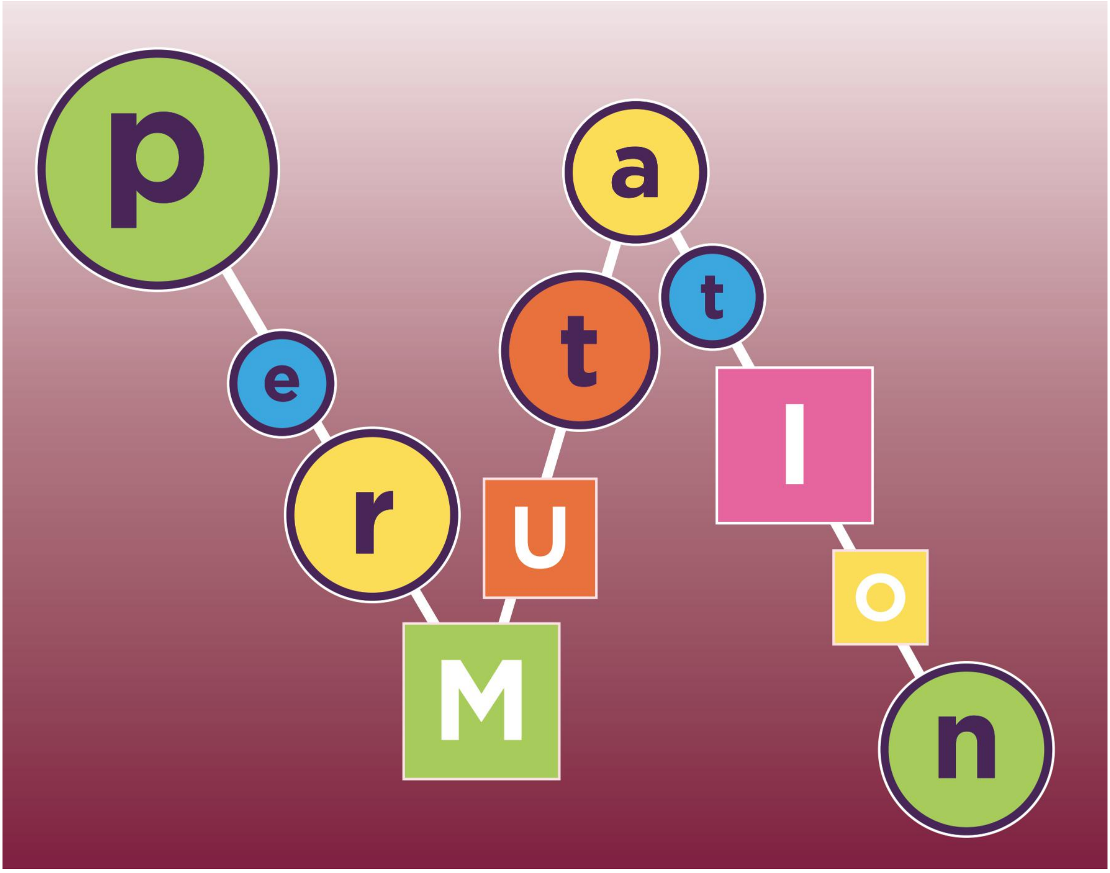


Image credit: Brandon Dzunda

Permutation Patterns 2022

June 20-24, 2022

Valparaiso University (Indiana, USA)

Contents

Welcome

Welcome to Valparaiso University for Permutation Patterns 2022! We hope that you enjoy the conference and the greater Valparaiso area.

We've included some suggestions of things to see this week in the Local Information section of this book.

If you need anything during your stay, please don't hesitate to reach out to one of our local organizers (Jon Beagley, Rick Gillman, or Lara Pudwell), who will be happy to help.

Whether you came from down the street or across the globe, thank you for joining us. We're glad you're here and we hope you have a fantastic week!

Lara Pudwell

(on behalf of the organizing committees)

Sponsors

Permutation Patterns 2022 is hosted by the Department of Mathematics and Statistics at Valparaiso University. It is also supported by National Science Foundation grant DMS-1901853 and by National Security Agency grant H98230-20-1-0286.

Important numbers

Harre Union front desk *(open 7:30-19:00)*	1-219-464-5415
Union/Conference Services Administration *(open 8:30-17:00)*	1-219-464-5413
On-call conference phone *(after hours)*	1-219-405-0105
Valparaiso University Police Department	1-219-464-5430

The nearest urgent care is at Northwest Health Urgent Care - Valparaiso. It is located at *809 LaPorte Ave, Valparaiso, IN 46383*, 0.5 miles west of Beacon Hall. The main number is 1-219-263-4977. They are open from 9:00-21:00 each day.

The nearest emergency room is at Northwest Health Porter. It is located at *85 E U.S. Hwy 6, Valparaiso, IN 46383*, 8 miles from campus. The main number is 1-219-983-8300.

You should call 911 if you have a medical emergency.

Campus Dining

Campus dining is available in the Harre Union.

Monday, Wednesday, and Friday meals

Meals are available in the *Founders' Table*, and our lunch breaks coincide with other groups on campus.

Meal times are:

Breakfast	8:00-9:45
Lunch	12:00-13:40
Dinner	17:45-18:30

If you prepaid for meals when you registered, use a voucher to pay for your meal in Founders'.

If you did not prepay, on Monday, Wednesday, and Friday you can pay cash in Founders at the following rates (*all prices include tax):

Breakfast	$8.03*
Lunch	$11.24*
Dinner	$13.64*

Tuesday and Thursday meals

Since we are the only group on campus these days, meals will be catered in the Founders' Wing of the Harre Union for those who prepaid for meal vouchers. Founders' Wing is the dining seating area adjacent to Founders', but the setup will be just for our group.

If you did not prepay, on Tuesday and Thursday, you will need to find off-campus alternatives for meals.

Local restaurants

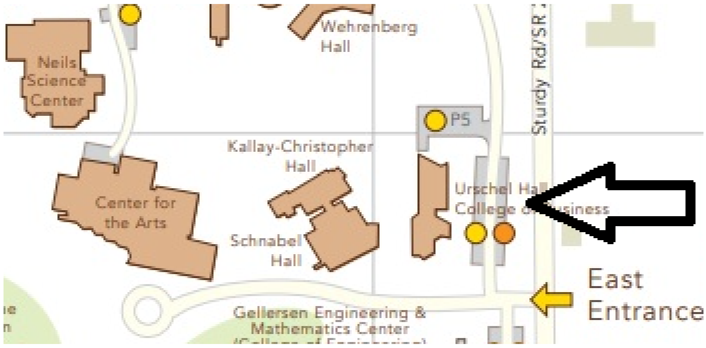
Valparaiso is a local hub of independent restaurants. In addition to many known national chains, we recommend checking out La Cabana, Santo Taco, Uptown Cafe, Prime Smoked Meats, Chunkys Tacos, Rise N Roll Bakery, Industrial Revolution, Louie Wingz & Catfish, Kelsey's Steakhouse, Tomato Bar, Kin Khao Thai and Sushi, Ricochet Tacos, Valparaiso Soup Company, Don Quixote, Blackbird Cafe, Radius, Meditrina, Brick Street Burrito, Blockhead Beerworks, Stacks, Pikks, Valpo Velvet, Bangkok Thai, Pestos, Burgerhaus, Le Peep, and more. (These are loosely ordered from close-to-campus to further afield.)

Parking

If you requested a parking pass during registration, it will be provided when you check in.

If you are staying in Beacon Hall, it is recommended that you park in the row of commuter parking spaces south of Beacon Hall (due east of the tennis courts) or the row of commuter parking spaces south of Scheele and Lankenau Halls.

If you are staying off campus, it is recommended that you park in the rows of commuter/staff parking spaces due east of Urschel Hall.



Things to do

While you're in Valparaiso, you may enjoy visiting some of the following:

- The **Chapel of the Resurrection** lies at the heart of campus. It was dedicated in 1959 and is one of the largest collegiate chapels in the world. You are welcome to walk in and explore this space. If you walk down the spiral stairs near the west entrance, you can access a hallway that includes pictures of the chapel while it was under construction. Also particularly recommended: walk to the middle of the space and look back up at the impressive Fred and Ella Reddel Memorial Organ. If you're lucky, you may even happen by the building while the organ is in use!

- The **Christopher Center Library (CCLIR)** has a fourth floor terrace that offers nice views of campus. Also, the Automated Storage and Retrieval System has interesting history and it's fun to see it in action. Ask one of the librarians at the main circulation desk if they're willing to show off the "storage robot" and tell you more about it.

- Valparaiso's downtown is home to **Central Park Plaza**, with many free community events throughout the summer. On Tuesday, June 21, they'll be showing the movie *Encanto* at dusk. There is also a farmers market in the park Tuesday and Saturday from 9:00-13:00. Even if you don't go for a particular event, look out for the bench where you can take your picture with a statue of Orville Redenbacher, the business magnate who chose Valparaiso for the popcorn factory that made him a household name.

  Fun fact: every September Valparaiso hosts a regionally-popular "Popcorn Festival" that was started to celebrate the Redenbacher connection to the city.

- The **Porter County Museum** is located at 20 E. Indiana Avenue. The original museum building across the street is the county's original 1870s era jail. The museum is free and open from 11:00-16:00 every day except for Monday. This could be an interesting way to spend a lunch break or Friday afternoon.

- Go on a **scavenger hunt**. In spring 2022, the city of Valparaiso commissioned local artists to make sculptures of 10 native birds, and they are hidden around downtown Valparaiso. You can learn more about the project (and download the one-page clue guide) at `http://tinyurl.com/ValpoBirds`.

- Go for a walk – in addition to campus, there are a number of nice parks around in town. If you have a car, **Coffee Creek Watershed**, 10 miles north of campus, and **Sunset Hill Farm**, 7 miles north of campus, are lovely free places to take a hike.

- If you're a conference regular, you may know Valparaiso is home to **Four Fathers Brewing**, a brewery that once made a small batch beer called *Permutation Pattern*. They're bringing the beer back for conference week. You'll be able to try it at the Tuesday reception and Thursday conference banquet. Want to sample more of the Four Fathers menu? Or want to buy some 4-packs of the conference beer to take home? Make sure to visit them at 3705 Bowman Dr, Suite B. They're open 12:00-20:00 Tuesday through Saturday.

A bit of history...

Valparaiso University was founded in 1859 by the Methodist Church and has been co-educational since its beginning. After going bankrupt in 1878, the university was purchased by Henry Brown and run as a for-profit institution, eventually becoming the second largest college in the country by 1914. After Brown's death, Valpo again fell on hard times and was eventually purchased by a group of Lutheran businessmen in 1925 during the height of the culture wars of the Roaring Twenties. (There is a long, long story about the why and how of this – ask Rick Gillman for more details.)

Fun fact: Valpo's university colors are brown and gold. Valparaiso is one of only a handful of colleges nationwide that use brown as a school color. In our case, it's a tribute to former University President Brown.

The Campanile...

A central figure in the Lutheran history of Valpo is President O.P. Kretzmann, who proclaimed that Valpo should uplift both "faith and reason on one fair campus," in contradiction to Tertullian's 4th Century claim that Athens and Jerusalem cannot be reconciled. This vision is embodied physically by the campanile, located in the physical center of campus and surrounded by the Chapel and the Library.

The Chapel...

Built in 1958, the Chapel of the Resurrection has been described as the largest university chapel in the United States, and one of the largest in the world. Take time to walk all the way up into the chancel to see the stained glass windows. As you do, notice how the building draws you into Christ resurrected. Then, as you turn to leave, notice how the building sends you out into the world.

The Library...

The Library (also known as the Christopher Center or the CCLIR) is a fun place to visit, from the coffee shop on the first floor, to the storage system on the main floor, to the university archives on the third floor, and the terrace on the fourth floor. Among the first students to enjoy this facility were participants in Valpo's VERUM program, a summer Research Experience for Undergraduates in mathematics targeting second year students at small colleges across the country.

The radio beacon...

As you walk by Schnabel Hall on your way to Urschel Hall from the center of campus, you walk over a very faded model of a radio beacon in the paving stones. This model represents the work of the both the Department of Communication and the Department of Meteorology which are housed in the building. You aren't likely to see it, but Valpo also has its own Doppler radar!

Speaking of beacons...

If you are staying on campus, you are staying in Beacon Hall. The yearbook is called the Beacon, and in 2021 we changed our school nickname to the Beacons. Ask Rick Gillman if you want to hear the full story.

The Solar Furnace...

If the day is sunny, you may see students and faculty working at the solar furnace, located on the east side of the Gellersen Engineering & Mathematics Center. Valpo is the only university in the United States with a research-focused solar furnace.

More About Gellersen...

Speaking of Gellersen, when our college of engineering, currently ranked 13th in the nation among undergraduate programs, built the building in 1969, they wanted a friendly housing partner and picked the math department. It was very much a tenant situation; engineering faculty had keys to all of the building, but math faculty only had keys to part of it! Take a few minutes to visit the west wing, housing the math faculty on the first and second floors and displaying posters of recent research work.

Wandering West...

If you head west from the Harre Union, you will first come to a statue "Homeless Jesus" who reminds us of our mission to serve others. Across the street, on the façade of the Arts & Sciences Building, you will see the university's motto "In thy light, we see light" shared in many different languages. A bit further on, you will see the burned remains of the Art/Psychology Building. This small building burned just a few months ago, but played a critical role in the history of the university as it was built in 1946 by engineering students who were returning soldiers from the just-ended war.

Campus Map

Urschel
Union
CCLIR
Beacon
North Entrance
East Entrance
Main Entrance
To Downtown Valpo
VALPARAISO UNIVERSITY
North
Inset Map
Sturdy Rd/SR 2
LaPorte Ave
Lincolnway
Campus Dr
Chapel Dr
Mound St
Union St
Urschel Hall (College of Business)
Harre Union
Christopher Center for Library & Information Services
Beacon Hall
Chapel of the Resurrection
Neils Science Center
Center for the Arts
Athletics-Recreation Center
Brown Field Football Stadium
Softball Field

Conference Social Events

Monday, June 20: Estimathon!

Have you participated in Estimathon at Mathfest or elsewhere? Andy Niedermaier (Jane Street Capital) is organizing a special Estimathon just for the PP community.

Estimathon will take place in the Brown and Gold room on the second floor of the Harre Union.

Tuesday, June 21: Poster session and reception

A reception and poster session will be held in the Community Room of the Christopher Center for Library and Information Sciences. Come see poster presentations and visit with conference attendees. Appetizers and drinks will be provided.

Wednesday, June 22: Excursion

Valparaiso is conveniently located near Indiana Dunes, one of the newest National Parks in the United States. A visit to the Dunes is planned for Wednesday afternoon. This excursion is included in the cost of registration for participants.

Thursday, June 23: Conference banquet

A conference banquet will be held in ballrooms on the second floor of the Harre Union. The banquet is included in the cost of registration for participants.

Conference Schedule

(All events are in Urschel 202 unless otherwise specified.)

Monday

8:15-9:00	Registration (Urschel Lobby)
9:00-9:30	Conference Welcome
9:30-10:00	**A Game of Darts** *–Andy Niedermaier*
10:00-10:30	**The shallow permutations are the unlinked permutations** *–Alexander Woo*
10:30-11:00	Break (Urschel Lobby)
11:00-11:30	**On Combinatorial Models of Affine Crystals** *–Adam Schultze*
11:30-12:00	**Stirling numbers of type** B *–Bruce E. Sagan*
12:00-14:00	Lunch Break
14:00-14:30	**A lifting of the Goulden–Jackson cluster method to the Malvenuto–Reutenauer algebra** *–Yan Zhuang*
14:30-15:00	**Connections between permutation clusters and generalized Stirling permutations** *–Justin M. Troyka*
15:00-15:30	Break (Urschel Lobby)
15:30-16:00	**Pattern-Avoiding Involutions and Brownian Bridge** *– Erik Slivken*
16:00-16:30	**The Expected Number of Distinct Patterns in a Random Permutation** *– Anant Godbole*
16:30-17:00	**The first occurrence of a pattern in a random sequence** *– Yixin (Kathy) Lin*
19:00-20:00	Estimathon! (Harre Union Brown & Gold Room)

Tuesday

9:00-9:30 **Using Constraint Programming to Enumerate Permutations avoiding Mesh Patterns**
– *Ruth Hoffmann*

9:30-10:00 **Combinatorial Exploration: An Algorithmic Framework for Enumeration**
– *Jay Pantone*

10:00-10:30 **The interval posets of permutations seen from the decomposition tree perspective**
– *Lapo Cioni*

10:30-11:00 Break (Urschel Lobby)

11:00-12:00 *Invited talk*
Limits of constrained permutations and graphs *via* decomposition trees
– *Mathilde Bouvel*

12:00-14:00 Lunch Break

14:00-14:30 **Preimages under the Bubblesort operator**
– *Luca Ferrari*

14:30-15:00 **Shuffle Sorting Permutations**
– *Rebecca Smith*

15:00-15:30 Break (Urschel Lobby)

15:30-16:00 **Restricted generating trees for weak orderings**
– *Juan B. Gil*

16:00-16:30 **Enumerating Orderings on Matched Product Graphs**
– *Daryl DeFord*

Tuesday (continued)

16:45-18:00 Poster Session and Reception
(Christopher Center for Library & Information Resources (CCLIR), Community Room)

Posters:

An analogue of direct sum, skew sum of permutations and chain permutational posets to words
– *Amrita Acharyya*

Recursive maps for derangements and nonderangements
– *Melanie Ferreri*

Restricted Grassmannian permutations
– *Juan B. Gil and Jessica A. Tomasko*

Unimodality of q-twotorials via alternating gamma vectors
– *Jordan Tirrell*

Wednesday

9:00-10:30	Virtual Posters (in Zoom)

Posters:

2-avoidance
– *Murray Elder*

Pattern-avoiding binary trees and permutations
– *Namrata*

Long paths, deep trees and dual cycles
– *Michal Opler*

10:30-11:00 Break (Urschel Lobby)

11:00-11:30 **Continuity of Major index on involutions**
– *Eli Bagno*

11:30-12:00 **Occurrences of a specific pattern in hypercube orientations, aka Statistics on reciprocal sign epistasis in fitness landscapes**
– *Manda Riehl*

12:00-13:00 Lunch Break

13:00-18:00 Conference Excursion

Thursday

9:00-9:30	**Descents on nonnesting multipermutations** – *Sergi Elizalde*
9:30-10:00	**Pattern avoidance in parking functions** – *Ayo Adeniran*
10:00-10:30	**Cycle structure of random parking functions** – *J. E. Paguyo*
10:30-11:00	Break (Urschel Lobby)
11:00-12:00	*Invited talk* **Generalizations of parking functions and a connection to pattern avoidance** – *Pamela Harris*
12:00-14:00	Lunch Break
14:00-14:30	**On the generating functions of pattern-avoiding Motzkin paths** – *Christian Bean*
14:30-15:00	**Transport of patterns by Burge transpose** – *Giulio Cerbai*
15:00-15:30	Break (Urschel Lobby)
15:30-16:30	Open Problem Session
18:00-20:00	Conference Banquet (Harre Union Ballrooms)

Friday

9:00-9:30 **Maximum Number of Common Increasing Subsequences of several Permutations**
– *Kento Nakada*

9:30-10:00 **A q-analogue and a symmetric function analogue of a result of Carlitz, Scoville and Vaughan**
– *Yifei Li*

10:00-10:30 **On permutation classes defined by pin sequences**
– *Ben Jarvis*

10:30-11:00 Break (Urschel Lobby)

11:00-11:30 **An extension of the Lindström-Gessel-Viennot theorem**
– *Yi-Lin Lee*

11:30-12:00 **Boolean** RSK **tableaux and fully commutative permutations**
– *Jianping Pan*

12:00 Farewell! Hope to see you at PP2023!

