# Boolean RSK Tableaux and Fully Commutative Permutations

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# Outline

# Basics of Permutations

We write $S_n$ for the set of permutations of $\{1, \ldots, n\}$.

The simple reflections in $S_n$ are $\{s_1, \ldots, s_{n-1}\}$, where $s_i$ swaps $i$ and $i+1$

- $s_i s_j = s_j s_i$ when $|i-j| > 1$ (commutation)
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (braid)

We will represent permutations in two ways:

- in one-line notation, as $w = w(1)w(2)\cdots w(n) \in S_n$, and
- as reduced words: (shortest) products of the $s_i$'s.

Ex. $51342$ (in one-line notation) has a reduced word $s_4 s_2 s_3 s_2 s_4 s_1$ or $[423241]$ for short.

# The RSK Correspondence

The RSK correspondence is a bijection

$$w \mapsto (P(w), Q(w))$$

from $S_n$ onto pairs of size-$n$ standard tableaux of identical shape $\lambda(w)$.

Build the insertion tableau $P(w)$ and the recording tableau $Q(w)$ using Schensted insertion.

Ex. $P(4132) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ , $Q(4132) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and $\lambda(4132) = (2, 1, 1)$.

Thm. [Schensted] The length of the first row (resp., first column) of $\lambda(w)$ is equal to the length of a longest increasing (resp., decreasing) subsequence in the one-line notation for $w$.

# The run statistic

Question. $\lambda_1(w)$ is the length of a longest increasing subsequence of $w$. Is there a concrete interpretation for $\lambda_2(w)$ and beyond?

A run is an increasing or decreasing sequence of *consecutive* integers.

For $w \in S_n$, let run$(w)$ be the fewest number of runs needed to form a reduced word for $w$.

Ex. The set of reduced words for $w = 4132$ is $\{[3231], [3213], [2321]\}$.

(audience participation)
From $[321\ 3]$ and $[23\ 21]$, we get run$(w) = 2$.

$[321\ 3]$ and $[23\ 21]$ are called optimal run word for $w$.

# Runs and RSK tableaux

Thm. [Tenner/Mazorchuk] For any boolean $w \in S_n$,

$$\mathsf{run}(w) = \lambda_2(w) = n - \lambda_1(w).$$

Our first theorem generalizes this to arbitrary permutations.

Thm. 1 [BGPRT] For any $w \in S_n$, $\mathsf{run}(w) = n - \lambda_1(w)$.

Ex. Let $w = 4132 \in S_4$.

$$P(4132) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \text{and} \quad Q(4132) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

So $\lambda_1(w) = 2$, and indeed $n - \mathsf{run}(w) = 4 - 2 = 2$.

# Fully Commutative and Boolean Permutations

We will focus on two classes of permutations:

- fully commutative (FC) permutations: all reduced words are related by commutations (not braids)
- boolean permutations: all reduced words use no repeated simple reflections

Thm. [Billey/Jockusch/Stanley; Tenner] A permutation is fully commutative iff it is $321$-avoiding.

Cor. $w$ is fully commutative iff $P(w)$ has at most two rows.

Thm. [Tenner] A permutation is boolean iff it is $321$- and $3412$-avoiding.

$$\{\text{boolean permutations}\} \subset \{\text{fully commutative permutations}\}$$

# Boolean permutations and RSK tableaux

Recall: If $w$ is a boolean permutation, run$(w) = \lambda_2(w)$.

Question. When $w$ is boolean, what is the 2nd row of $P(w)$?

Ex. Let $w = 412563$, a boolean permutation. The reduced words for $w$ are
$[364521], [634251], [364251], [321\ 6\ 45], [346521], \ldots$.

We define the canonical word of a boolean permutation as follows:

- Starting from the smallest word
- pushing decreasing runs to the left
- pushing increasing runs to the right

- $[34652\mathbf{1}]$
- $[321\ 6\mathbf{4}5]$
- $[321\ 6\ 45]$
- canon$(w) = [321645]$

# Boolean permutations and RSK tableaux

Let $\mathrm{Row}_2(P(w))$ denote the set of elements in the 2nd row of $P(w)$.

Thm. 2 [BGPRT] Let $w$ be boolean. Then

$$\mathrm{Row}_2(P(w)) = \{i+1 \mid i \text{ is the leftmost entry in a run of } \mathrm{canon}(w)\}.$$

Ex. Let $w = 4125736$, a boolean permutation.

- $\mathrm{canon}(w) = [321\ 6\ 45]$
- $P(w) =$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 6 |
| 4 | 5 | 7 | |

Cor. [BGPRT] Let $w$ be boolean. Then the canonical word for $w$ is optimal.

Cor. [BGPRT] If $w$ is boolean, then $\mathrm{Row}_2(P(w))$ can not contain 3 consecutive numbers.

# Characterizing Boolean RSK Tableaux

We say $T$ is a boolean (RSK) tableau if $T = P(w)$ for some boolean permutation $w$.

Question Every boolean tableau has at most two rows, but not vice versa.
Can we characterize boolean tableaux? Yes!

A set of integers $S$ is crowded if $S$ contains more than $z+1$ of the elements of some closed interval of even length $2z$.

Thm. 3 [BGPRT] A standard tableau $T$ with at most two rows is a boolean tableau iff $\text{Row}_2(T)$ is uncrowded.

