Maximal Number of Common Increasing Subsequences of Several Permutations

Permutation Pattern 2022 Valpo

Kento NAKADA
Okayama University.

2022/6/24

In this talk, we denote by $\mathfrak{S}_n$ the symmetric group of degree n, and by

$$\begin{aligned} w &= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ w(1) & w(2) & w(3) & \cdots & w(n) \end{pmatrix} && \text{(two-line notatione)} \\ &= w(1)w(2)w(3)\cdots w(n) && \text{(one-line notatione)} \end{aligned}$$

Example

$$\begin{aligned} w &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} && \text{(two-line notatione)} \\ &= 21354 && \text{(one-line notatione)} \end{aligned}$$

Denote by $\mathrm{Inc}_l(w)$ the set of increasing subsequences of length l of w, and by $\mathrm{inc}_l(w)$ its cardinality:

$$\mathrm{Inc}_l(w) := \left\{ w(i_1)w(i_2)\cdots w(i_l) \;\middle|\; \begin{array}{l} i_1 < i_2 < \cdots < i_l \text{ and} \\ w(i_1) < w(i_2) < \cdots < w(i_l) \end{array} \right\},$$

$$\mathrm{inc}_l(w) := \#\mathrm{Inc}_l(w).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$

Denote by $\mathrm{Inc}_l(w)$ the set of increasing subsequences of length l of w, and by $\mathrm{inc}_l(w)$ its cardinality:

$$\mathrm{Inc}_l(w) := \left\{ w(i_1)w(i_2)\cdots w(i_l) \,\middle|\, \begin{array}{l} i_1 < i_2 < \cdots < i_l \;\text{ and} \\ w(i_1) < w(i_2) < \cdots < w(i_l) \end{array} \right\},$$

$$\mathrm{inc}_l(w) := \#\mathrm{Inc}_l(w).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$

Denote by $\mathrm{Inc}_l(w)$ the set of increasing subsequences of length l of w, and by $\mathrm{inc}_l(w)$ its cardinality:

$$\mathrm{Inc}_l(w) := \left\{ w(i_1)w(i_2)\cdots w(i_l) \,\middle|\, \begin{array}{l} i_1 < i_2 < \cdots < i_l \ \text{ and} \\ w(i_1) < w(i_2) < \cdots < w(i_l) \end{array} \right\},$$

$$\mathrm{inc}_l(w) := \#\mathrm{Inc}_l(w).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$

Denote by $\mathrm{Inc}_l(w)$ the set of increasing subsequences of length l of w, and by $\mathrm{inc}_l(w)$ its cardinality:

$$\mathrm{Inc}_l(w) := \left\{ w(i_1)w(i_2)\cdots w(i_l) \;\middle|\; \begin{array}{l} i_1 < i_2 < \cdots < i_l \text{ and} \\ w(i_1) < w(i_2) < \cdots < w(i_l) \end{array} \right\},$$

$$\mathrm{inc}_l(w) := \#\mathrm{Inc}_l(w).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$

Denote by $\mathrm{Inc}_l(w)$ the set of increasing subsequences of length l of w, and by $\mathrm{inc}_l(w)$ its cardinality:

$$\mathrm{Inc}_l(w) := \left\{ w(i_1)w(i_2)\cdots w(i_l) \,\middle|\, \begin{array}{l} i_1 < i_2 < \cdots < i_l \text{ and} \\ w(i_1) < w(i_2) < \cdots < w(i_l) \end{array} \right\},$$

$$\mathrm{inc}_l(w) := \#\mathrm{Inc}_l(w).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$

Denote by $\mathrm{Inc}_l(w)$ the set of increasing subsequences of length l of w, and by $\mathrm{inc}_l(w)$ its cardinality:

$$\mathrm{Inc}_l(w) := \left\{ w(i_1)w(i_2)\cdots w(i_l) \;\middle|\; \begin{array}{l} i_1 < i_2 < \cdots < i_l \text{ and} \\ w(i_1) < w(i_2) < \cdots < w(i_l) \end{array} \right\},$$

$$\mathrm{inc}_l(w) := \#\mathrm{Inc}_l(w).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$
$$\mathrm{inc}_l(21354) = 4.$$

For an m-subset $S = \{w_1, w_2, \cdots, w_m\} \subseteq \mathfrak{S}_n$ of permutations, we denote by $\mathrm{Inc}_l(S) = \mathrm{Inc}_l(w_1, w_2, \cdots, w_m)$ the intersection of $\mathrm{Inc}_l(w_i)$'s, and by $\mathrm{inc}_l(S) = \mathrm{inc}_l(w_1, w_2, \cdots, w_m)$ its cardinality:

$$\mathrm{Inc}_l(S) := \bigcap_{w_i \in S} \mathrm{Inc}_l(w_i),$$

$$\mathrm{inc}_l(S) := \#\mathrm{Inc}_l(S).$$

Example

$$\mathrm{Inc}_l(21354) = \{235, 234, 135, 134\},$$
$$\mathrm{Inc}_l(12534) = \{125, 123, 124, 134, 234\},$$
$$\mathrm{Inc}_l(21354, 12534) = \{134, 234\},$$
$$\mathrm{inc}_l(21354, 12534) = 2.$$

In order to describe the properties of $\mathrm{Inc}_l(S)$, we use the *weak right Bruhat order* of $\mathfrak{S}_n$, In order to describe the properties of $\mathrm{Inc}_l(S)$, we use the *weak right Bruhat order* of $\mathfrak{S}_n$, as the Weyl group of type A_{n-1}.

