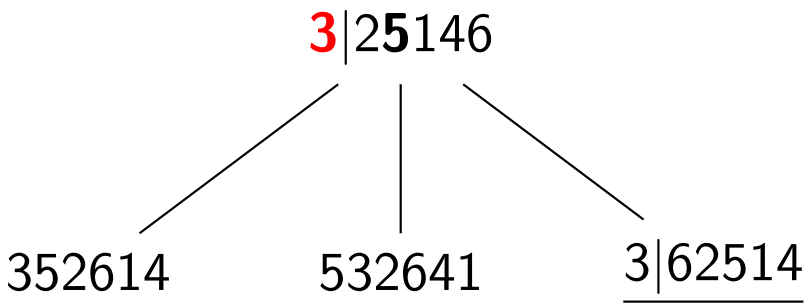

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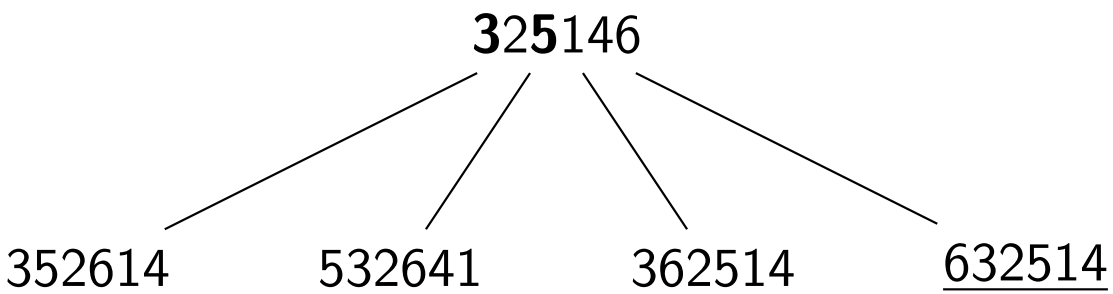

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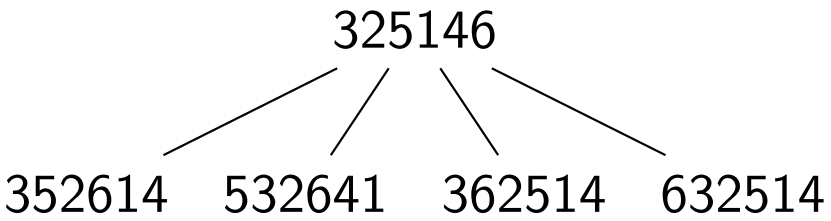

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$\sigma = \sigma_1 \cdots \sigma_{n-1} n \in S_n$ *with* k *LTR maxima*

- $|\mathbf{B}^{-1}(\sigma)| = 2^{k-1}$
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$|\{\sigma \in S_n \,|\, |\mathbf{B}^{-1}(\sigma)| = 2^{k-1}\}| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ (unsigned Stirling numbers of the I kind)

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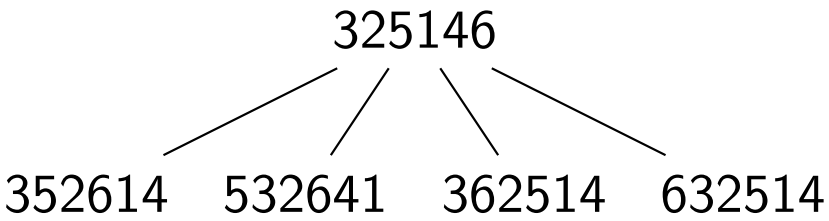

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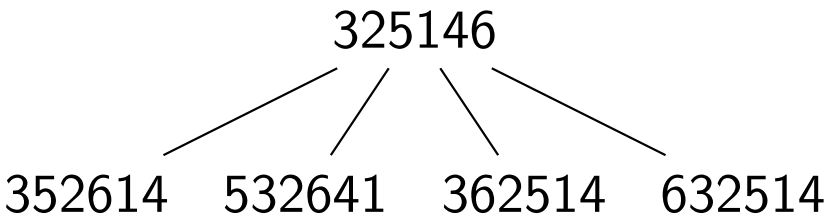

