RESTRICTED GRASSMANNIAN PERMUTATIONS

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Grassmannian & Related Permutations

A *Grassmannian permutation* is a permutation having at most one descent. If $\mathscr{G}_n$ denotes the set of Grassmannian permutations on $[n] = \{1, \dots, n\}$, then

- $\pi \in \mathscr{G}_n$ if and only if $\pi^{rc} \in \mathscr{G}_n$.
- $|\mathscr{G}_n| = 2^n - n$ for $n \ge 1$. $(1, 2, 5, 12, 27, 58, 121, 248, 503, 1014, \dots)$

A permutation, π, is called *biGrassmannian* if both $\pi, \pi^{-1} \in \mathscr{G}_n$. For $\pi \in \mathscr{G}_n$, the inverse π^{-1} has at most one *dip*, i.e. a pair (i,j) with $i < j$ such that $\pi(i) = \pi(j)+1$. A biGrassmannian permutation has at most one descent and at most one dip.

Proposition. A Grassmannian permutation is biGrassmannian if and only if it avoids the pattern 2413. In other words, $\mathscr{G}_n \cap \mathscr{G}_n^{-1} = \mathscr{G}_n(2413)$ for every n. Moreover,

$$|\mathscr{G}_n \cap \mathscr{G}_n^{-1}| = 1 + \binom{n+1}{3}.$$

We know $\mathscr{G}_n \subset S_n(3142)$, so $\mathscr{G}_n^{-1} \subset S_n(2413)$ and $\mathscr{G}_n \cap \mathscr{G}_n^{-1} \subset \mathscr{G}_n(2413)$. Conversely, suppose $\pi \in \mathscr{G}_n$ has two dips, say (i_1, j_1) and (i_2, j_2) with $i_1 < i_2$. Since π has at most one descent and avoids a 321 pattern, we must have

$$i_1 < i_2 < j_1 < j_2 \text{ and } \pi(j_1) < \pi(i_1) < \pi(j_2) < \pi(i_2),$$

giving a 2413 pattern. Thus, $\mathscr{G}_n(2413) \subset \mathscr{G}_n^{-1}$ and so $\mathscr{G}_n(2413) \subset \mathscr{G}_n \cap \mathscr{G}_n^{-1}$.

Proposition. For $n \in \mathbb{N}$, we have $\mathscr{G}_n \cup \mathscr{G}_n^{-1} = S_n(321, 2143)$. Moreover,

$$|\mathscr{G}_n \cup \mathscr{G}_n^{-1}| = 2^{n+1} - \binom{n+1}{3} - 2n - 1.$$

Proposition. $\pi \in \mathscr{G}_n$ is an involution if and only if it is of the form

$$\pi = \mathrm{id}_{k_1} \oplus \big(\mathrm{id}_{k_2} \ominus \mathrm{id}_{k_2}\big) \oplus \mathrm{id}_{k_3}$$

for some $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}$ with $k_1 + 2k_2 + k_3 = n$, where $\mathrm{id}_0 = \varepsilon$. Moreover, i_n, the number of Grassmannian involutions of size n is given by

$$i_n = \begin{cases} \frac{n^2+3}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2+4}{4} & \text{if } n \text{ is even.} \end{cases}$$

Grassmannian involutions correspond to standard Young tableaux of shape $(n-k_2, k_2)$ whose second row consists of the labels $k_1+k_2+1, \dots, k_1+2k_2$.

1 3 4 5 6 / 2 — $(1 \ominus 1) \oplus 1234$ — 2 1 3 4 5 6

1 2 4 5 6 / 3 — $1 \oplus (1 \ominus 1) \oplus 123$ — 1 3 2 4 5 6

1 2 3 5 6 / 4 — $12 \oplus (1 \ominus 1) \oplus 12$ — 1 2 4 3 5 6

1 2 3 4 6 / 5 — $123 \oplus (1 \ominus 1) \oplus 1$ — 1 2 3 5 4 6

1 2 3 4 5 / 6 — $1234 \oplus (1 \ominus 1)$ — 1 2 3 4 6 5

1 2 5 6 / 3 4 — $(12 \ominus 12) \oplus 12$ — 3 4 1 2 5 6

1 2 3 6 / 4 5 — $1 \oplus (12 \ominus 12) \oplus 1$ — 1 4 5 2 3 6

1 2 3 4 / 5 6 — $12 \oplus (12 \ominus 12)$ — 1 2 5 6 3 4

1 2 3 / 4 5 6 — $123 \ominus 123$ — 4 5 6 1 2 3

1 2 3 4 5 6 — 123456 — 1 2 3 4 5 6

Main Results

For $\sigma \in \{132, 213, 231, 312\}$ and $n \in \mathbb{N}$, we have

$$|\mathscr{G}_n(\sigma)| = n + \binom{n-1}{2} = 1 + \binom{n}{2}.$$

In $\mathscr{G}_n(132) \cap \mathscr{G}_n(231)$, there are n permutations:

$$\pi = 1 \cdots n, \quad \pi = n\,1 \cdots (n-1), \text{ and}$$
$$\pi = i\,1 \cdots (i-1)(i+1) \cdots n \text{ for } i \in \{2, \dots, n-1\}$$