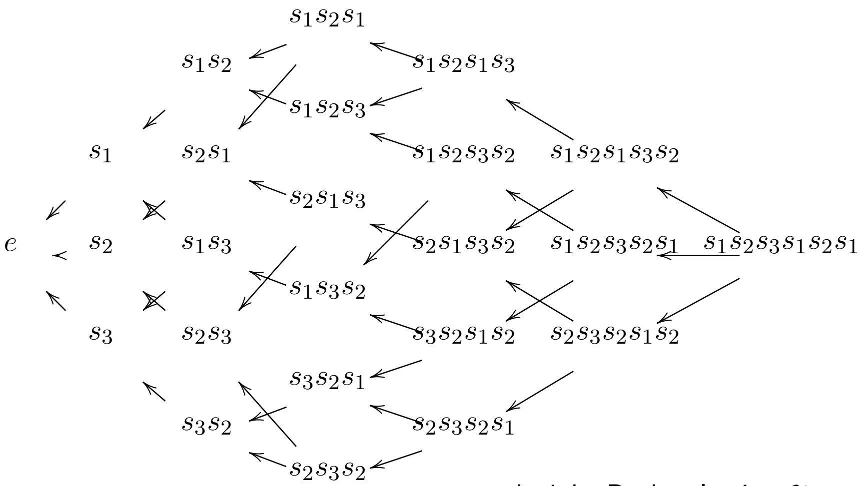
In order to describe the properties of $\mathrm{Inc}_l(S)$, we use the *weak right Bruhat order* of $\mathfrak{S}_n$, as the Weyl group of type A_{n-1}.

definition

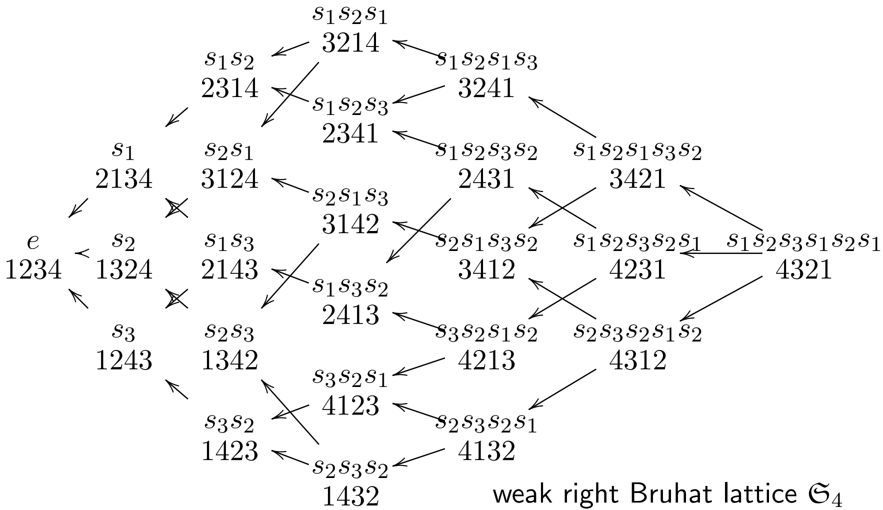
If $u, v \in \mathfrak{S}_n$ satisfy

$$us_i = v \text{ and } \ell(u) + 1 = \ell(v)$$

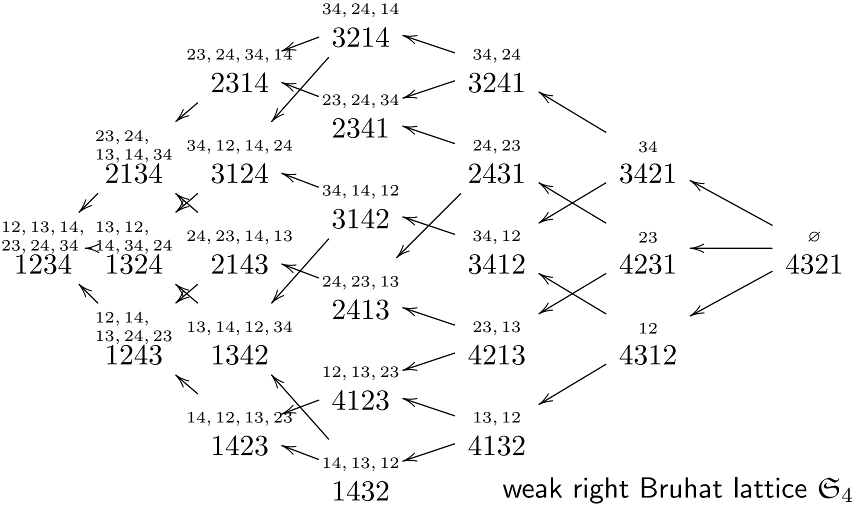
for some $1 \leq i \leq n-1$, we denote $u \dot{<} v$, where $s_i = (i, i+1)$ denotes the simple reflection (or the adjacent transposition), and $\ell(w)$ denotes the length of w (the number of inversions of w). The reflexive and transitive closure of $\dot{<}$ is denoted by $\leq$, which is called the *weak right Bruhat order of* $\mathfrak{S}_n$.