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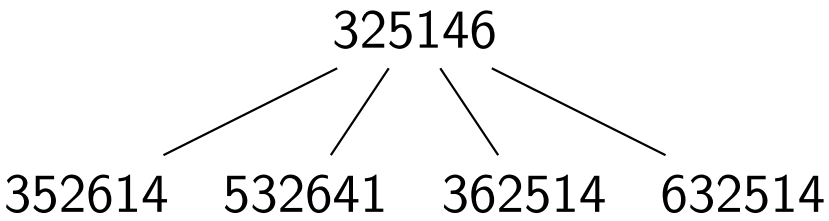

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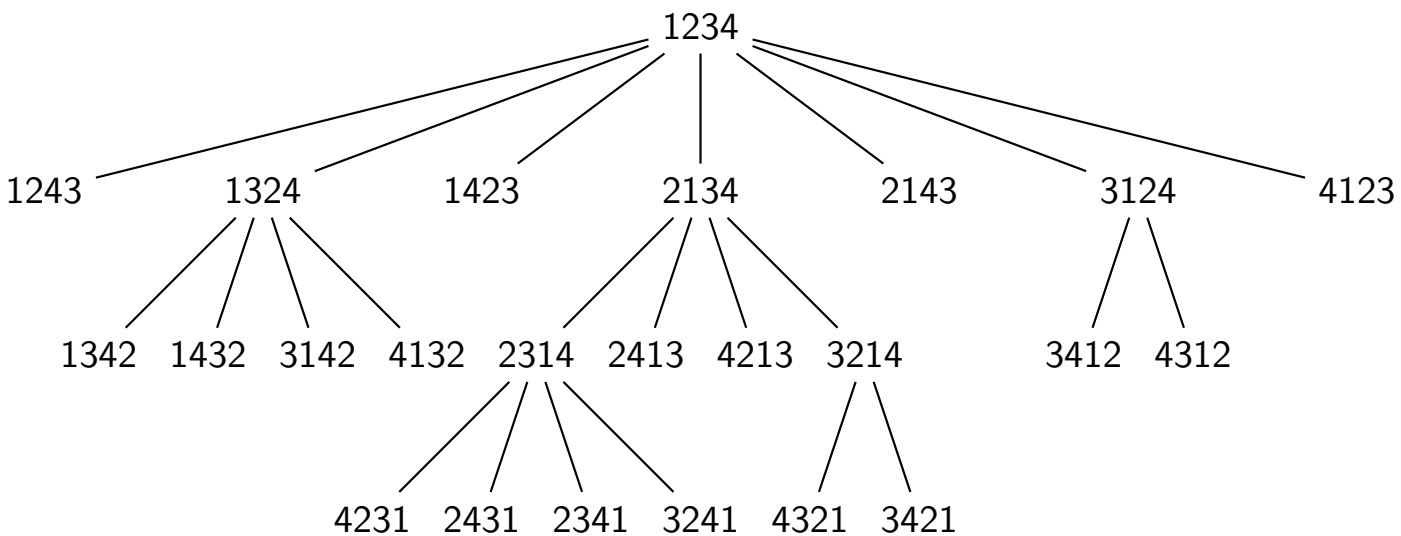
Trees and subtrees

Rooted tree T_n:

- nodes are permutations of size n;
- root is id_n;
- τ is a child of σ whenever $\mathbf{B}(\tau) = \sigma$ (and $\sigma \neq \tau$).



The tree T_4 of iterated preimages.