Moreover,

$$\pi \in \mathscr{G}_n(132) \setminus \mathscr{G}_n(231) \implies \pi = i(i+1) \cdots j\,1\,\tau,$$
$$\pi \in \mathscr{G}_n(231) \setminus \mathscr{G}_n(132) \implies \pi = 1 \cdots (i-1)\,j\,i\,\tau,$$

with $i, j \in \{2, \dots n\}$ and $i < j$.

Theorem. If $k \ge 3$ and $\sigma \in S_k$ with $\mathrm{des}(\sigma) = 1$, then

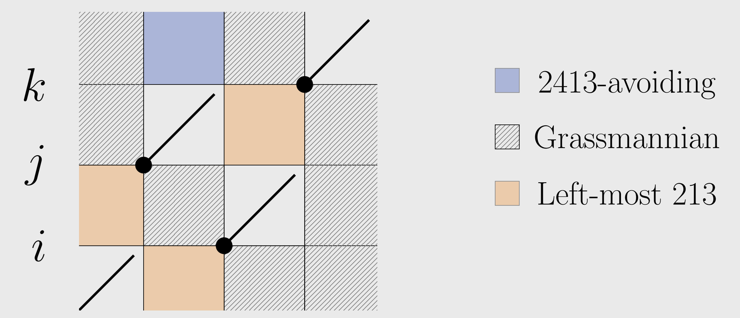
$$|\mathscr{G}_n(\sigma)| = 1 + \sum_{j=3}^{k} \binom{n}{j-1} \text{ for } n \in \mathbb{N}.$$

PROOF BY EXAMPLE. Let $\sigma = 2413$ and choose $\sigma' = 213$.

$$\mathscr{G}_n(2413) = \mathscr{G}_n(213) \,\dot\cup\, \big(\mathscr{G}_n(2413) \setminus \mathscr{G}_n(213)\big),$$

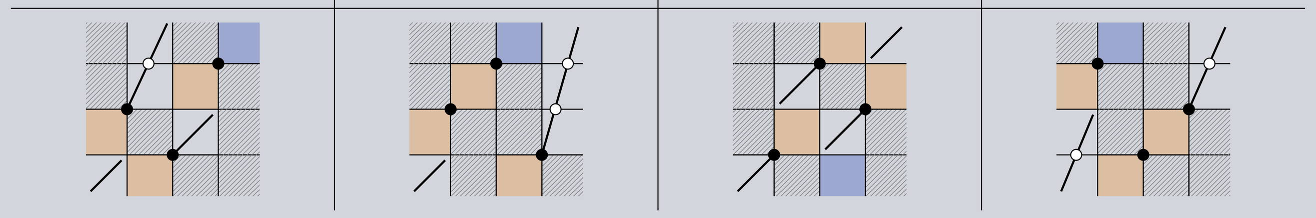
and every $\pi \in \mathscr{G}_n(2413) \setminus \mathscr{G}_n(213)$ must be of the form

$$\pi = \tau_0\, j\, \tau_1\, i\, \tau_2\, k\, \tau_3 \text{ with } 1 \le i < j < k \le n.$$



There are $\binom{n}{3}$ such permutations, so $|\mathscr{G}_n(2413)| = 1 + \binom{n}{2} + \binom{n}{3}$.

$\mathscr{G}_n(2134)\backslash\mathscr{G}_n(213)$ | $\mathscr{G}_n(2341)\backslash\mathscr{G}_n(231)$ | $\mathscr{G}_n(2413)\backslash\mathscr{G}_n(132)$ | $\mathscr{G}_n(3412)\backslash\mathscr{G}_n(312)$



If $\mathrm{des}(\sigma) > 1$, then $\mathscr{G}_n(\sigma) = \mathscr{G}_n$. Moreover,

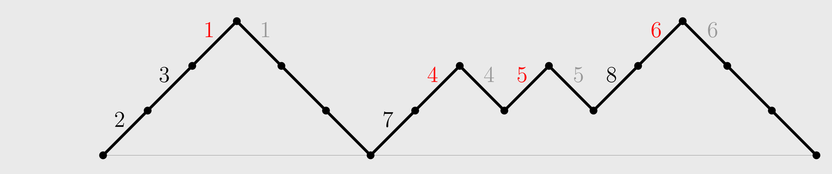
$$|\mathscr{G}_k(12 \cdots k)| = 2^k - k - 1 \text{ and } |\mathscr{G}_m(12 \cdots k)| = 2^m - m \text{ for } m < k.$$