weak right Bruhat lattice $\mathfrak{S}_4$

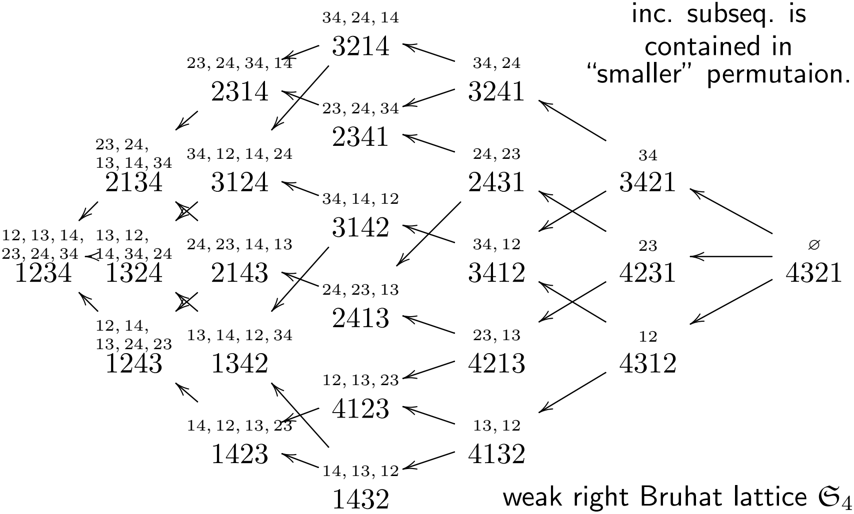


weak right Bruhat lattice $\mathfrak{S}_4$



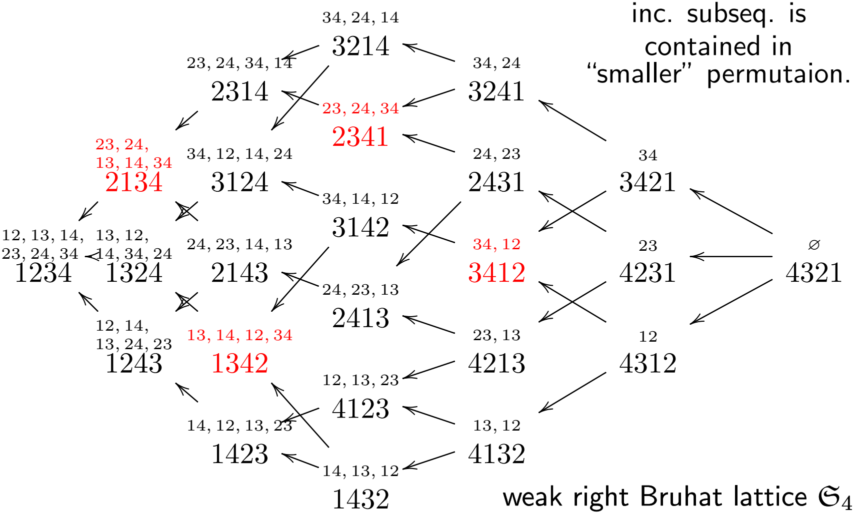
weak right Bruhat lattice $\mathfrak{S}_4$

inc. subseq. is contained in “smaller” permutaion.

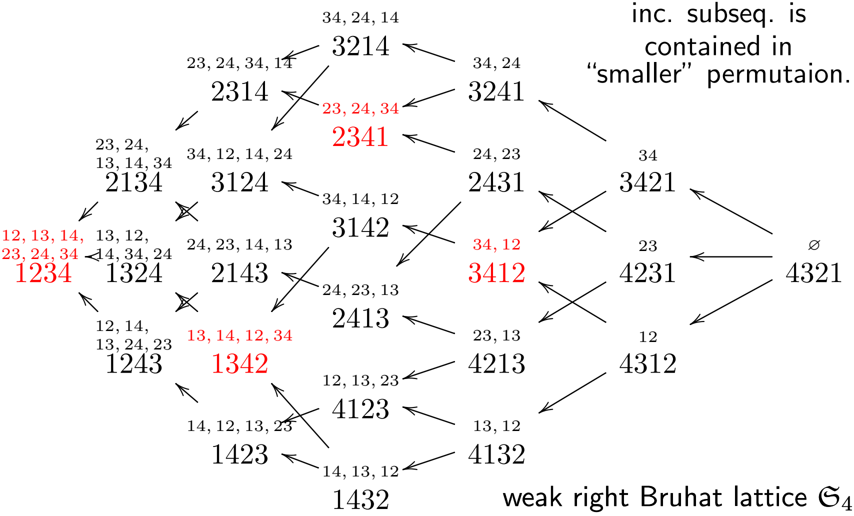


weak right Bruhat lattice $\mathfrak{S}_4$

inc. subseq. is contained in "smaller" permutaion.