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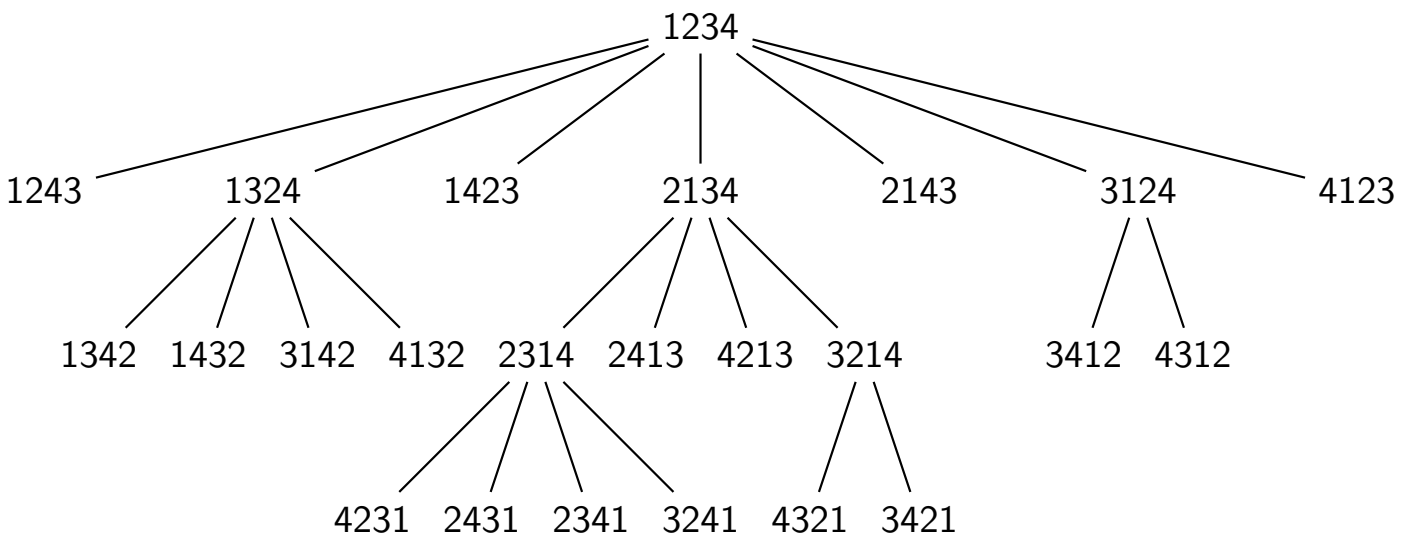

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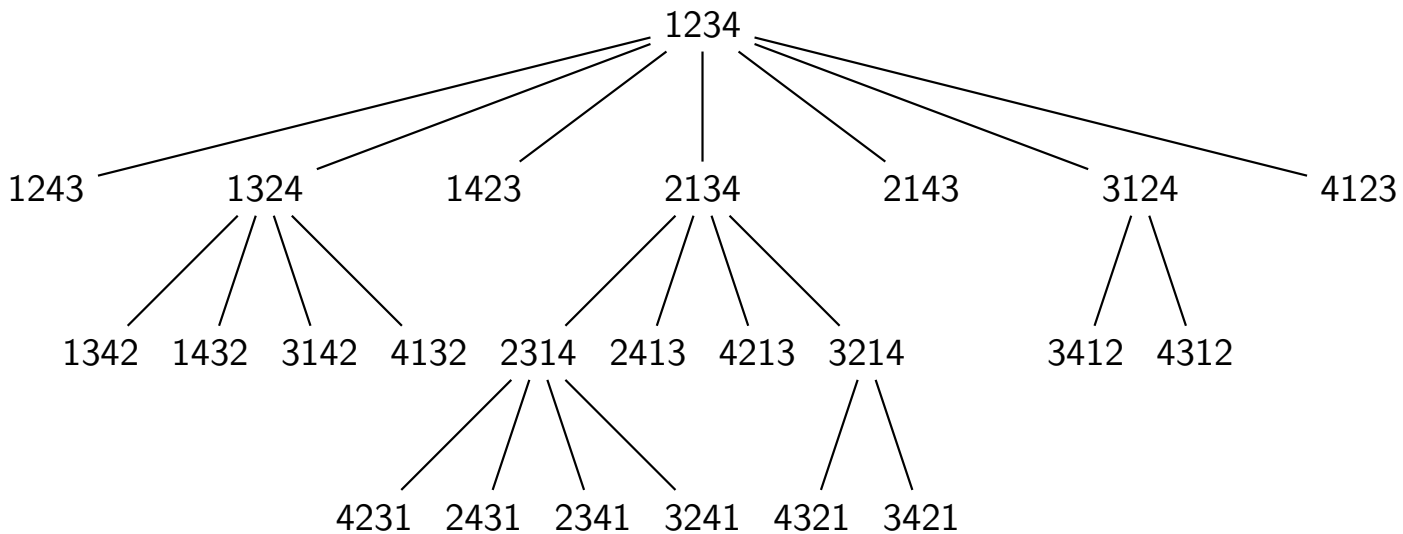