Ex. Are they boolean tableaux? (audience participation)

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}, \qquad T_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & 6 \\ \cline{1-3} \end{array}, \qquad T_3 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 9 \\ \hline 4 & 5 & 7 & 8 \\ \cline{1-4} \end{array}$$

$No \qquad\qquad Yes \qquad\qquad No$

Prop. 4 [BGPRT] The set of boolean tableaux with $n$ boxes are in bijection with the set of $01$-words of length $n-1$ in which all run-lengths of $1$s are odd. (A028495 in OEIS.)

# The (right) weak order

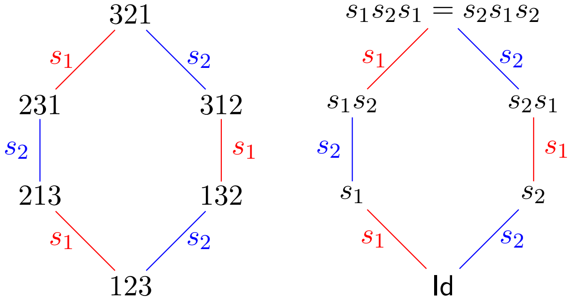
Goal.

- Analyze the poset of FC permutations under the right weak order
- and how boolean permutations live in it

The right weak order turns $S_n$ into a poset:

$v \lessdot w$ if $vs_i = w$ for some simple reflection $s_i$ and $\ell(v) + 1 = \ell(w)$.

Right weak order on $S_3$:



# Boolean core for a fully commutative Permutation

supp$(w)$ : the support of a permutation $w$; the set of simple reflections which appear in any reduced word for $w$.

Prop. 5 [BGPRT] Let $w$ be a FC permutation. Then we can write $w = bw'$, where $\ell(w) = \ell(b) + \ell(w')$, the permutation $b$ is boolean, and supp$(b) =$ supp$(w)$. Furthermore, this $b$ is uniquely determined by $w$.

Here we call $b$ the boolean core of $w$. It is also the maximal boolean permutation that is smaller than $w$ in the right weak order.

To find the boolean core of a FC permutation, select the leftmost appearance of each letter in its reduced word.

Ex. Let $w = 456123 = [\mathbf{3214}32\mathbf{5}43]$, a non-boolean FC element.

- $b = 412563 = [32145]$
- $w' = [3243]$

# Insertion tableaux under the weak order

Prop. 6 [BGPRT] If $v$ and $w$ are FC permutations with $v \lessdot w$ in the right weak order, then $\text{Row}_2(P(v)) \subseteq \text{Row}_2(P(w))$. When not equal, they differ by exactly one element.

Cor. If $b$ is the boolean core of a FC permutation $w$, then $\text{Row}_2(P(b)) \subseteq \text{Row}_2(P(w))$.

Ex.

- $v = 41623785 = [\mathbf{3215467}3]$
- $w = vs_5 = 41627385 = [\mathbf{321546}\mathbf{7}35]$
- boolean core $b = 41263785 = [3215467]$, $\quad b \lessdot v \lessdot w$
-

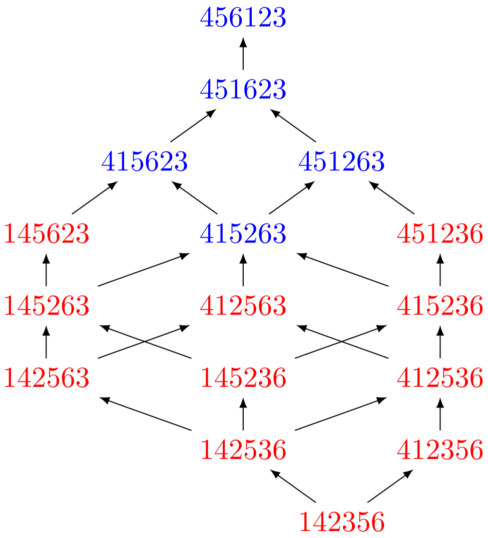
$$P(b) = P(v) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 8 \\ \hline 4 & 6 & 7 \\ \cline{1-3}\end{array} \quad \text{and} \quad P(w) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & 8 \\ \hline\end{array}$$

Thm. 7 [BGPRT] Suppose that $v$ and $w$ are FC permutations with $w = vs_i$, $\ell(w) = \ell(v) + 1$, and $s_i \in \text{supp}(v)$. Suppose, moreover, that $v$ and $w$ are uncrowded, then $P(v) = P(w)$.

# Order Ideal of Uncrowded Permutations

An FC permutation is uncrowded if its insertion tableau is boolean and is crowded otherwise.

- The crowded permutations form a dual order ideal of the poset.
- The uncrowded permutations form an order ideal of the poset.



# Characterizing Minimal Crowded Permutations

Thm. 8 [BGPRT] A FC permutation $w$ is a minimal crowded permutation iff $w$ satisfies:

1. $\text{des}(w) = \{a, a+2, \ldots, a+2k\}$ and $w(a), w(a+2), \ldots, w(a+2k)$ is crowded.
2. Contains the pattern $415263$. Every occurrence must be consecutive.
3. $w(a+2i)w(a+2i+1)w(a+2i+2)w(a+2i+3)w(a+2i+4)w(a+2i+5)$ is either of pattern $415263$ or $315264$ for $0 \leq i \leq k-2$.
4. If $a > 1$, then $w(a-1) < w(a+1)$.

When (1)-(4) hold, $\text{Row}_2(P(w)) = \{w(a), w(a+2), \ldots, w(a+2k)\}$.

Ex. $w = 41627385$,
(1) $\text{des}(w) = \{1, 3, 5, 7\}$, $4678$ is crowded;
(2) $416273$ and $627385$, both consecutive;
(3) $416273$ and $627385$ are both of the pattern $415263$;
(4) $a = 1$, so hold trivially.

*Thank you!*