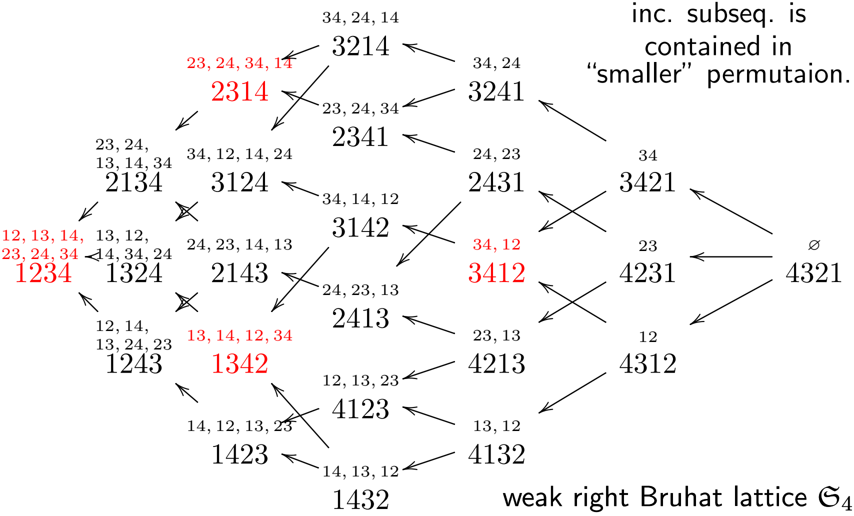


weak right Bruhat lattice $\mathfrak{S}_4$



inc. subseq. is contained in "smaller" permutaion.

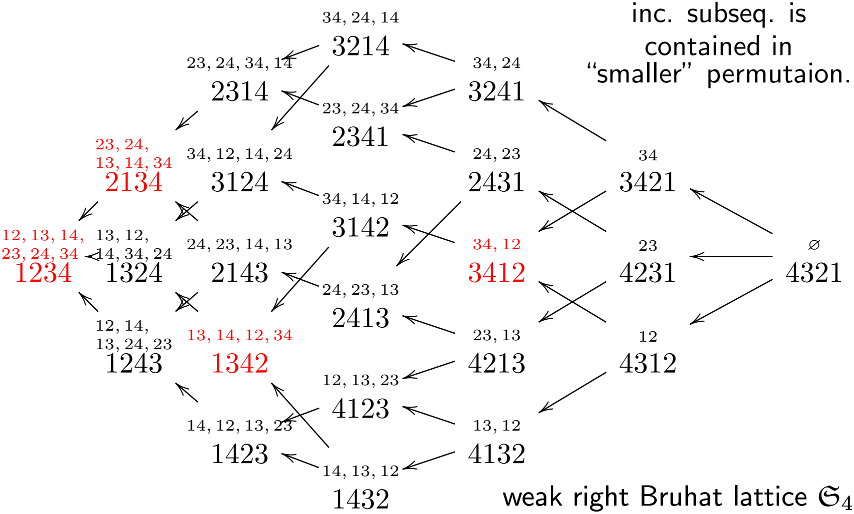
weak right Bruhat lattice $\mathfrak{S}_4$



inc. subseq. is contained in "smaller" permutaion.

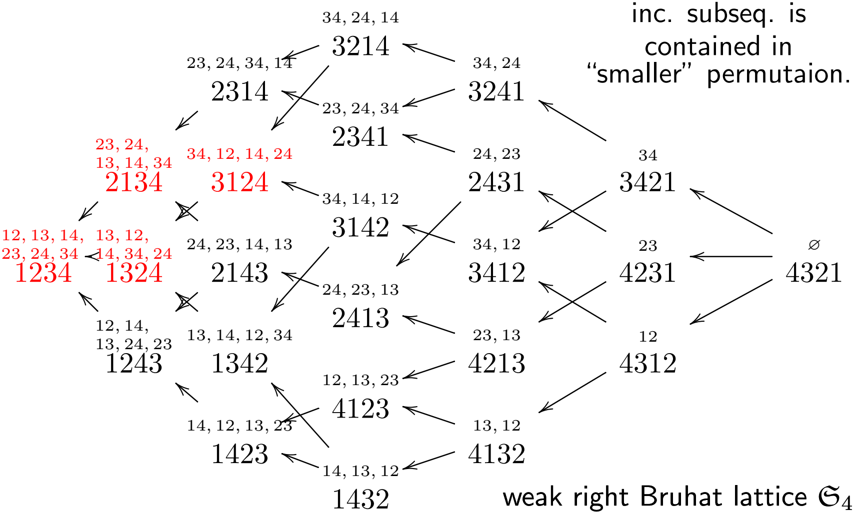
weak right Bruhat lattice $\mathfrak{S}_4$

inc. subseq. is contained in "smaller" permutaion.



weak right Bruhat lattice $\mathfrak{S}_4$

inc. subseq. is contained in "smaller" permutaion.



weak right Bruhat lattice $\mathfrak{S}_4$

Hence, we get:

Proposition

A maximal in $\left\{ \operatorname{Inc}_l(S) \,\middle|\, S \in \binom{\mathfrak{S}_n}{m} \right\}$ *with respect to set inclusion is achieved by some order ideal* S*.*

We desire to give an explicit formula for

$$\max_{S \in \binom{\mathfrak{S}_n}{m}} \operatorname{inc}_l(S).$$

First, we give two typical examples:

First, we give two typical examples:

For $m \geq 1$, we put

$w^{+}_{i;m} := s_i s_{i+1} \cdots s_{i+m-2}$ and $w^{-}_{i;m} := s_i s_{i-1} \cdots s_{j-(m-2)}$.

First, we give two typical examples:
For $m \geq 1$, we put
$w^{+}_{i;m} := s_i s_{i+1} \cdots s_{i+m-2}$ and $w^{-}_{i;m} := s_i s_{i-1} \cdots s_{j-(m-2)}$.
In particular, we have $w^{+}_{i;1} = w^{-}_{i;1} = e$ and $w^{+}_{i;2} = w^{-}_{i;2} = s_i$.