Contents

An analogue of direct and skew sum of permutations, chain permutational posets to words

Amrita Acharyya University of Toledo

In this presentation, I will describe a way to construct direct sum and skew sum of words that are not necessarily permutations in a similar method they are defined for permutations. Here I discuss some familiar statistics for example four fundamental Statistics lb,rb,ls,rs by Wachs and White towards direct and skew sum of restricted growth functions corresponding to set partitions and other words along with lsg,rsg. An analogue to chain permutational poset is defined replacing the set of permutations by set of words of certain finite lengths. Some examples of such posets in terms of familiar set partition posets, divisor posets for any positive integer n using their Hasse diagrams are given.

References

[1] Xin Chen, *A q=-1 Phenomenon for Pattern-Avoiding Permutations* Volume 12, Issue 2, Rose Hulman Undergraduate Mathematics Journal (2011)

[2] Rodica Simion, Frank W. Schmidts, *Restricted Permutations*, European J. Combin. 6 (1985), no. 4, 383-406.

[3] Lindsey R. Campbell, Samantha Dahlberg, Robert Dorward, Jonathan Gerhard,Thomas Grubb, Carlin Purcell, Bruce E Sagan, *Restricted growth function patterns and statistics*, Adv. in Appl. Math., 100 (2018), 1-42.

Ayo Adeniran Colby College

This talk is based on joint work with Lara Pudwell

Given a one-way street with n spots, in how many ways can n cars park on the street without any collisions or any car having to exit the street? This question is the crux of the classical parking problem. The solution to this problem is given by the concept of *parking functions*.

Parking functions and Dyck paths

Let us recall the following definitions:

Definition 1. A *parking function* is a sequence $a_1 \cdots a_n \in [n]^n$ such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is its increasing rearrangement, then $b_i \leq i$ for all $1 \leq i \leq n$.

Definition 2. A *Dyck path* is a lattice path from $(0,0)$ to (n,n) in the Cartesian plane consisting of n north-steps and n east-steps that never crosses below the line $y = x$.

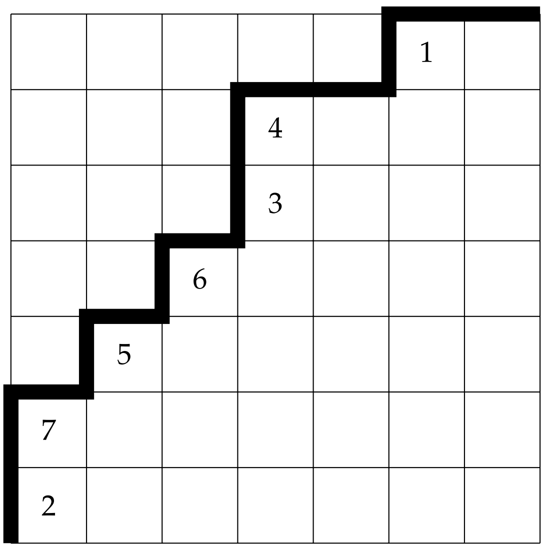


Figure 1: Labeled Dyck path representing 6144231

We can represent parking functions as Dyck paths with labeled north-steps. In such a path, we may label each of the n north-steps with a distinct integer from $\{1, \ldots, n\}$ such that consecutive north-steps must have their labels in increasing order. In this representation, the labels of north-steps along $y = k$ correspond to the cars who prefer spot $k+1$. For example, the parking function 6144231 corresponds to the representation:

$$\{2,7\}, \{5\}, \{6\}, \{3,4\}, \emptyset, \{1\}, \emptyset$$

with the associated permutation 2756341 (see Figure 1).

Patterns in Parking functions

We follow the definition of Remmel and Qiu in [1] to extend the classical definition of patterns in permutations to parking functions. In particular, we study parking functions that avoid permutations of length 3. For example, the parking function 6144231 avoids the pattern 213 since the permutation associated with its labeled Dyck path representation avoids the pattern 213.

We have enumerated parking functions avoiding any collection of two or more permutations of length 3. A number of well known combinatorial sequences arise in our analysis, and this talk will highlight several enumeration results and conjectures.

References

[1] J. Remmel and D. Qiu, Patterns in ordered set partitions and parking functions, Permutation Patterns 2016 (slides), available electronically at https://www.math.ucsd.edu/~duqiu/files/PP16.pdf.

Continuity of Major index on involutions

Eli Bagno Jerusalem College of Technology

This talk is based on joint work with Yisca Kares

We find the range of the major index on the various conjugacy classes of involutions in the symmetric group S_n. In addition to indicating the minimum and the maximum values, we show that except for the case of involutions without fixed points, all the values in the range are attained. For the conjugacy classes of involutions without fixed points we show that the only missing values are one more than the minimum and one less than the maximum.

Involutions, tableaux and the RSK

Let I_n be the set of all involutions in the symmetric group S_n. For a shape λ, let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ.

The *descent set* of a standard Young tableau T is

$$\mathrm{Des}(T) := \{i \mid i+1 \text{ appears in a lower row of } T \text{ than } i\}.$$

Define also the *major index* of a standard Young tableau T by

$$maj(T) = \sum_{i \in Des(T)} i.$$

The following are two crucial properties of the RSK correspondence which maps each permutation $\pi \in S_n$ to a pair of standard Young tableaux of the same shape (P_π, Q_π), on which we rely heavily in this work.

Fact 1. *For each $\pi \in S_n$, $Q_\pi = P_{\pi^{-1}}$.*

The RSK correspondence is a Des-preserving and hence also *maj*- preserving bijection in the following sense.

Fact 2. *For every permutation $\pi \in S_n$,*

$$\mathrm{Des}(P_\pi) = \mathrm{Des}(\pi^{-1}) \quad \textit{and} \quad \mathrm{Des}(Q_\pi) = \mathrm{Des}(\pi).$$

It follows from Fact 1 that π is an involution if and only if $P_\pi = Q_\pi$ so that by restricting the RSK correspondence to the set of involutions I_n, we get a $Des-$ preserving bijection from I_n to the set of standard Young tableaux of order n, $SYT(n)$.

Conjugacy classes in S_n are determined by their cycle structures, which are partitions of n. The conjugacy classes of involutions in S_n have the form $(2^k, 1^r)$ such that $0 \le r \le n$, $0 \le k \le \frac{n}{2}$ and $2k + r = n$. In other words, conjugacy classes of involutions are distinguished one from another by the number of fixed points.

The following known result by Schützenberger [3] gives a full description of the image of each conjugacy class of involutions under the RSK correspondence.

Proposition 3. *An involution $\pi \in I_n$ has r fixed points if and only if $P(\pi)$ has r columns of odd length.*

In light of this characterization, denote the set of Young diagrams of size n having exactly r odd columns by $D_n(r)$. The set of standard Young tableaux of shapes taken from $D_n(r)$ is denoted by $SYT_n(r)$.

The discussion above can now be concisely formulated as follows.

Proposition 4. *Let C_μ be the conjugacy class of the partition $\mu = (2^k, 1^r)$. Then the restriction of the RSK correspondence $R : C_\mu \to SYT_n(r)$ is a bijection which preserves the major index, i.e. for each $\pi \in C_\mu$ we have $maj(\pi) = maj(R(\pi))$.*

Definition 5. For a shape $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_u)$, let

$$b(\lambda) = \sum_{i=0}^{u} i\lambda_i.$$

The continuity of maj inside a single shape has been recently proven by Billey, Konvalinka and Swanson in [1] (see Theorem 1.1. there). The following is a reformulation of their result.

Proposition 6. *Let λ be a Young diagram. Then we have:*

$$m(\lambda) := Min\{maj(T) \mid T \in SYT(\lambda)\} = b(\lambda).$$

$$M(\lambda) := Max\{maj(T) \mid T \in SYT(\lambda)\} = \binom{n}{2} - b(\lambda').$$

Moreover, every value between $m(\lambda)$ and $M(\lambda)$ appears at least once except in the case when λ is a rectangle with at least two rows and columns, in which case the values $m(\lambda) + 1$ and $M(\lambda) - 1$ are missing.

Range of the major index on conjugacy classes of involutions

Our main result is the following.

Theorem 7. *Let $\mu = (2^k, 1^r)$ be a partition of n and let C_μ be the corresponding conjugacy class of involutions in S_n. Then*

- *If $r \neq 0$ then the major index on C_μ attains all values between k and $\binom{n}{2} - \binom{r}{2}$.*
- *If $r = 0$ then it attains all the values above, excluding $k + 1$ and $\binom{n}{2} - 1$.*

Any other value outside this range is not attained.

Before presenting the sketch of the proof, we need a lemma and a definition.

The following lemma determines the diagrams of $D_n(r)$ which attain the minimum and the maximum of the major index.

Lemma 8. *Let $n = 2k + r$.*

1. *The minimum value of the major index on $D_n(r)$ is k. It is attained by the diagram $\lambda = (n-k, k) = (k+r, r)$.*
2. *The maximum value of the major index on $D_n(r)$ is $\binom{n}{2} - \binom{r}{2}$. It is attained by the diagram $\lambda = (r, 1^{2k})$.*

Definition 9. Recall that a diagram of the form $(u, 1^{n-u})$ is called a hook. If the length of the leg of λ, $n - u + 1$ is odd then λ will be called an *odd hook*. It will be called an *even hook* otherwise.

Sketch of the proof of Theorem 7

By Propositions 3 and 4, it is sufficient to prove our results for the set $SYT_r(n)$ consisting of all standard tableaux of shapes having exactly r odd columns.

This will be done by ordering the set $D_n(r)$ of diagrams of size n with exactly r odd columns, by reverse dominant order and presenting an algorithm which starts with the diagram $\lambda^0 = (n-k, k) = (k+r, k)$, attaining the minimum value of *maj* over $D_n(r)$ which is k and ends with the odd hook diagram $\lambda^e = (r, 1^{2k})$, attaining the maximum value of *maj* over $D_n(r)$ which is $\binom{n}{2} - \binom{r}{2}$ (see Lemma 8.2). The algorithm traverses the set $D_n(r)$ in such a way that in each step, one or two squares of a diagram $\lambda \in D_n(r)$ are transferred to a new place to obtain a diagram $\nu \in D_n(r)$ such that following condition is satisfied:

$$M(\lambda) \geq m(\nu), \tag{1}$$

where $M(\lambda)$ ($m(\nu)$) is the maximum (minimum) value of *maj* on $SYT(\lambda)$ ($SYT(\nu)$), respectively as in Proposition 6. Together with Proposition 6, we are done.

The algorithm

Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_t)$. We add infinite number of zeroes at the end of λ and write λ as a Young diagram. Also, for each i, denote the last square of the row λ_i by λ_i^*. Now perform the following steps:

1. If λ is an odd hook then we are done by Lemma 8.2.

2. If λ is an even hook then we distinguish between two cases:
   - If $\lambda = (1^{2k})$, i.e. $r = 0$, then again we are done by Lemma 8.2.
   - Otherwise, let ν be the shape obtained from λ by removing the square λ_0^* and placing it at the end of the first column of λ. This ν is an odd hook so we are back in (1).
3. If there is some $0 \leq j \leq t$ such that $\lambda_j > \lambda_{j+1} > \lambda_{j+2}$ then let i be maximal with respect to this property. Since $\lambda_i > \lambda_{i+1} > \lambda_{i+2}$, we have in λ a column of length $i+1$ and a column of length $i+2$. Now, remove λ_i^* and place it as the last square of row $i+2$ and let ν be the resulting shape.

   Comment 10. *This case is illustrated in Figs. 2 and 3.*

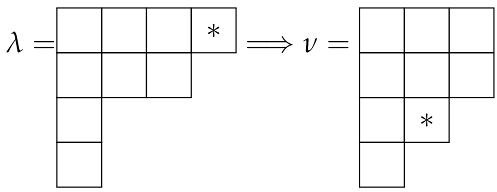


Figure 2: $\lambda = (4, 3, 1, 1)$ and $\nu = (3, 3, 2, 1), i = 0$.

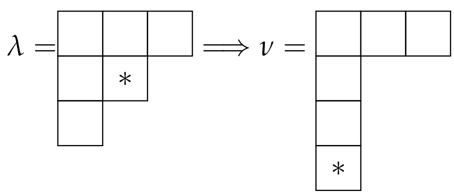


Figure 3: $\lambda = (3, 2, 1)$ and $\nu = (3, 1, 1, 1), i = 1$.

4. Otherwise, if there is no $0 \leq j \leq t$ such that $\lambda_j > \lambda_{j+1} > \lambda_{j+2}$ then there must exist some j such that $\lambda_j > \lambda_{j+1} = \lambda_{j+2}$ (we can always choose $j = t$ to get $\lambda_t > \lambda_{t+1} = 0 = \lambda_{t+2}$). Let i be maximal with respect to this property and such that $\lambda_i > 1$.

   Observe that we must have $\lambda_{i-1} = \lambda_i$ since otherwise we have already had treated this case in (3). This means that the squares λ_{i-1}^* and λ_i^* form a vertical domino.

   We distinguish between two sub-cases.

   (a) If we have $\lambda_i - \lambda_{i+1} = 1$ then by the maximality of i we must have $\lambda_i = 2$. In this case we put the domino at the end of the first column.

   (b) Now, if $\lambda_i - \lambda_{i+1} > 1$ then we move that domino to the pair of squares located right after λ_{i+1}^* and λ_{i+2}^*.
5. Back to step (1) with $\lambda = \nu$.

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On the generating functions of pattern-avoiding Motzkin paths

Christian Bean Reykjavik University

This talk is based on joint work with Antonio Bernini, Matteo Cervetti, and Luca Ferrari

A *Motzkin path* of length n is a lattice path starting at $(0,0)$ and ending at $(n,0)$ consisting of *up steps* ($U = (1,1)$), *down steps* ($D = (1,-1)$) and *horizontal steps* ($H = (1,0)$) that never goes below the x-axis. Let $\mathcal{M}$ be the set of all Motzkin paths, $\mathcal{M}_H$ be the set of Motzkin paths that start with a horizontal step, and $\mathcal{M}_U$ be the set of Motzkin paths starting with an up step. The "folklore" result on Motzkin paths says that every Motzkin path in $\mathcal{M}_H$ can be written as Hw for some w in $\mathcal{M}$, and every Motzkin path in $\mathcal{M}_U$ can be written $UxDy$ for some x, y in $\mathcal{M}$ as shown in Figure 4.

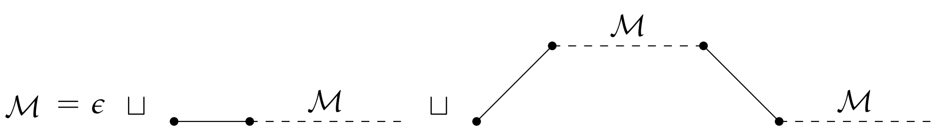

Figure 4: A pictorial representation of the structural decomposition of Motzkin paths.

Akin to investigations for other combinatorial structures (eminently for permutations, but also for graphs), there has been interest in studying properties related to the notion of *patterns* in the context of lattice paths. A Motzkin path p *contains* a *pattern* q in $\{U, H, D\}^*$, written $q \preceq p$, if q occurs as a subword in p. If p does not contain q we say p *avoids* q and write $q \not\preceq p$. For a set P of patterns, we say a path *avoids* P if it avoids all $q \in P$ and define the set of Motzkin paths avoiding P as

$$\text{Av}(P) = \{p \in \mathcal{M} \mid p \text{ avoids } P\}.$$

In this talk, we outline an algorithm using the decomposition in Figure 4 for computing a *combinatorial specification* for sets of Motzkin paths avoiding an arbitrary set of patterns. Such a specification then gives a method for computing the generating function but also the ability to sample uniformly from these sets.