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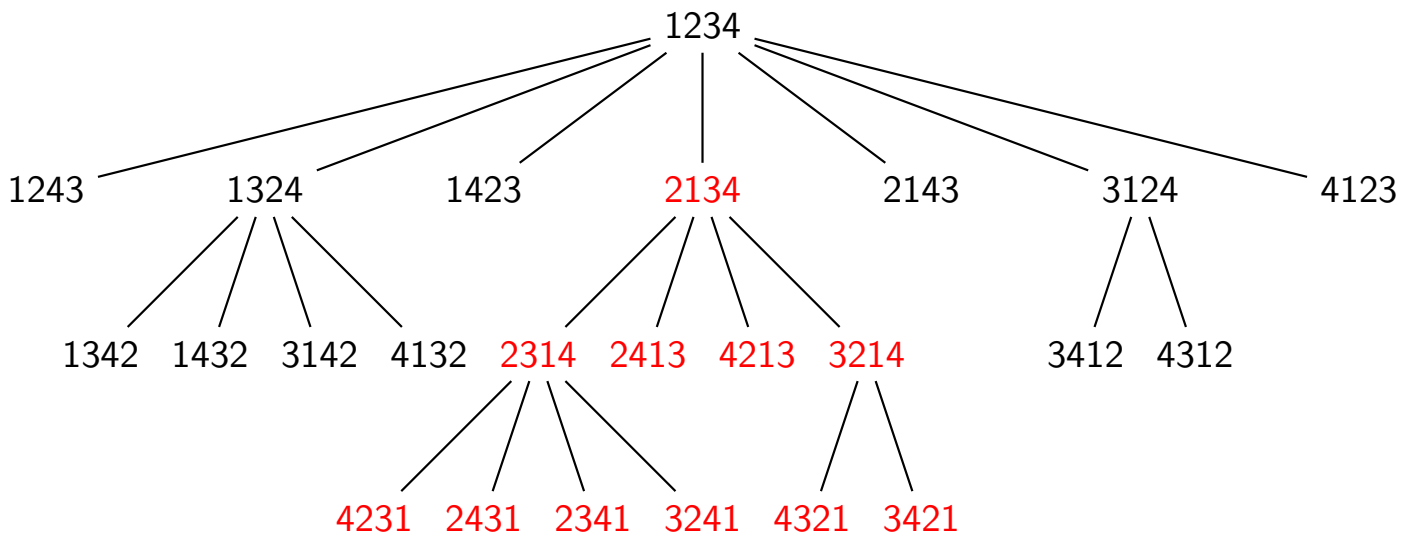
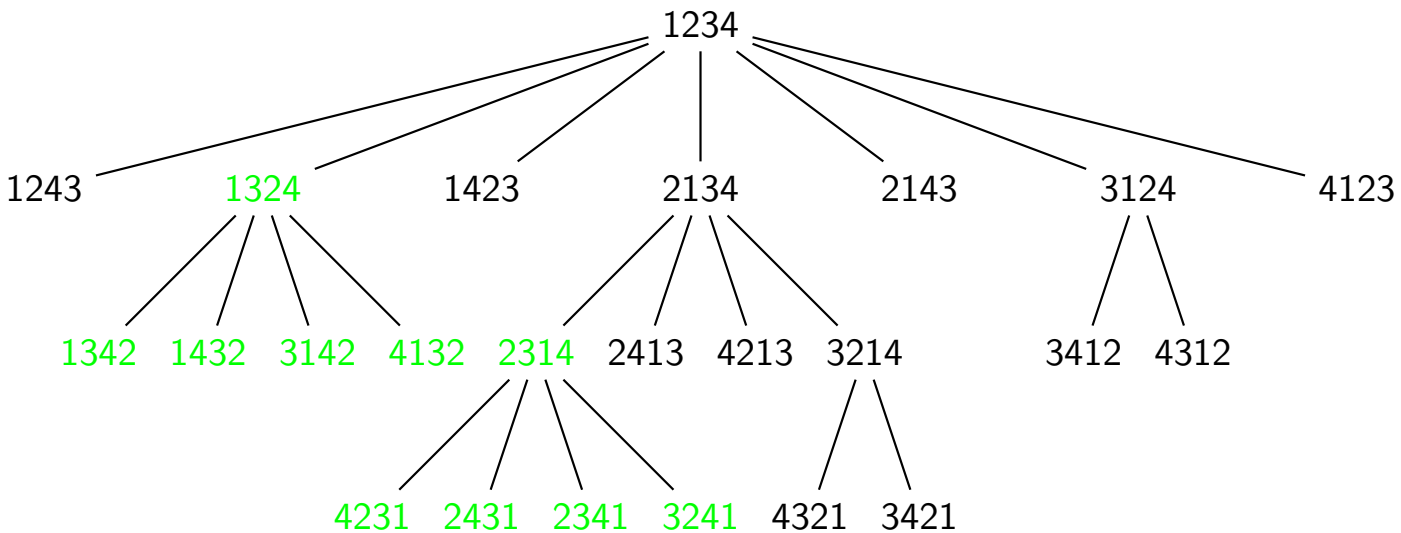


The subtree $T(2134)$.