First, we give two typical examples:
For $m \geq 1$, we put
$w^{+}_{i;m} := s_i s_{i+1} \cdots s_{i+m-2}$ and $w^{-}_{i;m} := s_i s_{i-1} \cdots s_{j-(m-2)}$.
In particular, we have $w^{+}_{i;1} = w^{-}_{i;1} = e$ and $w^{+}_{i;2} = w^{-}_{i;2} = s_i$.
If $1 \leq m' < m$, then we have $w^{\pm}_{i;m'} < w^{\pm}_{i;m}$ in weak right Bruhat order.

First, we give two typical examples:
For $m \geq 1$, we put
$w^{+}_{i;m} := s_i s_{i+1} \cdots s_{i+m-2}$ and $w^{-}_{i;m} := s_i s_{i-1} \cdots s_{j-(m-2)}$.
In particular, we have $w^{+}_{i;1} = w^{-}_{i;1} = e$ and $w^{+}_{i;2} = w^{-}_{i;2} = s_i$.
If $1 \leq m' < m$, then we have $w^{\pm}_{i;m'} < w^{\pm}_{i;m}$ in weak right Bruhat order. The order ideal $\langle w^{+}_{i;m} \rangle$ generated by $w^{+}_{i;m}$ is given by:

$$\langle w^{+}_{i;m} \rangle = \left\{ w^{+}_{i;m'} \in \mathfrak{S}_n \,\middle|\, 1 \leq m' \leq m \right\} \in \binom{\mathfrak{S}_n}{m}.$$

First, we give two typical examples:
For $m \geq 1$, we put
$w^{+}_{i;m} := s_i s_{i+1} \cdots s_{i+m-2}$ and $w^{-}_{i;m} := s_i s_{i-1} \cdots s_{j-(m-2)}$.
In particular, we have $w^{+}_{i;1} = w^{-}_{i;1} = e$ and $w^{+}_{i;2} = w^{-}_{i;2} = s_i$.
If $1 \leq m' < m$, then we have $w^{\pm}_{i;m'} < w^{\pm}_{i;m}$ in weak right Bruhat order. The order ideal $\langle w^{+}_{i;m} \rangle$ generated by $w^{+}_{i;m}$ is given by:

$$\langle w^{+}_{i;m} \rangle = \left\{ w^{+}_{i;m'} \in \mathfrak{S}_n \,\middle|\, 1 \leq m' \leq m \right\} \in \binom{\mathfrak{S}_n}{m}.$$

Hence we get:

$$\begin{aligned} \mathrm{Inc}_l\big(\langle w^{+}_{i;m} \rangle\big) &= \mathrm{Inc}_l\big(w^{+}_{i;m}\big). \\ \mathrm{inc}_l\big(\langle w^{+}_{i;m} \rangle\big) &= \mathrm{inc}_l\big(w^{+}_{i;m}\big) \\ &= \binom{n-1}{l} + \binom{n-m}{l-1}. \end{aligned}$$

Let $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$. Then the order ideal $\langle s_i s_j \rangle$ generated by $s_i s_j$ is given by:

$$\langle s_i s_j \rangle = \{e, s_i, s_j, s_i s_j\} \in \binom{\mathfrak{S}_n}{4}.$$

Let $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$. Then the order ideal $\langle s_i s_j \rangle$ generated by $s_i s_j$ is given by:

$$\langle s_i s_j \rangle = \{e, s_i, s_j, s_i s_j\} \in \binom{\mathfrak{S}_n}{4}.$$

Hence we get:

$$\begin{aligned} \mathrm{Inc}_l(\langle s_i s_j \rangle) &= \mathrm{Inc}_l(s_i s_j). \\ \mathrm{inc}_l(\langle s_i s_j \rangle) &= \mathrm{inc}_l(s_i s_j) \\ &= 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l}. \end{aligned}$$

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

1. *If* $m \leq 3$ *and* $n \geq m$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*
$$\binom{n-1}{l} + \binom{n-m}{l-1}.$$

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

1. *If* $m \leq 3$ *and* $n \geq m$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*

$$S = \langle w_{i;m}^{\pm} \rangle \longrightarrow \binom{n-1}{l} + \binom{n-m}{l-1}.$$

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

1. *If* $m \leq 3$ *and* $n \geq m$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*
$$\binom{n-1}{l} + \binom{n-m}{l-1}.$$

2. *If* $m = 4$ *and* $n \geq 3$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*
$$\begin{cases} 3\binom{n-3}{l-1} + \binom{n-3}{l} & \text{if } n = 3, \\ 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l} & \text{if } 4 \leq n \text{ and } n \geq 2l-1. \\ \binom{n-1}{l} + \binom{n-4}{l-1} & \text{if } 4 \leq n \leq 2l-1, \end{cases}$$

Note $n = 2l-1 \Rightarrow \binom{n-1}{l} + \binom{n-4}{l-1} = 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l}$.