Finally, we prove the following theorem by describing a recursive procedure to compute the generating function for Motzkin paths avoiding a single pattern.

Theorem 1. *Let q be a fixed pattern and let a_n be the number of q-avoiding Motzkin paths of length n. Then the generating function $\Delta_q(x) = \sum_{n\geq 0} a_n x^n$ is rational over x and $C(x) = \sum_{n\geq 0} C_n x^{2n}$, where C_n is the n-th Catalan number.*

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Limits of constrained permutations and graphs *via* decomposition trees

Mathilde Bouvel Université de Lorraine, CNRS, Inria, LORIA, France

This talk is based on joint work with Jacopo Borga, Frédérique Bassino, Michael Drmota, Valentin Féray, Lucas Gerin, Mickaël Maazoun, Adeline Pierrot, Benedikt Stufler.

In this talk, I would like to survey several recent results showing how the substitution decomposition of permutations can be used to answer (some instances of) the question: "for a permutation class $\mathcal{C}$, what does a large permutation of $\mathcal{C}$ typically look like?". I would also like to discuss the parallel approach on hereditary families of graphs using the modular decomposition.

Permutations

As reviewed in [1], every permutation (of size > 1) can be expressed as the inflation of a simple permutation. More precisely, every permutation (of size > 1) can be *uniquely* expressed as one of the following (see [2]):
• $\oplus[\alpha_1, \dots, \alpha_k]$ for $k \geq 2$ and permutations α_i which are $\oplus$-indecomposable;
• $\ominus[\alpha_1, \dots, \alpha_k]$ for $k \geq 2$ and permutations α_i which are $\ominus$-indecomposable;
• $\sigma[\alpha_1, \dots, \alpha_k]$ for σ a simple permutation of size $k \geq 4$.
Above, $\oplus[\dots]$ and $\ominus[\dots]$ denote respectively (direct) sums and skew sums, or equivalently increasing and decreasing permutations of any size at least 2.

This allows to recursively represent a permutation by a tree, whose root is $\oplus$, $\ominus$ or σ, to which the trees associated with the α_i are attached. It is not always the case that the family of trees associated with a permutation class $\mathcal{C}$ is conveniently described. But when it is, we will see that this encoding can be used to derive limiting results for uniform permutations in $\mathcal{C}$. More precisely, we will be able to describe the permuton limit of uniform permutations in $\mathcal{C}$.

Permutons are probability measures on the unit square with uniform projections on the axes. They can be understood as normalized permutation diagrams, of permutations which may be of infinite size. Consequently, when describing the permuton limit of a uniform permutation in a class $\mathcal{C}$, we are essentially describing the limit of the diagram of a large typical permutation in $\mathcal{C}$. Permutons are tightly related to patterns in permutations: indeed, the convergence in the permuton sense of a sequence of permutations $(\sigma_n)_n$ is characterized by the convergence, for any (classical) pattern τ, of the density of occurrences of τ in σ_n. This extends to the setting where the permutations $(\sigma_n)_n$ are random, in which case (perhaps surprisingly) convergence *in expectation* of the pattern densities is enough.

In all the classes $\mathcal{C}$ that we studied, the approach is similar. Using the tree encoding of the permutations in $\mathcal{C}$, we describe the generating function of $\mathcal{C}$. In addition, for every pattern τ, we describe the generating function of permutations of $\mathcal{C}$ with $|\tau|$ marked elements which form an occurrence of the pattern τ. Performing singularity

analysis of these generating functions, we derive the limit of the expected densities of occurrences of τ, hence the permuton limit.

This is the approach used in [4] for substitution-closed classes (essentially, the classes whose permutations are encoded by all possible trees obtained from a set of allowed simple permutations σ). It is extended in [6] to permutation classes whose trees are described by a finite combinatorial specification (and this includes all classes containing a finite number of simple permutations).

We note that an alternative approach is possible, proving the convergence of the trees encoding the permutations, and then deducing the limiting permuton result. This approach is used in our first article [3] on the topic, describing the limit of separable permutations (those where only $\oplus$ and $\ominus$ appear in the decomposition trees). See also the invited talk of Lucas Gerin at the on-line workshop PP2021. We have extended this approach in [5] to substitution-closed classes.

In many cases, the limiting permuton is a *biased Brownian separable permuton*, which is a simple deformation of the (unbiased) Brownian separable permuton which first appeared in [3] as the limit of uniform separable permutations. This is an instance of a *universality phenomenon*, with many families sharing essentially the same limit. Nevertheless, in other situations, the same methods allow to establish convergence to permutons which are different from Brownian separable permutons.

Graphs

Permutations are not the only objects which enjoy an encoding by trees through a recursive decomposition. In particular, the modular decomposition of graphs allows to encode each graph by a tree, similarly to the encoding of permutations described above. (Unlike for permutations, these trees are however non-plane, which is a framework slightly more complicated.) The role of $\oplus$ is played by the independent sets (graphs with no edges), $\ominus$ corresponds to cliques (a.k.a. complete graphs), and the analogue of simple permutations are the *prime* graphs (containing no *module*, the analogue of an interval in permutations).

In the first part of this abstract, we have seen how to use the substitution decomposition of permutations to prove permuton convergence of uniform permutations in permutation classes. We would like to extend this idea to the framework of graphs, making use of the analogy between substitution decomposition of permutations and modular decomposition of graphs. Of course, this requires to also have analogues of permutons and of permutation classes.

Permutons were actually defined as a permutation analogue to a corresponding notion in graphs, describing the limits of dense graphs: the *graphons*. Although the definition of graphons is less intuitive than for permutons, the graphon convergence enjoys a characterization analogous to the permuton case: by convergence of all densities of induced subgraphs.

Permutation classes are defined as sets of permutations closed downward for taking

patterns. The natural analogue of patterns in permutations are induced subgraphs in graphs. This leads us to consider hereditary classes of graphs, that is, families of graphs which are closed downward by taking induced subgraphs.

We start with the simplest case, which is the graph analogue of separable permutations: the family of cographs (defined, for instance, by the avoidance of induced P_4, the path with 4 vertices). As in the permutation case, there are two approaches to establish their graphon limit: using generating functions and singularity analysis [7], or using random trees [8]. Again, the limiting object is a Brownian object, called the *Brownian cographon*. Knowing the graphon limit of cographs, it is possible to show that a uniform random cograph contains no clique nor independent set of linear size. In other terms, this means that P_4 does not have the so-called asymptotic linear Erdős-Hajnal property [9], answering a question of Kang, McDiarmid, Reed and Scott in 2014.

The approach can be extended to other families of graphs, whose modular decomposition is well-behaved. This is the topic of the PhD thesis of Théo Lenoir (who started in September 2021 under the supervision of F. Bassino and L. Gerin), and I will try to present his first results.

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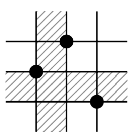
Transport of patterns by Burge transpose

Giulio Cerbai University of Iceland

This talk is based on joint work with Anders Claesson

Ascent sequences ($\mathcal{A}$) were introduced in 2010 [2] as an auxiliary set of objects that most transparently embodies the recursive structure of $(2+2)$-free posets, Stoimenow's matchings and Fishburn permutations ($\mathcal{F}$). Since then, pattern avoiding ascent sequences have been quite thoroughly investigated [1, 3], but a framework capable of producing general results is missing. Pattern avoidance on $\mathcal{F}$ has been studied in [4]. The main purpose of this work is to initiate the development of a theory of transport of patterns from Fishburn permutations to ascent sequences, and vice versa, aiming towards a more general understanding of pattern avoidance. Instead of ascent sequences we use their modified version, that is the bijective image $\hat{\mathcal{A}}$ of $\mathcal{A}$ under the $x \to \hat{x}$ mapping [2]. Our approach is in fact more general and can transport patterns between permutations ($\mathcal{S}$) and equivalence classes of so called Cayley permutations (Cay).

$\mathfrak{f}$ = 

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\varphi"] \arrow[d, "x \mapsto \hat{x}"'] & & \mathcal{F} \\ \hat{\mathcal{A}} \arrow[urr, "\psi"'] & & \end{tikzcd}$$

Figure 5: Bivincular pattern $\mathfrak{f}$ characterizing Fishburn permutations, on the left. Bijections relating $\mathcal{A}$, $\hat{\mathcal{A}}$ and $\mathcal{F}$, on the right.

Burge words and Burge transpose

A word consisting of positive integers that include at least one copy of each integer between one and its maximum value is called a *Cayley permutation*. Denote by Cay the set of Cayley permutations. Define the set of *Burge words*:

$$\mathrm{Bur}_n = \{ \binom{u}{v} : u \in I_n,\ v \in \mathrm{Cay}_n,\ D(u) \subseteq D(v)\},$$

where I_n is the subset of Cay_n consisting of the weakly increasing Cayley permutations and $D(v) = \{i : v(i) \geq v(i+1)\}$ is the set of weak descents of v. We shall define a transposition operation T on Bur_n as follows. Let $w \in \mathrm{Bur}_n$. To compute the *Burge transpose* w^T of w, turn each column of w upside down and then sort the columns in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. Observe that T is an involution on Bur_n. Define the map $\gamma : \mathrm{Cay}_n \to \mathcal{S}_n$ by

$$\binom{\mathrm{id}}{v}^T = \binom{\mathrm{sort}(v)}{\gamma(v)},$$

for any $v \in \mathrm{Cay}$, where id is the identity permutation and $\mathrm{sort}(v)$ is obtained by sorting v in weakly increasing order. If $\sigma \in \mathcal{S}$ is a permutation, then $\gamma(\sigma) = \sigma^{-1}$ (and thus γ is surjective). Moreover, if $\hat{x}$ is a modified ascent sequence then $\gamma(x) = \psi(\hat{x})$ is the Fishburn permutation corresponding to $\hat{x}$ (see Figure 5).

The transport theorem

We extend the notion of pattern containment on Burge words as follows. To ease notation we will often write biwords as pairs. Let $(v, y) \in \mathrm{Bur}_k$ and $(u, x) \in \mathrm{Bur}_n$. Then $(v, y) \leq (u, x)$ if there is an increasing injection $\alpha : [k] \to [n]$ such that $u \circ \alpha$ and $x \circ \alpha$ are order isomorphic to v and y, respectively. The next two results show that T behaves well with respect to pattern containment.

Lemma 1. *Let $(v, y) \in \mathrm{Bur}_k$ and $(u, x) \in \mathrm{Bur}_n$. Then:*

$$\binom{v}{y} \leq \binom{u}{x} \iff \binom{v}{y}^T \leq \binom{u}{x}^T.$$

Corollary 2. *Let $x \in \mathrm{Cay}_n$.*

1. *If $y \in \mathrm{Cay}_k$ and $y \leq x$, then $\gamma(y) \leq \gamma(x)$.*
2. *If $\sigma \in \mathcal{S}_k$ and $\sigma \leq \gamma(x)$, then there exists $y \in \mathrm{Cay}_k$ such that $y \leq x$ and $\gamma(y) = \sigma$.*

Corollary 2 can be reformulated in terms of equivalence classes of Cayley permutations. Let $y \sim y'$ if and only if $\gamma(y) = \gamma(y')$. Denote by $[y]$ the equivalence class of y, and denote by $[\mathrm{Cay}]$ the quotient set. Let us extend the notion of pattern containment to $[\mathrm{Cay}]$ by $[x] \geq [y]$ if $x' \geq y'$ for some $x' \in [x]$ and $y' \in [y]$.

Theorem 3 (The transport theorem)**.** *Let $x, y \in \mathrm{Cay}$. Then*

$$[x] \geq [y] \iff \gamma(x) \geq \gamma(y) \qquad \text{and} \qquad \gamma\big([\mathrm{Cay}][y]\big) = \mathcal{S}\big(\gamma(y)\big).$$

Since $\gamma(\sigma^{-1}) = \sigma$ for any $\sigma \in \mathcal{S}$, we can also write

$$\gamma\big([\mathrm{Cay}][\sigma^{-1}]\big) = \mathcal{S}(\sigma).$$

A remarkable consequence is that the sets $\mathcal{S}(\sigma)$ and $[\mathrm{Cay}][\sigma^{-1}]$ are equinumerous. We also found a constructive procedure for the set $[\sigma^{-1}]$ which we omit for lack of space.

Transport of patterns from $\mathcal{F}$ to $\hat{\mathcal{A}}$

Theorem 3 can be specialized by choosing a representative in each equivalence class of $[\mathrm{Cay}]$. Among the resulting examples, the most significant one is that of transport of patterns between Fishburn permutations and modified ascent sequences, which follows immediately since $\hat{\mathcal{A}} \subseteq \mathrm{Cay}$ and γ is injective on $\hat{\mathcal{A}}$.

Ω	$C(\Omega)$	Counting Sequence
21	11	$1, 1, \ldots$
12	12	$1, 1, \ldots$
213	112	2^{n-1}
312	121	2^{n-1}
132	122	2^{n-1}
123	123	2^{n-1}
231	212	Catalan
3142	1212	Catalan
2134	1123	Catalan
1423	132	Catalan
3412	312	$A202062$
$231, 4132$	$212, 221$	Odd indexed Fibonacci
$231, 4123$	$212, 231$	$A116703$
$231, 4312$	$212, 211, 321$	Odd indexed Fibonacci

Table 1: Sets of patterns Ω and $C(\Omega)$ such that $\mathcal{F}(\Omega) = \varphi\big(\mathcal{A}(C(\Omega))\big)$.

Theorem 4 (Transport of patterns from $\mathcal{F}$ to $\hat{\mathcal{A}}$)**.** *For any permutation σ and Cayley permutation y we have*

$$\mathcal{F}(\sigma) = \gamma\big(\hat{\mathcal{A}}[\sigma^{-1}]\big) \quad \textit{and} \quad \gamma\big(\hat{\mathcal{A}}[y]\big) = \mathcal{F}\big(\gamma(y)\big)$$

Therefore, for any permutation σ and $n \geq 1$ we have $|\mathcal{F}_n(\sigma)| = |\hat{\mathcal{A}}_n[\sigma^{-1}]|$.

Many examples where this approach can be pushed further by interpreting the correspondence described in Theorem 4 in terms of (plain) ascent sequences can be found in Table 1.

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The interval posets of permutations seen from the decomposition tree perspective

Lapo Cioni Università degli studi di Firenze

This talk is based on joint work with Mathilde Bouvel (LORIA, France) and Benjamin Izart (LORIA, France)

Recently, Bridget Tenner defined the *interval posets* associated with permutations, and described some properties of these posets in [2]. In this talk we will describe the interval poset of a permutation by its decomposition tree, and we will use this point of view to solve the open problems posed by Bridget Tenner and some other enumerative problems.

For our purpose, an interval of a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is an interval $[j, j+h]$ of values (for some $1 \leq j \leq n$ and some $0 \leq h \leq n-j$) which is the image by σ of an interval $[i, i+h]$ of positions (for some $1 \leq i \leq n-h$).

The inclusion relation naturally equips the set of intervals with a poset structure: the elements of this poset are the intervals, and the relation is the set inclusion. We can consider two versions of this poset: a first one in which the empty interval is an element, and a second one which excludes the empty interval.

While posets are essentially "unordered" objects, we follow [2] and consider a particular plane embedding of the poset of a permutation σ. The chosen embedding put an interval I to the left of an interval J if I appears to the left of J in σ. We denote $P(\sigma)$ (if we ignore the empty interval) and $P_\bullet(\sigma)$ (if we include the empty interval) this plane embedding of the interval poset of σ.

Finally, the *decomposition tree* $T(\sigma)$ of σ is the tree whose internal nodes are labeled by $\oplus$, $\ominus$ or a simple permutation, where the label corresponds to which decomposition is applied at that point in the recursive decomposition of σ in sums, skew-sums and inflations of simple permutations.

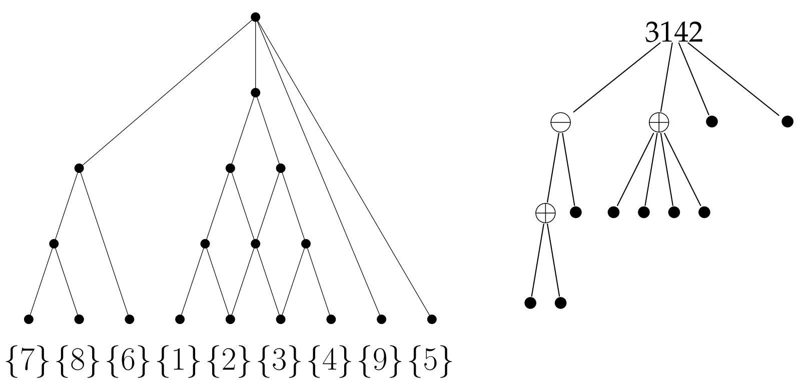

Figure 6: From left to right: The interval poset $P(\sigma)$, and the decomposition tree $T(\sigma)$, for $\sigma = 786123495$. The substitution decomposition of σ is indeed $\sigma = 3142[\ominus[\oplus[1,1],1], \oplus[1,1,1,1],1,1]$.

From decomposition trees to interval posets of permutations

In our talk, we will describe an algorithmic procedure to obtain $P(\sigma)$ (and $P_\bullet(\sigma)$) from $T(\sigma)$. This allow us to give alternative proofs of some of the results in [2] regarding the structure of $P(\sigma)$ and $P_\bullet(\sigma)$. For example, we have that $P_\bullet(\sigma)$ is a lattice for every σ, but it is modular if and only if σ is simple or has size at most 2, and it is also distributive only in the latter case.