Isomorphisms between subtrees

Lemma

Let π and τ be two permutations of the same size. If π and τ have their LTR maxima in the same positions, then $T(\pi)$ and $T(\tau)$ are isomorphic.

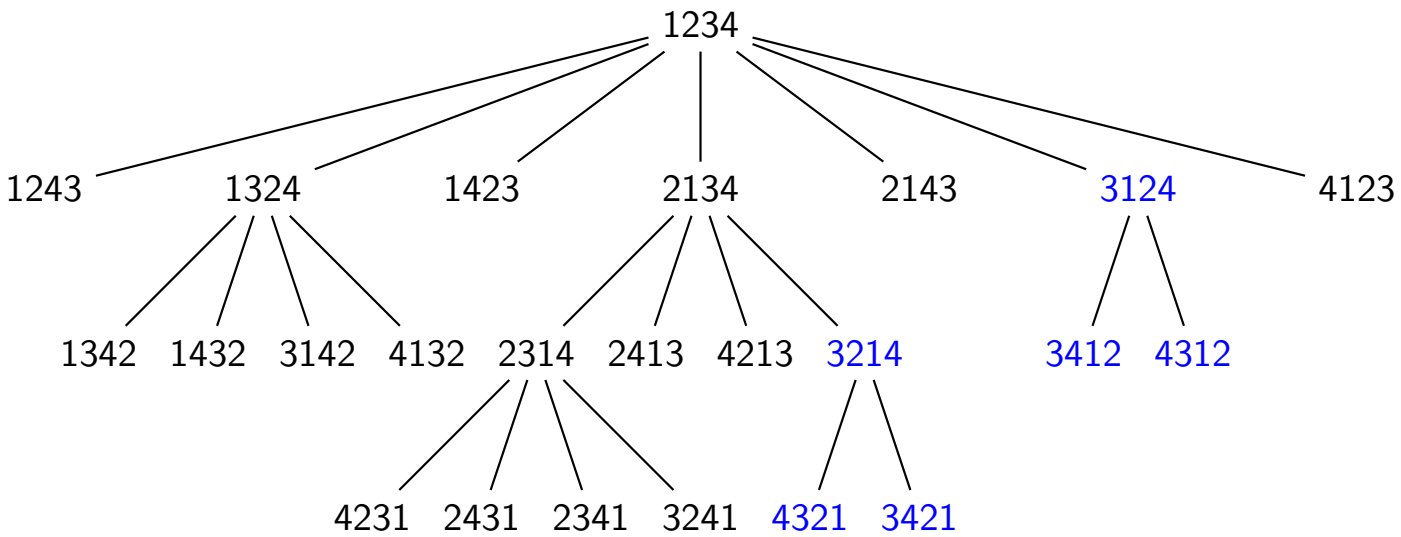


$T(1324)$ and $T(2314)$ are isomorphic.

Isomorphisms between subtrees

Proposition

For every permutation $\pi \in S_n$, $\pi \neq id_n$, there exists a child τ of id_n in T_n such that $T(\pi)$ and $T(\tau)$ are isomorphic.



$T(3214)$ is isomorphic to $T(3124)$.

The skeleton

- $\sigma \in S_n$ having k LTR maxima and whose last m_ℓ elements are LTR maxima
- (k, m_ℓ) is the *label* of σ
- *Skeleton* of $T(\sigma)$: obtained from $T(\sigma)$ by replacing each permutation at a node with its label.