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

1. *If* $m \leq 3$ *and* $n \geq m$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \operatorname{inc}_l(S)$ *is given by:*

$$\binom{n-1}{l} + \binom{n-m}{l-1}.$$

$$S = \langle w^{\pm}_{i;4} \rangle$$

2. *If* $m = 4$ *and* $n \geq 3$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \operatorname{inc}_l(S)$ *is given by:*

$$\begin{cases} 3\binom{n-3}{l-1} + \binom{n-3}{l} & \text{if } n = 3, \\ 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l} & \text{if } 4 \leq n \text{ and } n \geq 2l-1. \\ \binom{n-1}{l} + \binom{n-4}{l-1} & \text{if } 4 \leq n \leq 2l-1, \end{cases}$$

Note $n = 2l-1 \Rightarrow \binom{n-1}{l} + \binom{n-4}{l-1} = 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l}$.

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

1. *If* $m \leq 3$ *and* $n \geq m$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*

$$S = \langle s_i s_j \rangle \qquad \binom{n-1}{l} + \binom{n-m}{l-1}.$$

2. *If* $m = 4$ *and* $n \geq 3$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*

$$\begin{cases} 3\binom{n-3}{l-1} + \binom{n-3}{l} & \text{if } n = 3, \\ 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l} & \text{if } 4 \leq n \text{ and } n \geq 2l-1. \\ \binom{n-1}{l} + \binom{n-4}{l-1} & \text{if } 4 \leq n \leq 2l-1, \end{cases}$$

Note $n = 2l - 1 \Rightarrow \binom{n-1}{l} + \binom{n-4}{l-1} = 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l}$.

By classification of order ideals S with $m = \#S \leq 4$, we get:

Theorem (N.)

Let $n \geq 1$, $m \geq 1$ *and* $0 \leq l \leq n$. *Then:*

1. *If* $m \leq 3$ *and* $n \geq m$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*

$$S = \langle s_1, s_2 s_1 \rangle \qquad \binom{n-1}{l} + \binom{n-m}{l-1}.$$

2. *If* $m = 4$ *and* $n \geq 3$, *then* $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$ *is given by:*

$$\begin{cases} 3\binom{n-3}{l-1} + \binom{n-3}{l} & \text{if } n = 3, \\ 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l} & \text{if } 4 \leq n \text{ and } n \geq 2l-1. \\ \binom{n-1}{l} + \binom{n-4}{l-1} & \text{if } 4 \leq n \leq 2l-1, \end{cases}$$

Note $n = 2l - 1 \Rightarrow \binom{n-1}{l} + \binom{n-4}{l-1} = 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l}$.

When $l = n-2$, the values $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_{n-2}(S)$ for $m = 1, 2, 3, 4$ are:

$m=1$	$m=2$	$m=3$	$m=4$
$\binom{n-1}{n-2}+\binom{n-1}{n-3}$	$\binom{n-1}{n-2}+\binom{n-2}{n-3}$	$\binom{n-1}{n-2}+\binom{n-3}{n-3}$	$\begin{cases} 3\binom{n-3}{n-3}+\binom{n-3}{n-2} & n=3 \\ 4\binom{n-4}{n-4}+4\binom{n-4}{n-3}+\binom{n-4}{n-2} & n=4 \\ \binom{n-1}{n-2}+\binom{n-4}{n-3} & n\geq 5 \end{cases}$
$=\frac{n^2-n}{2}$	$=2n-3$	$=n$	$=\begin{cases} 3 & n=3 \\ 4 & n=4 \\ n-1 & n\geq 5 \end{cases}$

Back to Motivation.

In information theory, especially in coding theory, one of the subjects is error correction of inputs. Among various types of errors, deletion errors of t bits are known to be one of the most difficult problems to correct.

Figure: 3-Deletion channel

In information theory, especially in coding theory, one of the subjects is error correction of inputs. Among various types of errors, deletion errors of t bits are known to be one of the most difficult problems to correct.

Figure: 3-Deletion channel

In information theory, especially in coding theory, one of the subjects is error correction of inputs. Among various types of errors, deletion errors of t bits are known to be one of the most difficult problems to correct.

How difficult?

Figure: 3-Deletion channel

In information theory, especially in coding theory, one of the subjects is error correction of inputs. Among various types of errors, deletion errors of t bits are known to be one of the most difficult problems to correct.

How difficult?

If $t = 1$, then it is well-known. [Varshamov-Tenengolts code]

Figure: 3-Deletion channel

In information theory, especially in coding theory, one of the subjects is error correction of inputs. Among various types of errors, deletion errors of t bits are known to be one of the most difficult problems to correct.

How difficult?

If $t = 1$, then it is well-known. [Varshamov-Tenengolts code]
If $t = 2$, the problem is already difficult and no satisfactory theory has been developed.

Figure: 3-Deletion channel

In information theory, especially in coding theory, one of the subjects is error correction of inputs. Among various types of errors, deletion errors of t bits are known to be one of the most difficult problems to correct.

How difficult?

If $t = 1$, then it is well-known. [Varshamov-Tenengolts code]
If $t = 2$, the problem is already difficult and no satisfactory theory has been developed.
If $t \geq 3$, it is hopeless.