Also, using the approach of symbolic combinatorics (see [1, Part A] for example), we are able to exactly enumerate both the interval posets and the tree interval posets (that is, the interval posets that are also trees) with respect to the number of minimal elements, also finding the asymptotic behavior of these sequences.

Finally, we compute the Möbius function of any interval of the interval poset of a permutation, which is as follows.

Theorem 1. *Let σ be a permutation of size n whose substitution decomposition is $\pi[\alpha_1, ..., \alpha_k]$. For any $I \in P_\bullet(\sigma)$, it holds that*

$$\mu(I, [1,n]) = \begin{cases} 1 & \textit{if } I = [1,n], \\ -1 & \textit{if } I \textit{ is covered by } [1,n] \text{ (i.e., } I \textit{ is a coatom),} \\ k-1 & \textit{if } I = \emptyset \textit{ and } \pi \textit{ is either simple or } 12 \textit{ or } 21, \\ 1 & \textit{if } \pi \textit{ is } 12\dots k \textit{ or } k\dots 21 \textit{ for some } k \geq 3 \\ & \quad \textit{and } I \textit{ is covered by the two coatoms of } P_\bullet(\sigma), \\ 0 & \textit{otherwise.} \end{cases}$$

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Enumerating Orderings on Matched Product Graphs

Daryl DeFord Washington State University

This talk is based on joint work with Amir Barghi

In this talk we present enumerative results about the matched product for graphs, motivated by a popular construction for modeling multiplex networks. Results connecting these products to permutation patterns are obtained by relating reorderings of the nodes to consecutive-minima polygons.

The Matched Product

We begin by defining a formal graph product that generalizes the construction of the matched sum [4]. To this end, we define the **matched product** of a sequence of **layer graphs** $(G_1, G_2, \dots, G_k)$ with respect to a k node **structure graph** C as follows.

Definition 1 (Matched Product)**.** Let $G_1, G_2, \dots, G_k$ be an ordered list of graphs, each with n nodes and a common labeling of the nodes and let C be a graph with k ordered nodes. The matched product $\boxed{C}\,(G_1, G_2, \dots, G_k)$ is the graph with node set $\bigcup V_i$ and two nodes v_i^α and v_j^β in $\boxed{C}\,(G_1, G_2, \dots, G_k)$ are connected if and only if either

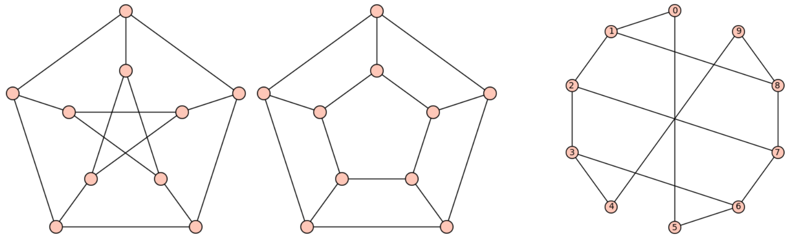
1. $c_\alpha \sim c_\beta$ and $i = j$
2. $\alpha = \beta$ and $v_i^\alpha \sim v_j^\alpha$

where c_α and c_β are nodes in C and v_i^α represents the copy of node i in G_α.

This definition allows for expressing several common multiplex models in compact form, as well as recovering several commonly studied combinatorial products as special cases:

Theorem 2. *There are labelings of the graphs below such that the following hold:*

1. *The cartesian product of G and H can be represented by* $\boxed{H}\,(G, G, \dots, G)$
2. *The rooted product of G and H can be represented by* $\boxed{H}\,(G, E_n, E_n, \dots, E_n)$
3. *The hierarchical product of G and H with subset $\{a_i\} \subset H$ can be represented by* $\boxed{H}\,(G_1, G_2, \dots, G_k)$ *where* $G_i = \begin{cases} G & \text{if } i \in \{a_i\} \\ E_n & \text{otherwise} \end{cases}$.
4. *For a multiplex defined on $G_1, G_2, \dots, G_k$ the disjoint layers, matched sum, and temporal matched sum can be represented by* $\boxed{E_k}\,(G_1, G_2, \dots, G_k)$, $\boxed{K_k}\,(G_1, G_2, \dots, G_k)$, *and* $\boxed{K_k}\,(G_1, G_2, \dots, G_k)$ *respectively.*



(a) Petersen Graph (b) Cylinder Graph (c) Not Planar

Figure 7: Examples of the matched product construction. Plots (a) and (b) show non-isomorphic orderings of $\boxed{P_2}\,(C_5, C_5)$ while (c) shows a non-planar labelling of $\boxed{P_2}\,(P_5, P_5)$.

Enumerating Relabellings

A distinguishing feature of the matched sum from other graph products is that it depends on the labelling of nodes. Figures 1(a) and 1(b) show two realizations of $\boxed{P_2}\,(C_5, C_5)$ with different node orderings, demonstrating that properties like planarity may not be preserved as the labellings are permuted. This gives rise to the interesting question of which permutations of the labels lead to planar embeddings, characterized for paths in Theorem 4 below.

Proposition 3. *Let G and H be connected graphs on n nodes. There exists a labeling so that of $\boxed{P_2}\,(G, H)$ is planar if and only if G and H are outerplanar.*

We will also state and prove similar results for other common graph-theoretic properties, such as the existence of Eulerian and Hamiltonian paths. The proof of the previous proposition gives a sense of what can go wrong, even in the outerplanar case, as the labeling may form a subgraph equivalent to $K_{3,3}$ if the ordering was chosen poorly. Figure 1(c) above shows a labeling of $\boxed{P_2}\,(P_5, P_5)$ that is not planar, since contracting vertex 0 to 1, 4 to 3, 9 to 8, and 5 to 6 gives a graph that is isomorphic to $K_{3,3}$. Further, for all $n > 5$ there is always such a labeling for connected graphs, via a similar construction. A natural related question is the enumerating the number of orderings that give rise to a non–planar graph. For P_n the sequence is 1, 2, 6, 24, 104, 464, 2088, 9392, 42064, 187296 and the permutations appear to be exactly the square [1, 5] permutations (OEIS A128652 [6]) enumerated by $2(n+2)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3}$. We are able to construct a bijection proving this relationship by embedding a path graph inside the convex minimal polygon.

Theorem 4. *There is a combinatorial bijection from labelings of P_n such that $\boxed{P_2}\,(P_n, P_n)$ is planar and the square permutations of the same order.*

As with the existence results mentioned above we compute similar enumerations for other common graph properties.

Finally, we use the matched product to formulate new families of graphs where the The matched product also allows us to enumerate graph Stirling numbers of the first

kind [2, 3] and graph factorials with combinatorial techniques. We will use $\widehat{\left[{G \atop j}\right]}$ to denote the number of decompositions of a given graph G into j disjoint cycles and $\left[{G \atop j}\right]$ to represent decompositions where we allow 1–cycles, which is equivalent to adding a self loop to each vertex in the graph. The total number of rearrangements is denoted by the graph factorials $G! = \sum_k \left[{G \atop k}\right]$ and $\hat{G}! = \sum_k \widehat{\left[{G \atop j}\right]}$, chosen to reflect the fact that $K_n! = n!$.

Throughout our combinatorial discussion we mostly focus on applications to two graphs with the P_2 product. Even this simple subcase leads to several interesting observations and applications and indeed many of the problems considered in [2, 3] can be constructed using the matched product. For example, Theorem 9 in [3] concerns $\boxed{P_2}\,(G, G)$ for bipartite G while examples 10 and 11 in the same paper compute $\sum_k \left[{\boxed{P_2}\,(C_n,C_n) \atop k}\right]$ and $\sum_k \widehat{\left[{\boxed{P_2}\,(C_n,C_n) \atop k}\right]}$ explicitly. We also provide some computational examples on matched products where the graphs are isomorphic regardless of labelling. The graph factorials are displayed in Table 1.

n	$\widehat{\boxed{P_2}\,(C_n, S_n)}!$	$\widehat{\boxed{P_2}\,(K_n, S_n)}!$	$\boxed{P_2}\,(P_n, K_n)!$	$\widehat{\boxed{P_2}\,(P_n, K_n)}!$	$\boxed{P_2}\,(C_n, K_n)!$	$\widehat{\boxed{P_2}\,(C_n, K_n)}!$
2	9	48	4	9	4	9
3	49	293	9	48	20	82
4	140	2022	49	345	121	577
5	394	15657	216	2994	589	4876
6	1093	135044	1773	30957	4820	49789
7	2986	1287813	12113	369132	35293	587182
8	8056	13480938	128036	4996761	365633	7887553
9	21504	153879977	1172341	75625710	3525212	118596664
10	56889	1903771512	14885241	1265833149	43894725	1974218701

Table 2: This table reports the number of rearrangements of several natural matched products whose structure are not determined by the ordering of the nodes on the layer graphs. None of these sequences currently appear in the OEIS.

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2-avoidance

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This talk is based on joint work with Yoong Kuan (Andrew) Goh

Motivated by a problem in stack sorting, we propose a different kind of pattern avoidance which we call 2-*avoidance*.

Let F, G be sets of permutations. A permutation 2-avoids (F, G) either if it avoids F in the usual sense of pattern avoidance, or if it contains some pattern $f \in F$, then the entries of the permutation that are order-isomorphic to f are themselves contained in a pattern order-isomorphic to some $g \in G$.

Informally, the patterns in G "save" a permutation from being forbidden on the basis of containing a pattern in F.

The poster will give a precise definition of 2-avoidance, explain how it is similar to and different from other popular non-classical avoidance notions, and give some (we believe) intriguing open questions about 2-avoidance classes.

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Descents on nonnesting multipermutations

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Motivated by recent results on quasi-Stirling permutations, which are permutations of the multiset $\{1,1,2,2,\dots,n,n\}$ that avoid the "crossing" patterns 1212 and 2121, we consider nonnesting permutations, defined as those that avoid the patterns 1221 and 2112 instead. We show that the polynomial giving the distribution of the number of descents on nonnesting permutations is a product of an Eulerian polynomial and a Narayana polynomial. It follows that, rather unexpectedly, this polynomial is palindromic. We provide bijective proofs of these facts by composing various transformations on Dyck paths, including the Lalanne–Kreweras involution.

Definitions

Given a sequence of positive integers $\pi = \pi_1\pi_2\dots\pi_m$, we say that i is a *descent* of π if $\pi_i > \pi_{i+1}$, that it is a *plateau* if $\pi_i = \pi_{i+1}$, and that it is a *weak descent* if $\pi_i \geq \pi_{i+1}$. Denote by $\operatorname{des}(\pi)$, $\operatorname{plat}(\pi)$ and $\operatorname{wdes}(\pi) = \operatorname{des}(\pi) + \operatorname{plat}(\pi)$ the number of descents, plateaus and weak descents of π, respectively.

The distribution of descents on the set $\mathcal{S}_n$ of permutations of $[n] := \{1,2,\dots,n\}$ is given by the Eulerian polynomials

$$A_n(t) = \sum_{\sigma\in\mathcal{S}_n} t^{\operatorname{des}(\sigma)},$$

whose generating function is

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{t-1}{t-e^{(t-1)z}}. \tag{1}$$

In 1978, Gessel and Stanley [3] introduced *Stirling permutations*, defined as permutations $\pi_1\pi_2\dots\pi_{2n}$ of the multiset $[n]\sqcup[n] := \{1,1,2,2,\dots,n,n\}$ satisfying that, if $i<j<k$ and $\pi_i = \pi_k$, then $\pi_j > \pi_i$; in other words, avoiding the pattern 212. They showed that the distribution of the number of descents on such permutations is related to the Stirling numbers of the second kind. There is an extensive literature on these permutations and their generalizations to other multisets.

In 2019, Archer et al. [1] introduced a variation of Stirling permutations, which they call *quasi-Stirling permutations*. These are permutations $\pi_1\pi_2\dots\pi_{2n}$ of $[n]\sqcup[n]$ that avoid 1212 and 2121, meaning that there do not exist $i<j<\ell<m$ such that $\pi_i = \pi_\ell$ and $\pi_j = \pi_m$. The number of such permutations is $n!\operatorname{Cat}_n = \frac{(2n)!}{(n+1)!}$, where Cat_n is the nth Catalan number. The generating function for these permutations with respect to the number of descents and plateaus was later found in [2], expressed as a compositional inverse of the generating function (1).

A permutation π of $[n]\sqcup[n]$ can be viewed as a labeled matching of $[2n]$, by placing an arc with label k between i with j if $\pi_i = \pi_j = k$. The condition that π avoids

1212 and 2121 is equivalent to the fact that this matching is *noncrossing*; see [4, Exer. 60]. With this perspective, it is natural to also consider permutations for which this matching is *nonnesting*; see [4, Exer. 64].

Figure 8: The matchings corresponding to the quasi-Stirling (noncrossing) permutation 4431152253 and the nonnesting permutation $3532521414 \in \mathcal{C}_5$.

Definition 1. A permutation π of the multiset $[n] \sqcup [n]$ is called *nonnesting* if it avoids the patterns 1221 and 2112; equivalently, if there do not exist $i < j < \ell < m$ such that $\pi_i = \pi_m$ and $\pi_j = \pi_\ell$. Denote by $\mathcal{C}_n$ the set of nonnesting permutations of $[n] \sqcup [n]$.

The above condition on π is equivalent to the requirement that the subsequence of π determined by the first copy of each entry coincides with the subsequence determined by the second copy of each entry. This subsequence, which is a permutation in $\mathcal{S}_n$, will be denoted by $\sigma(\pi)$. For example, if $\pi = 3532521414 \in \mathcal{C}_5$, then $\sigma(\pi) = 35214 \in \mathcal{S}_5$.

As in the noncrossing case, the number of nonnesting matchings of $[2n]$ is again the nth Catalan number [4, Exer. 64]. Since there are $n!$ ways to assign labels to the arcs of a nonnesting matching to form a nonnesting permutation, it follows that

$$|\mathcal{C}_n| = n!\,\mathrm{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

Motivated by the results on the distribution of the number of descents and plateaus on Stirling and quasi-Stirling permutations, here we describe the distribution of these statistics on nonnesting permutations. We are interested in the polynomial

$$C_n(t,u) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)}. \tag{2}$$

Let $\mathcal{D}_n$ be the set of lattice paths from $(0,0)$ to $(2n,0)$ with steps $\mathtt{u} = (1,1)$ and $\mathtt{d} = (1,-1)$ that do not go below the x-axis. Elements of $\mathcal{D}_n$ are called Dyck paths. A *peak* in a Dyck path is an occurrence of two adjacent steps $\mathtt{ud}$. A peak is called a *low peak* if these steps touch the x-axis, and a *high peak* otherwise. Denote the number of low peaks and the number of high peaks of $D \in \mathcal{D}_n$ by $\mathrm{lpea}(D)$ and $\mathrm{hpea}(D)$, respectively. Consider the *Narayana polynomials*

$$N_n(t,u) = \sum_{D \in \mathcal{D}_n} t^{\mathrm{hpea}(D)} u^{\mathrm{lpea}(D)}.$$

From the usual decomposition of Dyck paths by the first return, one can easily deduce that

$$\sum_{n \geq 0} N_n(t,u) z^n = \frac{1}{1 + (1+t-2u)z + \sqrt{1 - 2(1+t)z + (1-t)^2 z^2}}.$$

Main Results

We obtain the following strikingly simple expression for the polynomial $C_n(t,u)$ from Equation (2), as a product of an Eulerian polynomial and a Narayana polynomial.

Theorem 2. *For $n \geq 1$,*

$$C_n(t,u) = A_n(t)\, N_n(t,u).$$

As a consequence, using the palindromicity of $A_n(t)$, $N_n(t,t)$ and $N_n(t,1)$, we obtain the following two unexpected symmetries.

Corollary 3. *The distribution of the number of weak descents on $\mathcal{C}_n$ is symmetric, i.e.,*

$$|\{\pi \in \mathcal{C}_n : \mathrm{wdes}(\pi) = r\}| = |\{\pi \in \mathcal{C}_n : \mathrm{wdes}(\pi) = 2n - r\}|$$

for all r.

Corollary 4. *The distribution of the number of descents on $\mathcal{C}_n$ is symmetric, i.e.,*

$$|\{\pi \in \mathcal{C}_n : \mathrm{des}(\pi) = r\}| = |\{\pi \in \mathcal{C}_n : \mathrm{des}(\pi) = 2n - 2 - r\}|$$

for all r.

To establish Theorem 2, we prove a slightly stronger statement. For $\sigma \in \mathcal{S}_n$, define $\mathcal{C}_n^\sigma = \{\pi \in \mathcal{C}_n : \sigma(\pi) = \sigma\}$, so that $\mathcal{C}_n = \bigsqcup_{\sigma \in \mathcal{S}_n} \mathcal{C}_n^\sigma$. Letting

$$C_n^\sigma(t,u) = \sum_{\pi \in \mathcal{C}_n^\sigma} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

we prove the following refinement.

Theorem 5. *For all $\sigma \in \mathcal{S}_n$,*

$$C_n^\sigma(t,u) = t^{\mathrm{des}(\sigma)} N_n(t,u).$$

Corollary 6. *For each $\sigma \in \mathcal{S}_n$, the distribution of the number of weak descents on $\mathcal{C}_n^\sigma$ is symmetric, i.e.,*

$$|\{\pi \in \mathcal{C}_n^\sigma : \mathrm{wdes}(\pi) - \mathrm{des}(\sigma) = r\}| = |\{\pi \in \mathcal{C}_n^\sigma : \mathrm{wdes}(\pi) - \mathrm{des}(\sigma) = n + 1 - r\}|$$

for all r.