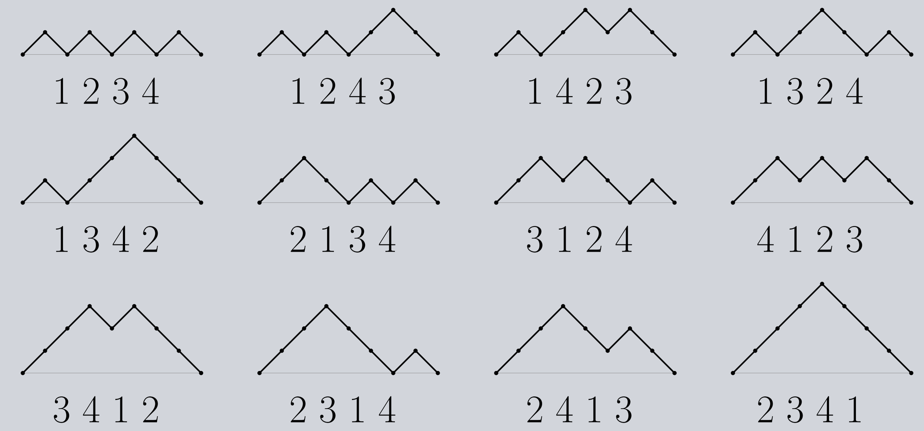
Conjecture. (Weiner) For $k \ge 2$ and $m \in \{k, \dots, 2k-2\}$,

$$|\mathscr{G}_m(12 \cdots k)| = \sum_{j=1}^{k - \lfloor m/2 \rfloor} (-1)^{j-1} j \cdot \tbinom{2k-m-j}{j} C_{k-j}.$$

Path Connections

Proposition. The set $\mathscr{G}_n$ of Grassmannian permutations on $[n]$ is in bijection with the set of Dyck paths of semilength n having at most one long ascent.





Lemma. We have $\pi \in \mathscr{G}_{n+1}(35124)$ if and only if its Lehmer code is of the form

$$L(\pi) = 0^{j_1} 1^{j_2} m^{j_3} 0^{j_4} = \underbrace{0 \cdots 0}_{j_1}\, \underbrace{1 \cdots 1}_{j_2}\, \underbrace{m \cdots m}_{j_3}\, \underbrace{0 \cdots 0}_{j_4},$$

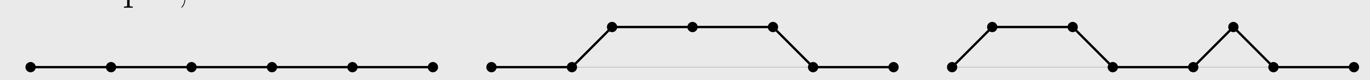
where $j_1 + \cdots + j_4 = n+1$, $j_4 > 0$, $m \in \{2, \dots, n\}$, and $m \le j_4$.

Proposition. The set $\mathrm{Schr}_n(\mathsf{UUDD})$ is in bijection with $\mathscr{G}_{n+1}(35124)$.

A Schröder word, w, corresponds to a Schröder path of semilength n when the number of letters in w satisfy $\#\mathsf{U} + \#\mathsf{D} + 2(\#\mathsf{H}) = 2n$. For $w \in \mathrm{Schr}_n(\mathsf{UUDD})$, w also satisfies $w = uv \Rightarrow \mathrm{val}(u) \in \{0, 1\}$ and $(\#\mathsf{U} \text{ in } w) \le 2$, and we define α to create a Lehmer code of $\mathscr{G}_{n+1}(35124)$ by

$$\alpha(w) = \begin{cases} 0^{n+1} & \text{if } w \text{ has no } \mathsf{U}, \\ \mathrm{bin}(w) & \text{if } w \text{ has only one } \mathsf{U}, \\ 0^{i_1} 1^{i_2 - 1} (i_3 + 1)^{i_4} 0^{i_3 + i_5} & \text{if } \mathrm{bin}(w) = 0^{i_1} 1^{i_2} 0^{i_3} 1^{i_4} 0^{i_5}. \end{cases}$$

For example,



$\alpha(\mathsf{HHHHH}) = 000000$ — 1 2 3 4 5 6

$\alpha(\mathsf{HUHHDH}) = 011100$ — 1 3 4 5 2 6

$\alpha(\mathsf{UHDHUDH}) = 130000$ — 2 5 1 3 4 6

References

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