Figure: 3-Deletion channel

The reconstruction model, first introduced by Levenshtein in 2001, assumes that an input x of some code C is transmitted over k identical t-deletion channels and that these channels generate k outputs $y_1, y_2, \cdots, y_k$ with distinct errors:

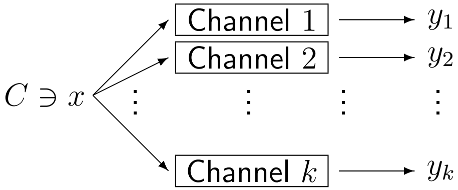
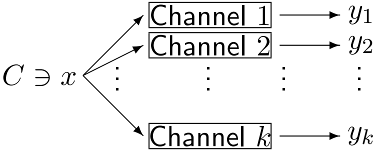
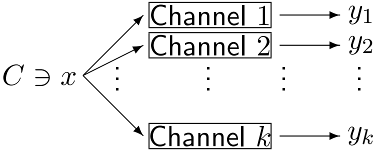


Figure: The Reconstruction Model

The transmitted word x is reconstructed using all of the channels' outputs $y_1, y_2, \cdots, y_k$.

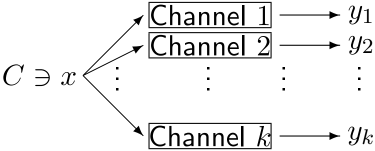


Let $\mathrm{B}_t^{\mathrm{D}}(x)$ be the set of possible channel outputs, i.e., the t-deletion error ball surrounding the word x.



Let $\mathrm{B}_t^{\mathrm{D}}(x)$ be the set of possible channel outputs, i.e., the t-deletion error ball surrounding the word x. Then, Levenshtein has proved in 2001, that unique decoding of the transmitted word is guaranteed to succeed if and only if

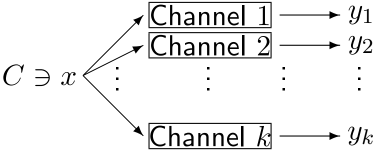
$$k > \max_{x_1, x_2 \in C, x_1 \neq x_2} |\mathrm{B}_t^{\mathrm{D}}(x_1) \cap \mathrm{B}_t^{\mathrm{D}}(x_2)|. \tag{2.1}$$



Let $\mathrm{B}_t^{\mathrm{D}}(x)$ be the set of possible channel outputs, i.e., the t-deletion error ball surrounding the word x. Then, Levenshtein has proved in 2001, that unique decoding of the transmitted word is guaranteed to succeed if and only if

$$k > \max_{x_1, x_2 \in C, x_1 \neq x_2} |\mathrm{B}_t^{\mathrm{D}}(x_1) \cap \mathrm{B}_t^{\mathrm{D}}(x_2)|. \tag{2.1}$$

However, if the number k of the channels does not satisfy the inequality in (2.1), then exact reconstruction of the transmitted word is not always possible as there may be several transmitted words leading to the same channels' outputs.



Let $\mathrm{B}_t^{\mathrm{D}}(x)$ be the set of possible channel outputs, i.e., the t-deletion error ball surrounding the word x.
Then, Levenshtein has proved in 2001, that unique decoding of the transmitted word is guaranteed to succeed if and only if

$$k > \max_{x_1, x_2 \in C, x_1 \neq x_2} |\mathrm{B}_t^{\mathrm{D}}(x_1) \cap \mathrm{B}_t^{\mathrm{D}}(x_2)|. \tag{2.1}$$

However, if the number k of the channels does not satisfy the inequality in (2.1), then exact reconstruction of the transmitted word is not always possible as there may be several transmitted words leading to the same channels' outputs. So, the value of the RHS of (2.1) is important as the threshold for the number k of channels.

However, even when inequality (2.1) does not hold, if there are only two candidates for input, the situation is much better than if there are too many candidates.

However, even when inequality (2.1) does not hold, if there are only two candidates for input, the situation is much better than if there are too many candidates.
Abu-Sini and Yaakobi considered in 2020 the case where there are only at most m candidates for input and considered the value

$$\mathrm{N}_t^{\mathrm{D}n,2}(m) := \max_{S \in \binom{C}{m}} \# \bigcap_{x \in S} \mathrm{B}_t^{\mathrm{D}}(x).$$

However, even when inequality (2.1) does not hold, if there are only two candidates for input, the situation is much better than if there are too many candidates.
Abu-Sini and Yaakobi considered in 2020 the case where there are only at most m candidates for input and considered the value

$$\mathrm{N}_t^{\mathrm{D}n,2}(m) := \max_{S \in \binom{C}{m}} \# \bigcap_{x \in S} \mathrm{B}_t^{\mathrm{D}}(x).$$

They has proved:

Theorem (Abu-Sini=Yaakobi, 2020)

If $\mathrm{N}_t^{\mathrm{D}n,2}(m+1) < k \leq \mathrm{N}_t^{\mathrm{D}n,2}(m)$, *then there are only at most* m *candidates for input.*

Theorem (Abu-Sini=Yaakobi, 2020)

If $\mathrm{N}_t^{\mathrm{D}n,2}(m+1) < k \leq \mathrm{N}_t^{\mathrm{D}n,2}(m)$, *then there are only at most* m *candidates for input.*

Theorem (Abu-Sini=Yaakobi, 2020)

If $\mathrm{N}_t^{\mathrm{D}n,2}(m+1) < k \leq \mathrm{N}_t^{\mathrm{D}n,2}(m)$*, then there are only at most* m *candidates for input.*

Proof of this theorem is easy. But we have no explicit formula of $\mathrm{N}_t^{\mathrm{D}n,2}(m)$.