Corollary 7. *For each $\sigma \in \mathcal{S}_n$, the distribution of the number of descents on $\mathcal{C}_n^\sigma$ is symmetric, i.e.,*

$$|\{\pi \in \mathcal{C}_n^\sigma : \mathrm{des}(\pi) - \mathrm{des}(\sigma) = r\}| = |\{\pi \in \mathcal{C}_n^\sigma : \mathrm{des}(\pi) - \mathrm{des}(\sigma) = n - 1 - r\}|$$

for all r.

All our proofs are bijective.

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Preimages under the Bubblesort operator

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This talk is based on joint work with Mathilde Bouvel, Lapo Cioni

`Bubblesort` is a very well known sorting algorithm for sequences of elements of a totally ordered set. Here we are interested in the Bubblesort operator $\mathbf{B}$, which corresponds to applying one pass of `Bubblesort` to a permutation. Specifically, $\mathbf{B}(\pi)$ is obtained from π by scanning its elements from left to right, each time exchanging an element with the one sitting to its right whenever the latter is smaller. Thereby the left-to-right maxima of the permutation "bubble up" to the right, until they are blocked by the next left-to-right maximum.

For example, for $\pi = \mathbf{4}21\mathbf{6}3\mathbf{78}5$, the left-to-right maxima are $4, 6, 7$ and 8 (shown in bold) and $\mathbf{B}(\pi) = 21436758$.

Similarly to what happens for other sorting operators, such as those associated with `Stacksort` and `Queuesort`, the set of bubble-sortable permutations can be characterized in terms of pattern avoidance: they are precisely the permutations avoiding the two patterns 231 and 321.

Following the footprints of what have been done for the sorting maps associated with `Stacksort` [2, 4] and `Queuesort` [3], in the present work we are interested in studying preimages of permutations under $\mathbf{B}$.

Our first result is a complete characterization and enumeration of the set of preimages of a given permutation.

Proposition 1. *Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation of size n ending with its maximum. Let k be the number of left-to-right maxima of σ. There is a bijective correspondence between the preimages of σ under $\mathbf{B}$ and the subsets of the $k-1$ left-to-right maxima of σ different from n. For any set $S = \{s_1 < \cdots < s_j\}$ of $j \leq k-1$ left-to-right maxima of σ different from n, writing $\sigma = B_0 s_1 B_1 s_2 B_2 \dots s_{j-1} B_{j-1} s_j B_j n$ (for the B_i possibly empty sequences of integers, which contain the $k-j$ left-to-right maxima not in S and the elements of σ which are not left-to-right maxima), the corresponding preimage of σ is $s_1 B_0 s_2 B_1 \dots s_j B_{j-1} n B_j$.*

Corollary 2. *The cardinality of $\mathbf{B}^{-1}(\sigma)$ is 2^{k-1}, and for any $1 \leq j \leq k$, the number of preimages of σ with j left-to right maxima is $\binom{k-1}{j-1}$.*

We can also characterize and enumerate permutations having a given number of preimages. This is an immediate consequence of the above propositions and of the classical Foata bijection, which maps permutations of size n with k cycles to permutations of size n with k left-to-right maxima.

Corollary 3. *For any $k \geq 1$, permutations having exactly 2^{k-1} preimages under $\mathbf{B}$ are those ending with their maximum and having k left-to-right maxima. In particular, there are $\left[{n-1 \atop k-1} \right]$ permutations of size n having 2^{k-1} preimages under $\mathbf{B}$, where $\left[{n \atop k} \right]$ are the (unsigned) Stirling numbers of the first kind.*

We further consider, for each n, the tree T_n recording all permutations of size n in its nodes, in which an edge from child to parent corresponds to an application of $\mathbf{B}$ (the root being the identity permutation). Also, given a permutation $\pi \in S_n$, we define the *tree of its preimages* $T(\pi)$ as the subtree of T_n with root π.

An interesting property of T_n is that, for $\pi \in S_n$, all possible shapes of the trees $T(\pi)$ can be found starting at depth 1 in T_n.

Proposition 4. *For every permutation $\pi \in S_n$, $\pi \neq id_n$, there exists a child τ of id_n in T_n such that $T(\pi)$ and $T(\tau)$ are isomorphic.*

Next we describe how the "shape" of any tree $T(\pi)$ is completely determined by a small piece of information about π, which we encapsulate in its label. The *label* of a permutation σ is the pair (k, m_ℓ), where k is the number of left-to-right maxima of π and m_ℓ is the size of the (possibly empty) maximal suffix of left-to-right maxima of π. Moreover, the *skeleton* of a tree $T(\pi)$ is obtained from $T(\pi)$ by replacing each permutation at a node with its label.

Given a permutation π, we can determine the skeleton of $T(\pi)$ using only the pair (k, m_ℓ). Specifically, it is the tree with root labeled by (k, m_ℓ), and whose children (and recursively, descendants) are obtained as described in the next proposition.

Proposition 5. *Let $\pi \in S_n$ with label (k, m_ℓ). Let T be the skeleton of $T(\pi)$. Then the root of T has label (k, m_ℓ) and its children have the following labels:*

- *for every $h = 0, \ldots, m_\ell - 2$:*
  - ♡ *for every $i = 1, \ldots, k-1-h$, there are $\binom{k-2-h}{i-1}$ children with label $(k-i, h)$;*
- *if $\pi \neq id_n$, we also have the case corresponding to $h = m_\ell - 1$:*
  - ♡ *for every $i = 0, \ldots, k - m_\ell$, there are $\binom{k-m_\ell}{i}$ children with label $(k-i, m_\ell - 1) = (k-i, h)$.*

The previous proposition provides a recursive description of T_n which is useful (among other things) to study enumerative properties of nodes and leaves, which is the last part of our work. The next proposition, which concerns nodes of T_n, is essentially a consequence of results in [1] and [5]. In particular, a crucial property is that permutations sorted by at most k applications of the operator $\mathbf{B}$ are those avoiding the set of patterns Γ_{k+2}, where Γ_k is the set of all permutations of size k whose last element is 1.

Proposition 6. *The number of nodes at height k in T_n is $k! \cdot ((k+1)^{n-k} - k^{n-k})$. The average height of a node in T_n is asymptotically equal to $n - \sqrt{\frac{\pi n}{2}} + O(1)$.*

Next we can find an analogous result for a subtree $T(\pi)$ by exploiting Proposition 5.

Proposition 7. *For a permutation π having label (k, m_ℓ), different from an identity permutation, the number of nodes of $T(\pi)$ is*

$$N(k, m_\ell) = \sum_{j=0}^{m_\ell} j!(j+1)^{k-j}. \tag{1}$$

Moreover, denoting with $N_j(k, m_\ell)$ the number of nodes at height j in $T(\pi)$, we have that $N_j(k, m_\ell) = j!(j+1)^{k-j}$.

The same information can be provided for leaves.

Proposition 8. *The number of leaves at height k in T_n is $k!(k(k+1)^{n-k-1} - (k-1)k^{n-k-1})$. The average height of a leaf in T_n is asymptotically equal to $n - \sqrt{\frac{\pi n}{2}} + O(1)$.*

Proposition 9. *Given a permutation π having label (k, m_ℓ), different from an identity permutation, the number of leaves of $T(\pi)$ is*

$$L(k, m_\ell) = \sum_{j=1}^{m_\ell - 1} j!j(j+1)^{k-j-1} + m_\ell!(m_\ell + 1)^{k-m_\ell}. \tag{2}$$

Moreover, denoting with $L_j(k, m_\ell)$ the number of leaves at height j in $T(\pi)$, we have that $L_j(k, m_\ell) = j!j(j+1)^{k-j-1}$ for $j < m_\ell$, and $L_{m_\ell}(k, m_\ell) = m_\ell!(m_\ell + 1)^{k-m_\ell}$.

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Recursive maps for derangements and nonderangements

Melanie Ferreri Dartmouth College

This talk is based on joint work with Peter Doyle

Derangements and Nonderangements

Definition 1. A *derangement* is a permutation with no fixed point. We denote by D_n the set of derangements of $[n]$ contained in the symmetric group S_n.

Definition 2. Similarly, a *nonderangement* is a permutation with at least one fixed point. We denote by $\overline{D_n}$ the set of nonderangements of $[n]$ contained in S_n.

Definition 3. We denote by E_n the set of permutations with exactly one fixed point, and we use $\overline{E}_n$ to denote the complement of E_n in S_n, i.e. the set of permutations that do not have exactly one fixed point.

It can be shown from well-known recurrence relations that

$$d_n = e_n + (-1)^n \tag{1}$$

where $d_n = |D_n|$ and $e_n = |E_n|$. It can also be shown that

$$\overline{d}_n = n\overline{d}_{n-1} - (-1)^n. \tag{2}$$

where $\overline{d}_n = |\overline{D}_n|$.

Recursive Maps

We present an involution on S_n which exchanges elements of D_n and E_n excluding one element, and describe a recursive map that gives a bijection exhibiting the identity (1). We then show the combinatorial interpretation of this map and how it compares with another known bijection [2]. This map can be used to obtain a map from $\overline{D}_n$ to $\overline{E}_n$ excluding one element, and can be combined with a bijection from $\overline{E}_n$ to $[n] \times \overline{D}_{n-1}$ to give a bijective proof of the one-term identity for nonderangements (2).

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Restricted generating trees for weak orderings

Juan B. Gil Penn State Altoona

This talk is based on joint work with D. Birmajer, D. Kenepp, and M. Weiner

Motivated by the study of pattern avoidance in the context of permutations and ordered partitions, we consider the enumeration of weak-ordering chains obtained as leaves of certain restricted rooted trees. A tree of order n is generated by inserting a new variable into each node at every step. A node becomes a leaf either after n steps or when a certain *stopping condition* is met. In this talk we will focus on conditions of size 3. Some of the cases considered here lead to the study of descent statistics of certain 'almost' pattern-avoiding permutations.

Introduction

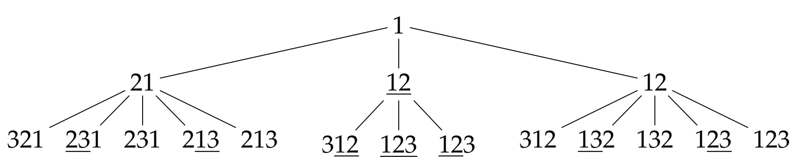
A *weak-ordering chain* in the variables $x_1, x_2, \ldots, x_n$ is an expression of the form

$$x_{i_1} \operatorname{op} x_{i_2} \operatorname{op} \cdots \operatorname{op} x_{i_n},$$

where op is either $<$ or $=$. We let $\mathcal{WOC}(n)$ denote the set of all weak-ordering chains in n variables. Every $w \in \mathcal{WOC}(n)$ corresponds to an ordered partition of $[n] = \{1, \ldots, n\}$ obtained from the indices of the variables in w, where the numbers i and j are in the same block of the partition whenever $x_i = x_j$. For example,

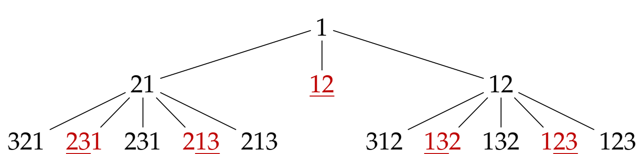
$$x_2 < x_4 = x_5 < x_1 < x_3 \quad \longleftrightarrow \quad \{\{2\}, \{4,5\}, \{1\}, \{3\}\}.$$

Every element $w \in \mathcal{WOC}(n)$ can be recursively generated starting with x_1, and then inserting x_i (together with either $<$ or $=$) into a previously constructed weak-ordering chain of length $i-1$. This process generates a rooted labeled tree whose nodes at level i are labeled by the elements of $\mathcal{WOC}(i)$. For example, for $n = 3$, we get the tree



where ij is a shortcut for $x_i < x_j$ and $\underline{ij}$ represents $x_i = x_j$.

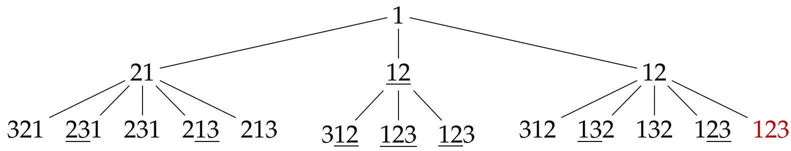
Now, suppose that we wish to stop the above generating process as soon as we have a tie. In other words, suppose that we do not allow nodes with $x_i = x_j$ for some $i > j$ to have descendants. Then, the above tree would take the form



(1)

with only 11 leaves instead of 13. We call (1) a restricted generating tree of weak-ordering chains subject to the *stopping condition* $x_i = x_j$.

As another example, consider the stopping condition $x_i < x_j < x_k$ with $i < j < k$. In this case, the generating tree at level 3 looks like the tree for $\mathcal{WOC}(3)$:



but the node with label 123 will have no descendants as the generating tree grows.

The enumeration of weak-ordering chains subject to a stopping condition is equivalent to counting the number of leaves of the corresponding restricted generating subtree.

Stopping conditions of size 3

Let $e_{n,d}$ be the number of 123-avoiding permutations on $[n]$ having exactly d descents. Let $g_{1,d} = g_{2,d} = 0$, $g_{3,0} = 1$, and for $n > 3$ and $1 \le d \le n-3$,

$$g_{n,d} = \#\{\sigma \in S_n \mid \sigma \text{ has a 123 pattern, } d \text{ descents, and } \sigma' \in S_{n-1}(123)\},$$

where $\sigma' \in S_{n-1}$ denotes the permutation obtained from $\sigma \in S_n$ by removing n.

Theorem 1. *If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_{i_1} < x_{i_2} < x_{i_3}$ with $i_1 < i_2 < i_3$, then*

$$w_n = \sum_{d=0}^{n-1} 2^d e_{n,d} + \sum_{j=3}^{n} \sum_{d=0}^{j-3} 2^d g_{j,d}.$$

The generating function $W(x) = \sum_{n=1}^{\infty} w_n x^n$ satisfies

$$W(x) = \frac{x + x\sqrt{1-8x+8x^2}}{2(1-x)\sqrt{1-8x+8x^2}}.$$

Theorem 2. *If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_{i_1} \le x_{i_2} \le x_{i_3}$ with $i_1 < i_2 < i_3$, then*

$$w_n = \sum_{d=0}^{n-1} 2^{n-1-d} e_{n,d} + \sum_{j=3}^{n} \sum_{d=2}^{j-1} 2^{j-1-d} g_{j,d}.$$

The generating function $W(x) = \sum_{n=1}^{\infty} w_n x^n$ satisfies

$$1 + W(x) = \frac{x}{1-x^2} + \frac{1-2x-2x^2}{(1-x^2)\sqrt{1-4x-4x^2}}.$$

Let $N_{n,d}$ be the number of 213-avoiding permutations on $[n]$ having exactly d descents. Let $\ell_{1,d} = \ell_{2,d} = 0$, $\ell_{3,1} = 1$, and for $n > 3$ and $1 \leq d \leq n-3$,

$$\ell_{n,d} = \#\{\sigma \in S_n \mid \sigma \text{ has a 213 pattern, } d \text{ descents, and } \sigma' \in S_{n-1}(213)\},$$

where $\sigma' \in S_{n-1}$ denotes the permutation obtained from $\sigma \in S_n$ by removing n.

Theorem 3. *If w_n is the number of weak-ordering chains in $\mathcal{WOC}(n)$, subject to the stopping condition $x_{i_1} \leq x_{i_2} < x_{i_3}$ with $i_1 < i_2 < i_3$, then*

$$w_n = \sum_{d=0}^{n-1} 2^d N_{n,d} + \sum_{j=3}^{n} \sum_{d=1}^{j-2} 2^d \ell_{j,d}.$$

The generating function $W(x) = \sum_{n=1}^{\infty} w_n x^n$ satisfies

$$W(x) = \frac{(1-x)^2 - (1-3x)\sqrt{1-6x+x^2}}{4(1-x)\sqrt{1-6x+x^2}}.$$

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Juan B. Gil and Jessica A. Tomasko Penn State Altoona

A permutation is called Grassmannian if it has at most one descent. In this poster, we present recent results [1] on pattern avoidance and parity restrictions for such permutations. We derive formulas for the enumeration of Grassmannian permutations that avoid a classical pattern of arbitrary size. In addition, for patterns of the form $k12\cdots(k-1)$ and $23\cdots k1$, we provide combinatorial interpretations in terms of Dyck paths, and for 35124-avoiding Grassmannian permutations, we give an explicit bijection to certain pattern-avoiding Schröder paths. .

Grassmannian & related permutations

If $\mathscr{G}_n$ denotes the set of Grassmannian permutations on $[n]=\{1,\dots,n\}$, then

- $\pi \in \mathscr{G}_n$ if and only if $\pi^{rc} \in \mathscr{G}_n$.
- $|\mathscr{G}_n| = 2^n - n$ for $n \ge 1$. $(1,2,5,12,27,58,121,248,503,1014,\dots)$

A permutation, π, is called *biGrassmannian* if both $\pi, \pi^{-1} \in \mathscr{G}_n$. For $\pi \in \mathscr{G}_n$, the inverse π^{-1} has at most one *dip*, i.e. a pair (i,j) with $i<j$ such that $\pi(i)=\pi(j)+1$. A biGrassmannian permutation has at most one descent and at most one dip.