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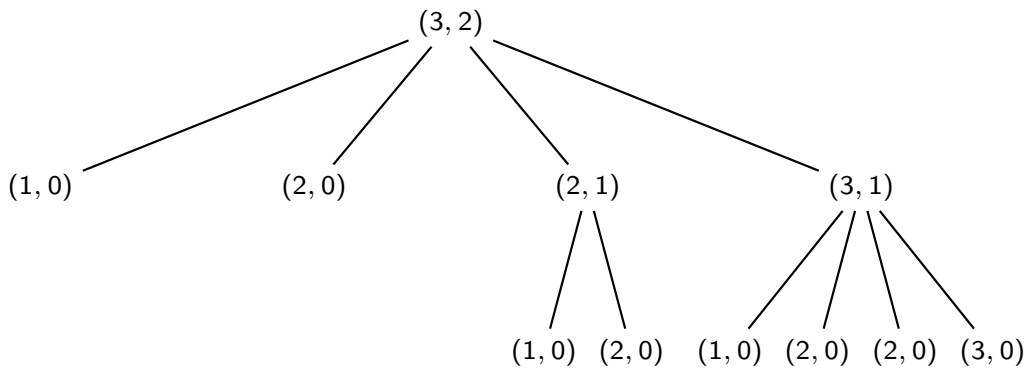

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The skeleton

The skeleton of $T(\pi)$ is completely determined by the label of its root π.

Proposition

Let $\pi \in S_n$ with label (k, m_ℓ). Let T be the skeleton of $T(\pi)$. Then the root of T has label (k, m_ℓ) and its children have the following labels:

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Some consequences:

Corollary

Let π be a permutation of size n with label (k, m_ℓ) such that $\pi \neq id_n$. Then $T(\pi)$ has height m_ℓ. In addition, for every $n \geq 1$, T_n has height $n - 1$.

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Height of a node is equal to the number of passes of Bubblesort needed to sort the permutation.

Proposition (Albert, Atkinson, Bouvel, Claesson, and Dukes, 2011)

The set of permutations of size n sorted by at most k passes of Bubblesort is $\mathrm{Av}_n(\Gamma_{k+2})$, *where* Γ_k *is the set of all permutations of size k whose final element is 1. As a consequence, setting* $\varphi_n^{(k)} = |\mathrm{Av}_n(\Gamma_k)|$, *the number of nodes at height at most k in* T_n *is given by* $\varphi_n^{(k+2)} = (k+1)^{n-k-1}(k+1)!$

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The number $f_n^{(k)}$ *of nodes at height k in* T_n *is given by*

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3	1	3	2			
4	1	7	10	6		
5	1	15	38	42	24	
6	1	31	130	222	216	120

Sequence A056151: number of nodes in T_n having height k.

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The average height of a node in T_n is asymptotically equal to $n - \sqrt{\frac{\pi n}{2}} + O(1)$.

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Height of nodes in $T(\pi)$

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$$N(k, m_\ell) = \sum_{j=0}^{m_\ell} j!(j+1)^{k-j}. \tag{1}$$

Moreover, each summand records the contribution of each level of $T(\pi)$. In other words, denoting with $N_j(k, m_\ell)$ the number of nodes at height j in $T(\pi)$, we have that $N_j(k, m_\ell) = j!(j+1)^{k-j}$.

Heights of leaves in T_n

- $\mathrm{Av}_n^*(\Gamma_k)$: permutations of size n avoiding Γ_k whose last element is different from n
- $\gamma_n^{(k)} = |\mathrm{Av}_n^*(\Gamma_k)|$
- $\gamma_n^{(k+2)}$ is the number of leaves at height at most k in T_n

Proposition

$$\gamma_n^{(k)} = \begin{cases} (n-1)(n-1)! & n < k, \\ (k-2)(k-1)^{n-k}(k-1)! & n \geq k. \end{cases}$$

Corollary

The number $g_n^{(k)}$ of leaves of T_n at height k is given by

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Heights of leaves in T_n

The argument by Flajolet and Sedgewick to compute the average height of a node in T_n can be adapted to the case of leaves.

Proposition

The average height of a leaf in T_n is asymptotically equal to $n - \sqrt{\frac{\pi n}{2}} + O(1)$.

Height of leaves in $T(\pi)$

Proposition

For a permutation π having label (k, m_ℓ), different from an identity permutation, the number of leaves of $T(\pi)$ is

$$L(k, m_\ell) = \sum_{j=1}^{m_\ell - 1} j!j(j+1)^{k-j-1} + m_\ell!(m_\ell+1)^{k-m_\ell}. \tag{2}$$

Moreover, each summand in (2) records the contribution of each level of $T(\pi)$. In other words, denoting with $L_j(k, m_\ell)$ the number of leaves at height j in $T(\pi)$, we have that $L_j(k, m_\ell) = j!j(j+1)^{k-j-1}$ for $j < m_\ell$, and $L_{m_\ell}(k, m_\ell) = m_\ell!(m_\ell+1)^{k-m_\ell}$.

Further work

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