Theorem (Abu-Sini=Yaakobi, 2020)

If $\mathrm{N}_t^{\mathrm{D}n,2}(m+1) < k \leq \mathrm{N}_t^{\mathrm{D}n,2}(m)$*, then there are only at most* m *candidates for input.*

Proof of this theorem is easy. But we have no explicit formula of $\mathrm{N}_t^{\mathrm{D}n,2}(m)$.
Abu-Sini=Yaakobi gave the formula for $t = 2$ and $m \leq 4$:

Theorem (Abu-Sini=Yaakobi, 2020)

$\mathrm{N}_2^{\mathrm{D}n,2}(1)$	$\mathrm{N}_2^{\mathrm{D}n,2}(2)$	$\mathrm{N}_2^{\mathrm{D}n,2}(3)$	$\mathrm{N}_2^{\mathrm{D}n,2}(4)$
$\frac{n^2-3n+4}{2}$	$2n-4$	n	$\begin{cases} 2 & n=3 \\ 4 & n=4 \\ n-1 & n \geq 5 \end{cases}$

Generalize to q-ary code case.

Generalize to q-ary code case.

Remark

If $q = 4$, there is an application to DNA codes.
Deletion error is a typical error of copy of DNA strands.

definition

Let $q \geq 2$. For $x \in (\mathbb{Z}/q\mathbb{Z})^n$, the t-deletion ball $\mathrm{B}_t^{\mathrm{D}}(x)$ is similarly defined. We put:

$$\mathrm{N}_t^{\mathrm{D}n,q}(m) := \max_{S \in \binom{Q^n}{m}} \# \bigcap_{x \in S} \mathrm{B}_t^{\mathrm{D}}(x).$$

definition

Let $q \geq 2$. For $x \in (\mathbb{Z}/q\mathbb{Z})^n$, the t-deletion ball $\mathrm{B}_t^{\mathrm{D}}(x)$ is similarly defined. We put:

$$\mathrm{N}_t^{\mathrm{D}n,q}(m) := \max_{S \in \binom{Q^n}{m}} \# \bigcap_{x \in S} \mathrm{B}_t^{\mathrm{D}}(x).$$

Then, similarly we get:

definition

Let $q \geq 2$. For $x \in (\mathbb{Z}/q\mathbb{Z})^n$, the t-deletion ball $\mathrm{B}_t^{\mathrm{D}}(x)$ is similarly defined. We put:

$$\mathrm{N}_t^{\mathrm{D}n,q}(m) := \max_{S \in \binom{Q^n}{m}} \# \bigcap_{x \in S} \mathrm{B}_t^{\mathrm{D}}(x).$$

Then, similarly we get:

Theorem (N.)

If $\mathrm{N}_t^{\mathrm{D}n,q}(m+1) < k \leq \mathrm{N}_t^{\mathrm{D}n,q}(m)$, *then there are only at most* m *candidates for input.*

When $t = 2$, we have

Theorem (N. for $q \geq 3$)

	$N_2^{Dn,q}(1)$	$N_2^{Dn,q}(2)$	$N_2^{Dn,q}(3)$	$N_2^{Dn,q}(4)$
$q \geq 3$	$\frac{n^2-n}{2}$	$2n-3$	n	$\begin{cases} 3 & n=3 \\ 4 & n=4 \\ n-1 & n \geq 5 \end{cases}$

When $t = 2$, we have

Theorem (N. for $q \geq 3$ (A=Y for $q = 2$))

	$\mathrm{N}_2^{\mathrm{D}n,q}(1)$	$\mathrm{N}_2^{\mathrm{D}n,q}(2)$	$\mathrm{N}_2^{\mathrm{D}n,q}(3)$	$\mathrm{N}_2^{\mathrm{D}n,q}(4)$
$q = 2$	$\frac{n^2-3n+4}{2}$	$2n-4$	n	$\begin{cases} 2 & n = 3 \\ 4 & n = 4 \\ n-1 & n \geq 5 \end{cases}$
$q \geq 3$	$\frac{n^2-n}{2}$	$2n-3$	n	$\begin{cases} 3 & n = 3 \\ 4 & n = 4 \\ n-1 & n \geq 5 \end{cases}$

When $t = 2$, we have

Theorem (N. for $q \geq 3$ (A=Y for $q = 2$))

	$N_2^{Dn,q}(1)$	$N_2^{Dn,q}(2)$	$N_2^{Dn,q}(3)$	$N_2^{Dn,q}(4)$
$q = 2$	$\frac{n^2-3n+4}{2}$	$2n-4$	n	$\begin{cases} 2 & n=3 \\ 4 & n=4 \\ n-1 & n \geq 5 \end{cases}$
$q \geq 3$	$\frac{n^2-n}{2}$	$2n-3$	n	$\begin{cases} 3 & n=3 \\ 4 & n=4 \\ n-1 & n \geq 5 \end{cases}$

Same table as $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_{n-2}(S)$ for $q \geq 3$!!!

In general, we have:

In general, we have:

Theorem (N.)

When we fix n, t and m, the value $\mathrm{N}_t^{\mathrm{D}n,q}(m)$ is weakly increasing for q and constant for $q \geq n$.
The constant is given by $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_{n-t}(S)$:

In general, we have:

Theorem (N.)

When we fix n, t and m, the value $\mathrm{N}_t^{\mathrm{D}n,q}(m)$ is weakly increasing for q and constant for $q \geq n$.
The constant is given by $\max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_{n-t}(S)$:

$$\lim_{q \to \infty (\text{ or } n)} \mathrm{N}_t^{\mathrm{D}n,q}(m) = \max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_{n-t}(S).$$

Thank You !!