Proposition 1. *A Grassmannian permutation is biGrassmannian if and only if it avoids the pattern* 2413*. In other words,* $\mathscr{G}_n \cap \mathscr{G}_n^{-1} = \mathscr{G}_n(2413)$ *for every n. Moreover,*

$$|\mathscr{G}_n \cap \mathscr{G}_n^{-1}| = 1 + \binom{n+1}{3}.$$

Proposition 2. *For* $n \in \mathbb{N}$*, we have* $\mathscr{G}_n \cup \mathscr{G}_n^{-1} = S_n(321, 2143)$*. Moreover,*

$$|\mathscr{G}_n \cup \mathscr{G}_n^{-1}| = 2^{n+1} - \binom{n+1}{3} - 2n - 1.$$

Proposition 3. $\pi \in \mathscr{G}_n$ *is an involution if and only if it is of the form*

$$\pi = \mathrm{id}_{k_1} \oplus \big(\mathrm{id}_{k_2} \ominus \mathrm{id}_{k_2}\big) \oplus \mathrm{id}_{k_3}$$

for some $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}$ *with* $k_1 + 2k_2 + k_3 = n$*, where* $\mathrm{id}_0 = \varepsilon$*.*
Moreover, i_n*, the number of Grassmannian involutions of size n is given by*

$$i_n = \begin{cases} \frac{n^2+3}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2+4}{4} & \text{if } n \text{ is even.} \end{cases}$$

Pattern avoidance

Theorem 4. *If $k \geq 3$ and $\sigma \in S_k$ with* $\operatorname{des}(\sigma) = 1$, *then*

$$|\mathscr{G}_n(\sigma)| = 1 + \sum_{j=3}^{k} \binom{n}{j-1} \text{ for } n \in \mathbb{N}.$$

$\lvert\sigma\rvert$	Sequence $\lvert\mathscr{G}_n(\sigma)\rvert$	OEIS
3	$1, 2, 4, 7, 11, 16, 22, 29, 37, 46, \ldots$	A000124
4	$1, 2, 5, 11, 21, 36, 57, 85, 121, 166, \ldots$	A050407
5	$1, 2, 5, 12, 26, 51, 92, 155, 247, 376, \ldots$	A027927
6	$1, 2, 5, 12, 27, 57, 113, 211, 373, 628, \ldots$	n/a
7	$1, 2, 5, 12, 27, 58, 120, 239, 457, 838, \ldots$	n/a
8	$1, 2, 5, 12, 27, 58, 121, 247, 493, 958, \ldots$	n/a

Table 3: Enumeration of $\mathscr{G}_n(\sigma)$ for a pattern σ with $\operatorname{des}(\sigma) = 1$.

Proposition 5. *For $k \geq 3$, the elements of $\mathscr{G}_n(k12\cdots(k-1))$ are in one-to-one correspondence with the Grassmannian Dyck paths of semilength n having at most $k-2$ peaks at height greater than 1.*

Proposition 6. *For $k \geq 3$, the elements of $\mathscr{G}_n(23\cdots k1)$ are in one-to-one correspondence with the Grassmannian Dyck paths of semilength n and height at most $k-1$.*

Proposition 7. *The set $\mathscr{G}_{n+1}(35124)$ is in bijection with the set of Schröder paths of semilength n that avoid the pattern* UUDD.

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The Expected Number of Distinct Patterns in a Random Permutation

Anant Godbole East Tennessee State University

This talk is based on joint work with Verónica Borrás-Serrano, Isabel Byrne, Nathaniel Veimau.

This talk will focus on how perfectly random permutations pack distinct patterns.

The Consecutive Case

Let π_n be a uniformly chosen random permutation on $[n]$. Using an analysis of the probability that two overlapping consecutive k-permutations are order isomorphic, the authors of Allen et al [2] showed that the expected number of distinct *consecutive* patterns of all lengths $k \in \{1, 2, \ldots, n\}$ in π_n is $\frac{n^2}{2}(1 - o(1))$ as $n \to \infty$. This exhibits the fact that random permutations pack consecutive patterns near-perfectly.

The Non-Consecutive Case

Beginning with a question asked by Herb Wilf at the inaugural Permutation Patterns Conference held in Dunedin in 2003, several authors have studied the maximum value $\psi(\pi_n)$ of the number of distinct patterns in a permutation π_n on $[n]$. This includes the successive work of Coleman [3], Albert et al [1], and Miller who showed in [4] that

$$2^n - O(n^2 2^{n-\sqrt{2n}}) \le \max_{\pi_n \in S_n} \psi(\pi_n) \le 2^n - \Theta(n 2^{n-\sqrt{2n}}). \tag{1}$$

In this talk we obtain results for both the number of pairs of non-isomorphic patterns and the number of distinct patterns in a random permutation.

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Generalizations of parking functions and a connection to pattern avoidance

Pamela Harris Williams College

We begin this talk by introducing parking functions and results related to their enumeration and related statistics. One result will establish a bijection between parking functions with displacement one and the set of ideal states on the Tower of Hanoi game. We then consider a variety of generalizations of parking functions including L-interval (rational) parking functions, which are related to Fubini competitions, and MVP parking functions, which yield a connection to permutations avoiding certain patterns. The talk will conclude with an opportunity to discuss further directions and open problems in this area.

Using Constraint Programming to Enumerate Permutations avoiding Mesh Patterns

Ruth Hoffmann University of St Andrews

This talk is based on joint work with Özgür Akgün, Chris Jefferson

Constraint programming is a proven technology for solving complex combinatorial decision, optimisation or enumeration problems. Constraints are a natural, powerful means of representing and reasoning about complex problems that impact all of our lives. Constraint programming offers a means by which solutions to such problems can be found automatically, and proceeds in two phases. First, the problem is modelled as a set of decision variables, and a set of constraints on those variables that a solution must satisfy. A decision variable represents a choice that must be made in order to solve the problem. The domain of potential values associated with each decision variable corresponds to the options for that choice.

Enumerating permutations avoiding mesh (or other) patterns lends itself perfectly to constraint programming. We have created a model which represents the definition of a mesh pattern [1] in `Essence` [3]. We then use `Conjure` [2] to solve a range of permutation pattern problems. We can find if a pattern is present inside a target permutation at all, or count how many permutations of a certain length (or range of lengths) avoid the pattern or a set of patterns.

`Essence` is a high-level problem specification language: it natively supports decision variables with abstract domains like set, multi-set, function, relation, partition domains and operations defined on these domains. `Conjure` translates problem specifications into concrete models suitable as input to standard constraint programming toolkits. `Conjure` allows practitioners to explore alternative approaches to convert problem specifications to concrete models and allows the use of different state-of-the-art black box solvers. The solver that we use in this work are Minion[4] and an AllSAT solver `nbc_minisat_all` [5].

The main benefit of modelling permutation patterns as a set of constraints is that constraint models are highly modular. This means it is easy to use the same (or only slightly modified) code to find the permutations which satisfy a set of patterns, or find permutations which contain one mesh pattern, but do not contain a second mesh pattern.

Modelling Mesh Patterns

We will present our model for finding all permutations in S_n which avoid a set of mesh patterns (code in Figure 9. This model can easily be extended to consider many similar problems. Our models are given in `Essence` , a high-level constraint modelling language.

```
1  language Essence 1.3
2
3  given avoid : set of (sequence(injective) of int, relation of (int*int))
4
5  given n : int
6  find perm : matrix indexed by [int(0..n+1)] of int(0..n+1)
7
8  such that
9    perm[0] = 0, perm[n+1] = n+1,
10   allDiff(perm)
11
12 such that
13   forAll (av, mesh) in avoid .
14   exists avinv: matrix [int(0..|av|+1)] of int(0..|av|+1),
15     and([avinv[0] = 0, avinv[|av|+1] = |av|+1,
16          (forAll i: int(1..|av|) . avinv[av(i)] = i)].
17       forAll ix : matrix indexed by [int(0..|av|+1)] of int(0..n+1),
18         and([ ix[0]=0 /\ ix[|av|+1]=n+1
19             , forAll i : int(0..|av|) . ix[i] < ix[i+1]
20             , forAll n1, n2 : int(1..|av|) , n1 < n2 .
21                av(n1) < av(n2) <-> perm[ix[n1]] < perm[ix[n2]]
22             ]) .
23           ( exists i,j: int(0..|av|).
24             (i,j) in mesh /\
25             exists z: int(ix[i]+1..ix[i+1]-1).
26               (perm[ix[avinv[j]]] <= perm[z] /\ perm[z] <= perm[ix[avinv[j+1]]])
27           )
```

Figure 9: The Essence specification of the mesh pattern avoidance.

One of the most difficult part of modelling mesh patterns is edge conditions. The cells in the mesh which are around the edge and represent "all other values". We want to avoid having to handle these values specially. To avoid having to special case the edges of the mesh, we extend the permutation perm we search for from a permutation p in S_n to a permutation on $\{0, \ldots, n+1\}$, where $0^p = 0$ and $(n+1)^p = n+1$ (Lines 6–10).

The mesh pattern is stored as a set of pairs (av, mesh) (Line 3), where av is an injective sequence (representing the permutation) and mesh is a relation representing the cells in the mesh which cannot contain values. The mesh pattern is defined in a separate file which represents the given information, alongside n which represents the length of target permutations in S_n. An example file of a mesh pattern is found in Figure 10, which represents the mesh pattern in Figure 11.

We dynamically calculate avinv, the inverse of av. Similarly to the permutation, we extend this with an extra value, where avinv[0]=0 and avinv[x+1] = n+1 (where x is the length of the pattern av) (Lines 14–16).

We then search for all occurrences of the pattern in the permutation (Lines 17–21. Similarly when searching for the pattern in a permutation, we add an extra fixed value to the start and end of the pattern, which map to 0 and $n+1$ respectively.

Finally, when we find an occurrence of the pattern, we check that at least one member of the mesh contains a value (which means the mesh is not actually present) (Lines 23–27).

```
language Essence 1.3

letting n be 4

letting avoid be {(sequence(1,3,2), relation((1,2),(2,3),(3,0),(3,1)))}
```

Figure 10: The `Essence` parameter file of a particular mesh pattern and the length of permutations that will be enumerated.

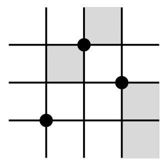

Figure 11: Mesh pattern as defined in the parameter file.

We will talk about the model, give more insights into how it and the solving of it works. We will present results to show how competitive using a general purpose declarative method (constraint programming) can be in comparison to bespoke algorithms when solving NP-complete enumeration problems and specifically when enumeration mesh pattern avoiding permutations.

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On permutation classes defined by pin sequences

Ben Jarvis Open University

This talk is based on joint work with Robert Brignall

Pin sequences were introduced by Brignall, Huczynska and Vatter [1] as a means of studying simple permutations. Since then they have attracted interest as a method of constructing permutation classes with a large number of simples, and in connection with monotone griddability - see [3]. In this talk we consider so-called 'pin classes' - permutation classes consisting of all finite permutations contained in a given infinite pin sequence - with a focus on the smallest possible growth rates that these classes can have. We will conclude with a brief discussion of the application of this theory to the study of growth rates of permutation classes with bounded oscillations.

Classifying small pin classes

We begin with a definition, following Bassino, Bouvel, and Rossin [2]:

Definition 1. A **pin sequence** is an word (finite or infinite) over the language

$$\{1,2,3,4\}(\{l,r\}\{u,d\})^* \cup \{1,2,3,4\}(\{u,d\}\{l,r\})^*$$

A finite pin sequence can be converted into a 2-by-2 gridded permutation by the following procedure (see Fig. 12 for an illustration of this process):

1. Place an initial point in the quadrant specified by the initial number (counting anti-clockwise from the top-right);
2. At all subsequent steps, place a point either up, down, left or right (depending on the letter u, d, l, or r) of the bounding rectangle of all previous points (including a 'ghost point' at the origin) at the end of a 'pin' which separates the last point from all points before.

Note that this definition almost guarantees that a permutation produced from a pin sequence will be simple - and it is in fact this connection with simple permutations that has motivated much of the study of pin sequences. Given an (infinite) pin sequence we can define the corresponding **pin class** as the downward closure of the set of all permutations produced by a finite initial subsequence. In this talk we develop the theory of pin sequences and apply this to classify the growth rates of 'small' pin classes.

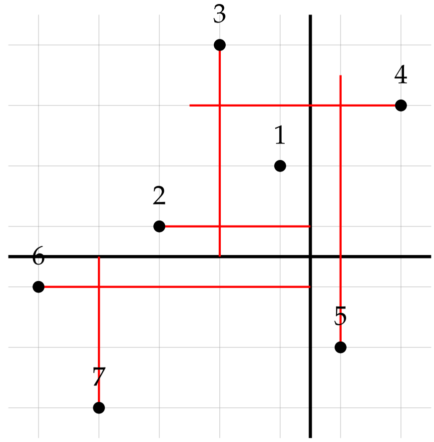

Figure 12: The permutation $3, 1, 4, 7, 5, 2, 6$ (in a 2-by-2 grid), constructed from the pin sequence 2lurdld. The numbers refer to the order in which the points were placed: the first point was placed in quadrant 2 (due to the 2 at the start of the pin sequence); then, the second point was placed to the left (due to the *l*) of the bounding rectangle of the first point and the origin, at the end of a pin separating point 1 from the origin; next, point 3 was placed above (or 'up', due ot the u) the bounding rectangle of the first two points and the origin, at the end of a pin separating point 2 from point 1 and the origin; and so on...

We begin with the class $\mathcal{O}$, the downwards closure of the increasing oscillations, which is also the pin class defined by the sequence $1(ur)^*$. The growth rate of this class is $\kappa \approx 2.20557$; it is known that this is the smallest possible growth rate of a pin class and that $\mathcal{O}$ is 'essentially' the only pin class that achieves it.

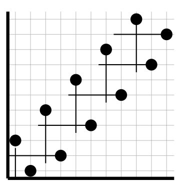

Figure 13: The first 11 points given by the pin sequence $1(ur)^*$, defining the permutation $3, 1, 5, 2, 7, 4, 9, 6, 11, 8, 10$. The downward closure of this pin sequence is the pin class $\mathcal{O}$, the class of **increasing oscillations**.

We shall show that, somewhat surprisingly, the next smallest pin class does not appear until the (significantly larger) growth rate $\nu \approx 3.069$, achieved by the class $\mathcal{V}$, defined by the pin sequence $1(ulur)^*$; see Figure 14. This is the first pin class to visit two quadrants infinitely often - though much more exotic behaviour is possible in two quadrants; for example, non-periodic pin sequences.

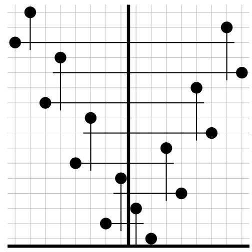

Figure 14: The first 16 points given by the pin sequence $1(ulur)^*$; the downward closure of this pin sequence is the pin class $\mathcal{V}$

In seeking to find the next possible growth rate of a pin class, we are naturally led to define the class $\mathcal{Y}$ from the pin sequence $1(uldlur)^*$; this has growth rate $\gamma \approx 3.366$ and can be shown to be the smallest pin class that visits three quadrants infinitely often. This leaves open the question of what happens between $\mathcal{V}$ and $\mathcal{Y}$; we will address this problem, as well as briefly looking beyond γ at the smaller pin classes in three and four quadrants, concluding with a classification of what we shall call the 'small' pin classes.

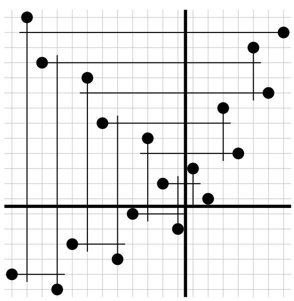

Figure 15: The first 17 points given by the pin sequence $1(uldlur)^*$; the downward closure of this pin sequence is the pin class $\mathcal{Y}$

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An extension of the Lindström-Gessel-Viennot theorem

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Consider a weighted directed acyclic graph G having an upward planar drawing. We give a formula for the total weight of the families of non-intersecting paths on G with arbitrary starting and ending points. While the Lindström-Gessel-Viennot theorem ([3],[2]) gives the signed enumeration of these weights (according to the permutation generated by these paths), our result provides the straight count, expressing it as a determinant whose entries are signed counts of lattice paths with given starting and ending points.

Statement of main results

We consider a directed acyclic graph (simply called a graph) G with a weight function $\mathrm{wt} : E(G) \to \mathcal{R}$ that assigns elements in some commutative ring $\mathcal{R}$ to each edge of G. On the graph G, the *weight* of a path p is the product $\mathrm{wt}(p) = \prod_e \mathrm{wt}(e)$, where the product is over all edges e of the path p. A path with length zero has weight 1 by convention. The weight of an n-tuple of paths $P = (p_1, \dots, p_n)$ is the product of the weights of each path: $\mathrm{wt}(P) = \prod_{i=1}^n \mathrm{wt}(p_i)$. We say two paths are *non-intersecting* if they do not pass through the same vertices.

An *upward planar drawing* of a graph G is a drawing of G on the Euclidean plane such that each edge is drawn as a line segment that is either horizontal or up-pointing, and no two edges may intersect except at vertices of G. In [1], a graph G has an upward planar drawing if and only if G is a subgraph of an st-planar graph on the same vertex set.

Given an st-planar graph $\widetilde{G}$ and its subgraph G having the same vertex set. Let $U = \{u_1, \dots, u_n\}$ be the set of n distinct starting points and $V = \{v_1, \dots, v_n\}$ be the set of n distinct ending points; these points will be called the *marked points*. We introduce the notations for the set of directed paths going from U to V as follows.

- $\mathscr{P}(u_i, v_j)$ denotes the set of paths going from $u_i \in U$ to $v_j \in V$.
- $\mathscr{P}^\pi(U, V)$ denotes the set of n-tuples of paths $(p_1, \dots, p_n)$, where $p_i \in \mathscr{P}(u_i, v_{\pi(i)})$ for $1 \le i \le n$. The permutation π is called the *connection type*.
- $\mathscr{P}(U, V)$ is the set of all n-tuples of paths connecting U to V; i.e., $\mathscr{P}(U, V)$ is the union of $\mathscr{P}^\pi(U, V)$ over all the permutations $\pi \in \mathfrak{S}_n$.
- $\mathscr{P}_0(U, V)$ (resp., $\mathscr{P}_0^\pi(U, V)$) is the subset of $\mathscr{P}(U, V)$ (resp., $\mathscr{P}^\pi(U, V)$) consisting of non-intersecting n-tuples of paths.
- The generating function of the sets of paths $\mathscr{P}$ according to the weight wt is given by $GF(\mathscr{P}) = \sum_{P \in \mathscr{P}} \mathrm{wt}(P)$.

Definition 1. Let s be the source and t be the sink of the st-planar graph $\widetilde{G}$, and let $p \in \mathscr{P}(u,v)$ be a path in the subgraph G. The *left side of the path* p is the closed region of the plane bounded by the following paths in $\widetilde{G}$:

- the leftmost path (i.e., the path obtained by taking the leftmost step at each stage) from s to u,
- the path p itself,
- the leftmost path from v to t, and
- the left boundary of $\widetilde{G}$ going from s to t.

We write $L(p)$ for the collection of marked points of $U \cup V$ which are on the left side of the path p; this includes the starting point u and the ending point v of the path p.

Definition 2. The *path sign* of a path $p \in \mathscr{P}(u,v)$ is defined to be

$$\operatorname{sgn}(p) = (-1)^{|L(p)|}.$$

The path sign of an n-tuple of paths $P = (p_1, \ldots, p_n) \in \mathscr{P}(U,V)$ is defined to be the product of all path signs of the p_i's:

$$\operatorname{sgn}(P) = \prod_{i=1}^{n} \operatorname{sgn}(p_i).$$

Theorem 3 (Main theorem). *Given an st-planar graph $\widetilde{G}$ and a subgraph G having the same vertex set. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be two sets of n marked points of G. Let M be the $n \times n$ matrix whose (i,j)-entry is*

$$\sum_{p \in \mathscr{P}(u_i, v_j)} \operatorname{sgn}(p) \operatorname{wt}(p).$$

Then the total weight of families of non-intersecting paths connecting U to V is given by

$$GF(\mathscr{P}_0(U,V)) = \sum_{\pi \in \mathfrak{S}_n} GF(\mathscr{P}_0^{\pi}(U,V)) = |\det M|\,.$$

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A q-analogue and a symmetric function analogue of a result of Carlitz, Scoville and Vaughan

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Let $f(z) = \sum_{n=0}^{\infty}(-1)^n z^n/n!n!$. In their 1975 paper, Carlitz, Scoville and Vaughan provided a combinatorial interpretation of the coefficients in the power series $1/f(z) = \sum_{n=0}^{\infty} \omega_n z^n/n!n!$. They proved that ω_n counts the number of pairs of permutations of $\mathcal{S}_n$ with no common ascent. In this talk, I will give a combinatorial interpretation of a natural q-analogue of ω_n.

Theorem 1. *Let $\mathcal{D}_n$ denote the set $\{(\sigma,\tau) \in \mathcal{S}_n \times \mathcal{S}_n \mid \sigma$ and τ have no common ascent$\}$, and let $W_n(q) = \sum_{(\sigma,\tau)\in\mathcal{D}_n} q^{\mathrm{inv}(\sigma)+\mathrm{inv}(\tau)}$. Then for $n \geq 1$,*

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q) = 0, \tag{1}$$

where $\begin{bmatrix} n \\ i \end{bmatrix}_q$ is the q-analogue of the binomial coefficient $\binom{n}{i}$.

Theorem 1 gives a combinatorial interpretation to the coefficients of a reciprocal q-Bessel function. This result is obtained by studying the top homology of the Segre product of the subspace lattice $B_n(q)$ with itself. We also derive an equation that is analogous to a well-known symmetric function identity: $\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$, which then generalizes our q-analogue to a symmetric group representation result.

THE FIRST OCCURRENCE OF A PATTERN IN A RANDOM SEQUENCE

Yixin (Kathy) Lin Dartmouth College

This talk is based on joint work with Sergi Elizalde

We introduce an analogue for permutations of the famous Penney's ante game for coin tosses.

Introduction

In Penney's game, player A selects a binary word of length $n \geq 3$, then player B selects another binary word of the same length. A fair coin is tossed repeatedly to determine a random binary word, until one of the players' words appears as a consecutive subword, making that player the winner. It is known that, for any word picked by player A, player B can always pick a word will be more likely to appear first. The exact odds of winning can be computed using Conway's algorithm [4], in terms of the overlaps of the two words with themselves and with each other. The same method applies to words over any finite alphabet.

In this talk, we consider the analogue of Penney's game for permutations instead of words. Let $\mathbf{X} = X_1, X_2, \ldots$ be a sequence of i.i.d. continuous random variables. A *consecutive occurrence* of a permutation $\sigma \in \mathcal{S}_k$ is a subsequence $X_i, X_{i+1}, \ldots, X_{i+k-1}$ whose entries are in the same relative order as $\sigma_1, \sigma_2, \ldots, \sigma_k$. Now player A chooses a permutation σ, and then player B chooses a permutation τ (one may consider the version where σ and τ have different lengths). Then the random variables $X_1, X_2, \ldots$ are drawn until a consecutive occurrence of σ or τ appears, which determines the winner.

Expected time to see the first occurrence of a pattern

For $\sigma \in \mathcal{S}_k$, define the random variable T_σ as the smallest j such that $X_1, \ldots, X_j$ contains a consecutive occurrence of σ. This definition, as well as the next theorem, can be easily extended to vincular and classical patterns, but here we will focus on the consecutive case, which is the analogue of Penney's game.

Let $\alpha_n(\sigma)$ be the number of permutations in $\mathcal{S}_n$ that avoid the consecutive pattern σ, and denote the corresponding exponential generating function by

$$P_\sigma(z) = \sum_{n \geq 0} \alpha_n(\sigma) \frac{z^n}{n!}.$$

The expectation $\mathbb{E}T_\sigma$ has a surprisingly simple expression in terms of this generating function.

Theorem 1. *For every σ,*

$$\mathbb{E}T_\sigma = P_\sigma(1).$$

Expressions for $P_\sigma(z)$ for various σ have been obtained by Elizalde and Noy [2, 3]. For example, it follows from Theorem 1 that

$$\mathbb{E}T_{12} = e, \quad \mathbb{E}T_{123} = \frac{\sqrt{3e}}{2\cos(\frac{\sqrt{3}}{2}+\frac{\pi}{6})} \approx 7.924, \quad \mathbb{E}T_{132} = \frac{1}{1-\int_0^1 e^{-t^2/2}dt} \approx 6.926.$$

Adapting Theorem 1 to vincular patterns, one can compute that the expected number of tosses before the first occurrence of the vincular pattern $1\text{—}23$ is $e^{e-1} \approx 5.575$, and that this expected number is $\sum_{n\geq 0} \frac{C_n}{n!} \approx 5.091$ for any classical pattern of length 3. The value $\mathbb{E}T_\sigma$ gives a measure of how easy it is to avoid σ in a random permutation. It is interesting to compare these values for different patterns σ.

Probability of seeing one pattern before another

Given two permutations σ and τ, we would like to obtain an analogue of Conway's formula to compute the probability that σ appears before τ in $\mathbf{X}$, which we denote by $\Pr(\sigma \prec \tau)$. While it is difficult to find a general formula for arbitrary σ and τ, we can compute these probabilities in specific cases.

For example, we have the following result for the decreasing permutation $\delta_k = k(k-1)\ldots 21$.

Proposition 2.

$$\Pr(12 \prec \delta_k) = \frac{1}{k!}.$$

We also have expressions for $\Pr(\sigma \prec \tau)$ for some patterns of length 3.

Proposition 3.

$$\Pr(123 \prec 132) = \Pr(213 \prec 231) = \Pr(312 \prec 321) = \frac{1}{2}.$$

Theorem 4.

$$\Pr(132 \prec 231) = \frac{e^2-2e-1}{2} \approx 0.476.$$

Other expressions follow from these using that $\Pr(\tau \prec \sigma) = 1 - \Pr(\sigma \prec \tau)$ assuming that neither of the patterns contains the other. Additionally, if $\overline{\sigma}$ denotes the permutation such that $\overline{\sigma}_i = k+1-\sigma_i$ for $1 \leq i \leq k$, then $\Pr(\sigma \prec \overline{\sigma}) = \frac{1}{2}$ and $\Pr(\sigma \prec \tau) = \Pr(\overline{\sigma} \prec \overline{\tau})$.

In the case of words, computing the probability that one word occurs before another is closely related to the expected number of additional tosses to see one word assuming that the sequence of tosses starts with the other word. For permutations $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{S}_\ell$, there are multiple different ways to define an analogue of this notion.

One is to define $B_{\sigma\to\tau}$ as the smallest j such that $X_1, \ldots, X_{k+j}$ contains a consecutive occurrence of τ, conditioning on the fact that $X_1, \ldots, X_k$ is an occurrence of σ. In other

words, $B_{\sigma\to\tau}$ is the number of further steps needed to see τ, assuming that σ occurs at the beginning of $\mathbf{X}$. For example, one can show that $\mathbb{E}B_{\sigma\to\sigma} = k!$ for any $\sigma \in \mathcal{S}_k$. We can adapt the cluster method from [3] to obtain formulas for $\mathbb{E}B_{12\to\delta_k}$ and $\mathbb{E}B_{\delta_k\to 12}$.

On the other hand, a related (but different) random variable, denoted by $F_{\sigma\to\tau}$, is defined as the number of further steps to see the pattern τ after the first occurrence of σ, assuming that σ occurs before τ in $\mathbf{X}$. We following result shows that the expectation of the random variables $F_{\sigma\to\tau}$ is closely related to the probability that one pattern occurs before another, in analogy to Collings' formula [1].

Theorem 5. *For any two consecutive patterns σ and τ,*

$$\Pr(\sigma \prec \tau) = \frac{\mathbb{E}F_{\tau\to\sigma} + \mathbb{E}T_\tau - \mathbb{E}T_\sigma}{\mathbb{E}F_{\tau\to\sigma} + \mathbb{E}F_{\sigma\to\tau}}.$$

Giving an expression for $\mathbb{E}F_{\sigma\to\tau}$ for arbitrary patterns is difficult, but we can compute it in some cases.

Proposition 6.

$$\mathbb{E}F_{\delta_k\to 12} = k!\left(e - \sum_{i=0}^{k-1}\frac{1}{i!}\right),$$

$$\mathbb{E}F_{12\to\delta_k} = \frac{k!}{k!-1}\left(\sum_{i\geq 0}\frac{1}{(ik)!} - \sum_{i\geq 0}\frac{1}{(ik+1)!} - \sum_{i=0}^{k-1}\frac{1}{i!}\right).$$

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Maximum Number of Common Increasing Subsequences of several Permutations

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For given distinct permutations $w_1, w_2, \cdots, w_m \in \mathfrak{S}_n$ of degree n, we denote by $\mathrm{Inc}_l(w_1, w_2, \cdots, w_m)$ the set of common increasing subsequences of length l of them, and by $\mathrm{inc}_l(w_1, w_2, \cdots, w_m)$ its cardinality.

Example 1. For $w_1 = 2134, w_2 = 3124 \in \mathfrak{S}_4$, we have

$$\begin{aligned} \mathrm{Inc}_2(w_1) &= \{23, 24, 13, 14\}, & \mathrm{inc}_2(w_1) &= 4, \\ \mathrm{Inc}_2(w_2) &= \{34, 12, 14, 24\}, \text{ and} & \mathrm{inc}_2(w_2) &= 4, \text{ and} \\ \mathrm{Inc}_2(w_1, w_2) &= \{14, 24\}. & \mathrm{inc}_2(w_1, w_2) &= 2. \end{aligned}$$

Definition 2. For $m, n \geq 1$ and $0 \leq l \leq n$, we put

$$\mathrm{inc}_l(n; m) := \max_{S \in \binom{\mathfrak{S}_n}{m}} \mathrm{inc}_l(S)$$

if $\binom{\mathfrak{S}_n}{m}$ is nonempty. If, on the other hand, $\binom{\mathfrak{S}_n}{m}$ is empty, then we do not define the value $\mathrm{inc}_l(n; m)$.

In this talk, we give explicit formulae of $\mathrm{inc}_l(n; m)$ for $m \leq 4$. These formulae are determined by calculation on the weak right Bruhat order over $\mathfrak{S}_n$.

Theorem 3. *Let $n \geq 1, m \geq 1$ and $0 \leq l \leq n$. Then:*

1. *For $m \leq 3$ and $n \geq m$, we have*

$$\mathrm{inc}_l(n; m) = \binom{n-1}{l} + \binom{n-m}{l-1}.$$

2. *For $m = 4$ and $n \geq 3$, we have*

$$\mathrm{inc}_l(n; m) = \begin{cases} 3\binom{n-3}{l-1} + \binom{n-3}{l} & \text{if } n = 3, \\ 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l} & \text{if } 4 \leq n \text{ and } n \geq 2l-1. \\ \binom{n-1}{l} + \binom{n-4}{l-1} & \text{if } 4 \leq n \leq 2l-1, \end{cases}$$

Note that $n + 1 = 2l$ implies $\binom{n-1}{l} + \binom{n-4}{l-1} = 4\binom{n-4}{l-2} + 4\binom{n-4}{l-1} + \binom{n-4}{l}$.

The motivation for this study lies in coding theory. In coding theory on deletion error correction, the reconstruction model is first introduced by V. I. Levenshtein in [2] and developed by M. Abu-Sini and E. Yaakobi [1], which assumes that a codeword of some code $\mathcal{C}$ is transmitted over m identical noisy deletion channels that output distinct erroneous words. Our theorem can be used to compute a certain function that gives the accuracy limit for reconstructing a codeword in their study.

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Pattern-avoiding binary trees and permutations

Namrata University of Warwick

This talk is based on joint work with Torsten Mütze (University of Warwick)

Pattern-avoidance is a fundamental topic in combinatorics, and in this work we consider pattern-avoidance in Catalan structures, specifically, in binary trees. The study of pattern-avoidance in binary trees was initiated by Rowland [7], who considered contiguous tree patterns, i.e., in a pattern match, the tree pattern appears as an induced subtree of the host tree; see Figure 16 (a). Dairyko, Pudwell, Tyner and Wynn [2] considered non-contiguous tree patterns, i.e., in a pattern match, the tree pattern appears as a minor of the host tree; see Figure 16 (b). Non-contiguous tree patterns are analogous to classical permutation patterns, where matched entries can be arbitrarily far apart, whereas contiguous tree patterns are analogous to consecutive permutation patterns, where matched entries must all be at consecutive positions.

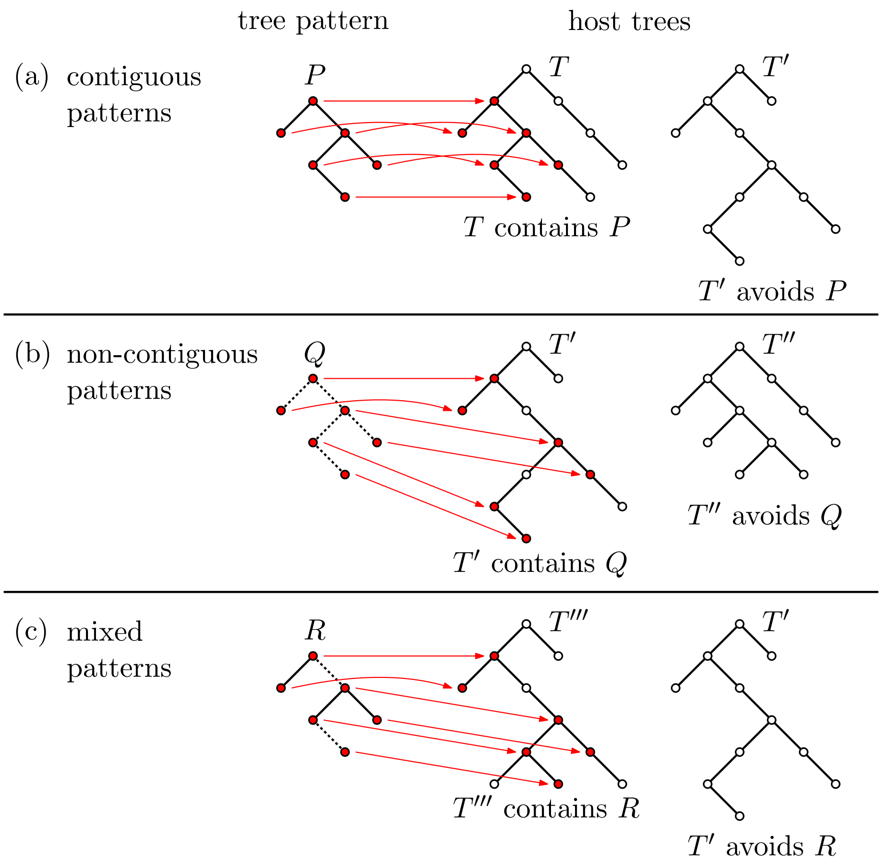

Figure 16: Different notions of tree pattern avoidance.

We generalize the two aforementioned types of tree patterns, by considering an arbitrary mix of both types, i.e., each individual edge of the tree pattern can be considered either contiguous or non-contiguous, independently of the other edges; see Figure 16 (c). This is analogous to vincular permutation patterns, where some pairs

of entries are required to be consecutive, and some other pairs not.

Our first result is a bijection between all binary trees with n nodes that avoid any given set of such generalized tree patterns, and a set of pattern-avoiding permutations of length n. This avoidance characterization uses mesh patterns introduced by Brändén and Claesson [1], and the mesh pattern corresponding to a tree pattern is derived from a simple recursive procedure based on a pre-order traversal of the tree; see Figure 17. This generalizes the earlier bijection of Pudwell, Scholten, Schrock and Serrato [6] for non-contiguous tree patterns.

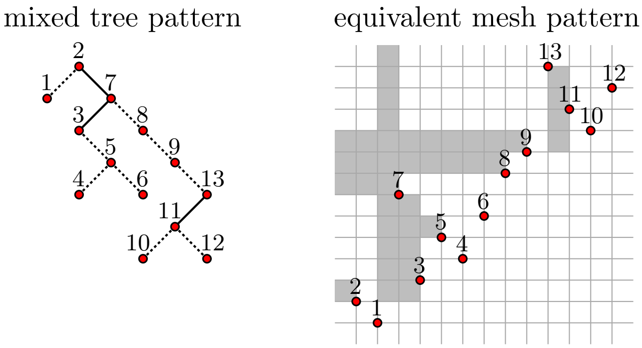

Figure 17: Mixed tree pattern (left) and equivalent mesh pattern (right).

Our main contribution is to apply this bijection to provide exhaustive generation algorithms for a large variety of pattern-avoiding binary trees, based on our permutation language framework [3]. In particular, we discover many sequences new to the OEIS [5], along with finding several known sequences. We also provide efficient implementations of our generation algorithm in C++. This is a continuation of our earlier work on exhaustively generating pattern-avoiding rectangulations [4].

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A Game of Darts

Andy Niedermaier Jane Street Capital

We introduce a new problem related to permutation patterns and statistics: "mathematical darts", a game featuring $n \geq 2$ equally-skilled players who take turns throwing darts at a dartboard. Initially, the players $P_1, \ldots, P_n$ stand in line before the dartboard, with P_1 at the front. When it becomes player P_k's turn, she throws one dart. If it is closer to the center of the board than *every* previous dart, she **remains** in the game, and goes to the back of the line. Otherwise, she **loses immediately** and leaves the line. The last player remaining in line is the winner.

Games of darts can be represented as permutations in S'_n – namely, the set of $n! - 1$ permutations of $\{1, 2, \ldots, n\}$ with at least one descent. We will present some results of the game for $2, 3$, and 4 players; introduce some statistics and generating functions on S'_n; and conclude with conjectures on asymptotic bounds for player P_k's chances of winning.

(No actual darts will be thrown during the talk, we swear.)

Long paths, deep trees and dual cycles

Michal Opler Charles University

This talk is based on joint work with Vít Jelínek, Jakub Pekárek

In this talk, we will introduce structural properties of permutations classes defined via containment of certain grid classes. We show how these properties influence the structural complexity of permutations from the given class and moreover, that they imply conditional lower bounds on the hardness of counting permutation patterns.

Definitions

Recall that a *monotone gridding matrix* is any matrix $\mathcal{M}$ with entries from the set $\{\mathrm{Av}(21), \mathrm{Av}(12), \emptyset\}$. We say that a permutation π has an $\mathcal{M}$*-gridding* if its plot can be partitioned, by horizontal and vertical cuts, into an array of rectangles, where each rectangle induces in π a subpermutation from the permutation class in the corresponding cell of $\mathcal{M}$. The permutation class $\mathrm{Grid}(\mathcal{M})$ then consists of all the permutations that have an $\mathcal{M}$-gridding.

Furthermore, we associate to a monotone gridding matrix $\mathcal{M}$ a *cell graph*, denoted by $G_{\mathcal{M}}$, whose vertices are the non-empty entries in $\mathcal{M}$, with two vertices being adjacent if they belong to the same row or column of $\mathcal{M}$ and there is no other non-empty entry of $\mathcal{M}$ between them.

We say that a permutation class $\mathcal{C}$ has:

the long path property (LPP) if for every k, $\mathcal{C}$ contains a monotone grid subclass whose cell graph is a path of length k,

the deep tree property (DTP) if there is a constant c such that for every d, $\mathcal{C}$ contains a monotone grid subclass whose cell graph is obtained from a binary tree of depth d by subdividing every edge at most c times,

the bicycle property (BP) if $\mathcal{C}$ contains a monotone grid subclass whose cell graph is connected and contains at least two cycles.

Structural complexity

We measure the structural complexity of permutations using the tree-width of a particular graph associated to each permutation. The *incidence graph* G_π of a permutation $\pi = \pi_1, \ldots, \pi_n$ is the graph whose vertices are the n entries $\pi_1, \ldots, \pi_n$, with two entries π_i and π_j connected by an edge if $|i - j| = 1$ or $|\pi_i - \pi_j| = 1$. In particular, the graph G_π is a union of two paths, one of them visiting the entries of π in left-to-right order, and the other in top-to-bottom order. The *tree-width of* π, denoted by $\mathrm{tw}(\pi)$, is defined as the tree-width of the incidence graph G_π.

We remark that tree-width greatly influences the hardness of pattern matching, justifying our choice of tree-width as the measure of structural complexity. In particular, Berendsohn et al. [1] showed that patterns with bounded tree-width can be found in polynomial time.

Theorem 1 ([1])**.** *Given a permutation π of length k and a permutation τ of length n, we can decide if τ contains π in time $O(n^{\mathrm{tw}(\pi)+1})$.*

We are interested in the worst-case behavior in a given class $\mathcal{C}$. To that end, we define the *tree-width growth function* of a class $\mathcal{C}$ as

$$\mathrm{tw}_{\mathcal{C}}(n) = \max\{\mathrm{tw}(\pi);\ \pi \in \mathcal{C} \wedge |\pi| = n\}.$$

We show that each of the three properties implies a different lower bound on the tree-width growth function of the given class.

Theorem 2. *For a permutation class $\mathcal{C}$ we have*

- $\mathrm{tw}_{\mathcal{C}}(n) \in \Omega(\sqrt{n})$ *if $\mathcal{C}$ has the long path property,*
- $\mathrm{tw}_{\mathcal{C}}(n) \in \Omega(n/\log n)$ *if $\mathcal{C}$ has the deep tree property, and*
- $\mathrm{tw}_{\mathcal{C}}(n) \in \Theta(n)$ *if $\mathcal{C}$ has the bicycle property.*

We remark that all of these bounds are asymptotically tight. On top of that, we conjecture that the long path property exactly characterizes the classes with bounded tree-width.

Conjecture 3. *A permutation class $\mathcal{C}$ has unbounded tree-width if and only if it has the long path property.*

Principal classes

We have a full understanding of which properties are attained by principal classes, i.e., classes defined by a single avoidance pattern, as summarized in the following table. Note that we list only a single pattern from each equivalence class under the usual permutation symmetries.

σ	LPP	DTP	BP	$\mathrm{tw}_{\mathrm{Av}(\sigma)}$
1, 21, 132	✗	✗	✗	$\Theta(1)$
321	✓	✗	✗	$\Theta(\sqrt{n})$
3142, 4213, 3412, 4123, 41352	✓	✗	✗	$\Omega(\sqrt{n})$
All other	✓	✓	✓	$\Theta(n)$

Pattern counting

We show that under standard computer-theoretic assumptions there cannot be fast algorithms for counting patterns from classes with the long path and deep tree properties. Our lower bounds are based on the well-known and studied *Exponential Time Hypothesis (ETH)* of Impagliazzo and Paturi [2] which states that 3-SAT cannot be solved in subexponential time in the number of variables. Previously, Berendsohn et al. [1] showed that assuming ETH, there cannot be an efficient algorithm for counting permutation patterns in general.

Theorem 4 ([1]). *Assuming the exponential-time hypothesis (ETH), there is no algorithm that counts the number of occurrences of π in τ in time $f(k) \cdot n^{o(k/\log k)}$ where n is the length of τ and k is the length of π, for any function f.*

We show that similar lower bounds hold even when we restrict the patterns to a class with the long path or deep tree property.

Theorem 5. *Let $\mathcal{C}$ be a fixed permutation class. Assuming the exponential-time hypothesis (ETH), there is no algorithm that counts the number of occurrences of $\pi \in \mathcal{C}$ in τ in time*

- *$f(k) \cdot n^{o(\sqrt{k})}$ if $\mathcal{C}$ has the long path property, and*
- *$f(k) \cdot n^{o(k/\log^2 k)}$ if $\mathcal{C}$ has the deep tree property,*

where n is again the length of τ and k is the length of π, for any function f.

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Cycle structure of random parking functions

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Consider n parking spots placed sequentially on a one-way street. A line of n cars enter the street one at a time, with each car having a preferred parking spot. The ith car drives to its preferred spot, π_i, and parks if the spot is available. If the spot is already occupied, the car parks in the first available spot after π_i. If the car is unable to find any available spots, the car exits the street without parking. A sequence of preferences $\pi = (\pi_1, \dots, \pi_n)$ is a *parking function* if all n cars are able to park.

More precisely, a sequence $\pi = (\pi_1, \dots, \pi_n) \in [n]^n$ is a parking function of size n if and only if $|\{k : \pi_k \leq i\}| \geq i$ for all $i \in [n]$. Equivalently, $\pi = (\pi_1, \dots, \pi_n) \in [n]^n$ is a parking function if and only if $\pi_{(i)} \leq i$ for all $i \in [n]$, where $(\pi_{(1)}, \dots, \pi_{(n)})$ is π sorted in a weakly increasing order $\pi_{(1)} \leq \cdots \leq \pi_{(n)}$. Let PF_n denote the set of parking functions of size n.

Parking functions were introduced by Konheim and Weiss [7] in their study of the hash storage structure. It has since found many applications to combinatorics, probability, and computer science, with connections to other combinatorial objects such as noncrossing set partitions [11], hyperplane arrangements [12], and volume polynomials of certain polytopes [9].

Probabilistic questions have also been considered, but tend to be more complicated than enumeration problems. Connections between parking functions, empirical processes, and the Brownian bridge were discovered in [2]. The asymptotic distribution of the area statistic was studied in [5] and [6], where it was shown to converge to normal, Poisson, or Airy distributions, depending on the ratio between the number of cars and spots. More recently, the distribution of coordinates, descent pattern, area, and other statistics of random parking functions were studied in [4].

Let μ and ν be probability distributions. The *total variation distance* between μ and ν is

$$d_{TV}(\mu, \nu) := \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)|,$$

where Ω is a measurable space. If X and Y are random variables with distributions μ and ν, respectively, then we write $d_{TV}(X, Y)$ in place of $d_{TV}(\mu, \nu)$.

Let $C_k(\pi)$ be the number of k-cycles in the parking function $\pi \in \mathrm{PF}_n$. Our main result gives an upper bound on the total variation distance between the joint distribution of cycle counts $(C_1, \dots, C_d)$ of a random parking function and a Poisson process $(Z_1, \dots, Z_d)$, where the Z_k are independent Poisson random variables with rate $\frac{1}{k}$. This partially answers a question posed by Diaconis and Hicks [4].

Theorem 1. *Let $\pi \in \mathrm{PF}_n$ be a parking function chosen uniformly at random. Let $C_k = C_k(\pi)$ be the number of k-cycles in π and let $W = (C_1, C_2, \dots, C_d)$. Let $Y = (Y_1, Y_2, \dots, Y_d)$, where $\{Y_k\}$ are independent Poisson random variables with rate $\frac{1}{k}$. Suppose $d = o(n^{1/5})$. Then*

$$d_{TV}(W, Y) = O\left(\frac{d^5}{n - d}\right)$$

and the process of cycle counts converges in distribution to a process of independent Poisson random variables,

$$(C_1, C_2, \ldots) \xrightarrow{D} (Y_1, Y_2, \ldots)$$

as $n \to \infty$.

The proof uses a multivariate Stein's method with exchangeable pairs. Stein's method via exchangeable pairs has previously been used to prove limit theorems in a wide range of settings. Our limit theorem parallels the result of Arratia and Tavare [1] on the cycle structure of uniformly random permutations. There is a vast probabilistic literature on the cycle structure of random permutations, which includes the works on ordered cycle lengths [10] and a functional central limit theorem for cycle lengths with connections to Brownian motion [3]. Our work initiates the parallel study of the cycle structure in random parking functions [8], but further study is fully warranted.

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Jianping Pan North Carolina State University

This talk is based on joint work with Emily Gunawan, Heather Russell and Bridget Tenner

Permutations whose reduced words use no repeated simple reflections are called boolean; they avoid the pattern 321 and 3412 [3]. Fully commutative permutations are permutations that avoid the pattern 321. We present results on boolean and fully commutative permutations and on their RSK tableaux.

The Run Statistic and RSK

We represent permutations of $\mathfrak{S}_n$ in *one-line notation*, as $w = w(1)w(2)\cdots w(n)$, and as *reduced words*, that is, as shortest products of the $\{s_i\}$. For example, 51342 (in one-line notation) in $\mathfrak{S}_5$ has a reduced word $s_4s_2s_3s_2s_4s_1$ or $[423241]$ for short.

The RSK (Robinson–Schensted–Knuth) correspondence is a bijective map between permutations and pairs of standard Young tableaux of identical shape. Given a permutation w, we denote its RSK insertion tableau by $P(w)$, and its shape by $\lambda(w) = (\lambda_1(w), \lambda_2(w), \dots)$. For example, if $w = 4132$, we have

$$P(w) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \text{and} \quad Q(w) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \quad \text{so } \lambda_1(w) = 2,\ \lambda_2(w) = 1,\ \lambda_3(w) = 1.$$

A permutation w is fully commutative if and only if $\lambda(w)$ has at most two rows [2].

Our first main result is a concrete interpretation of $n - \lambda_1(w)$ of a permutation $w \in \mathfrak{S}_n$. An increasing or decreasing sequence of consecutive integers is a *run*. We define $\mathsf{run}(w)$ to be the fewest number of runs needed to form a reduced word for w.

Theorem 1. *For any $w \in \mathfrak{S}_n$, $\mathsf{run}(w) = n - \lambda_1(w)$.*

For example, the set of reduced words for $w = 4132$ is $\{[3231], [3213], [2321]\}$. From $[321\ 3]$ and $[23\ 21]$, we get $\mathsf{run}(w) = 2$, and indeed $n - \lambda_1(w) = 4 - 2 = 2$.

Boolean Permutations and Canonical Reduced Words

Given a boolean permutation, the following algorithm from [1] will produce a reduced word with fewest number of runs.

Definition 2. Let w be a boolean permutation, and let $[s]$ be an arbitrary reduced word for w. The following process will produce the *canonical reduced word* for w.

1. Let a be the smallest value appearing in $[s]$.

(a) If $a+1$ does not appear in $[s]$, write $w = [a]w'$.

(b) If $a+1$ is to the left of a, let $b \geq a+1$ be the maximal value such that $i+1$ is to the left of i for all $i < b$. Using commutation relations, write $w = [b(b-1)\cdots a]w'$.

(c) If $a+1$ is to the right of a, let $b \geq a+1$ be the maximal value such that $i+1$ is to the right of i for all $i < b$. Using commutation relations, write $w = w'[a\cdots(b-1)b]$.

2. If w' is not the identity permutation, repeat step 1 on an arbitrary reduced word for w'. If w' is the identity, we are done.

In fact, the above process recovers the second row of $P(w)$, which we denote by $\mathsf{Row}_2(P(w))$.

Theorem 3. *If w is boolean, then* $\mathsf{Row}_2(P(w)) = \{i+1 \mid i$ *is the leftmost entry in a run of the canonical word of* $w\}$.

For example, consider the boolean permutation $w = 51237486$. The canonical word of w is $[4321\ 65\ 7]$, and $P(w) =$

1	2	3	4	6
5	7	8		

.

Boolean RSK tableaux

The insertion tableau of a boolean permutation has at most two rows, but not every such standard tableau is the insertion tableau of a boolean permutation. We characterize the 2-row standard tableaux which are insertion tableaux of boolean permutations.

Definition 4. Let S be a set of integers. If, for all integers $x > 0$ and $y \geq 0$, we have $|[y, y+2x] \cap S| \leq x+1$ (i.e., for every interval I of length $2x$, S contains at most $x+1$ elements of the interval I), then we will say that S is *uncrowded*. Otherwise, we say that S is *crowded*.

For example, if S contains three consecutive integers, then S is crowded, since all three integers live in a length-2 interval. (Here $x = 1$.)

Theorem 5. *Let X be a subset of $[2, n]$, and set $L := \{x-1 : x \in X\}$. The set X is equal to* $\mathsf{Row}_2(P(v))$ *for some boolean permutation $v \in \mathfrak{S}_n$ if and only if $L \cup \{0\}$ is uncrowded.*

We call a 2-row standard tableau an *uncrowded tableau* if its second row is uncrowded, and a *crowded tableau* if its second row is crowded.

Proposition 6. *Let U_n be the set of uncrowded tableaux and the one-row tableau with n boxes, and let X_n be the set of 01-words of length $n-1$ in which all run-lengths of 1s are odd. Then U_n and X_n are in bijection, and they are enumerated by the sequence A028495 in* https://oeis.org/.

Fully Commutative Permutations and the Weak Order

Next, we analyze the fully commutative permutations as a subposet of the right weak order.

The support of a permutation w, denoted by $\mathsf{supp}(w)$, is the set of simple reflections which appear in any reduced word for w.

Proposition 7 (Boolean core for a fully commutative element)**.** *Let w be a fully commutative permutation. Then we can write $w = bw'$, where $\ell(w) = \ell(b) + \ell(w')$, the permutation b is boolean, and $\mathsf{supp}(b) = \mathsf{supp}(w)$.*

For example, consider a a non-boolean fully commutative permutation $w = 456123 = [321453423]$. Then $b = 412563 = [32145]$ and $w' = [3423]$.

Definition 8. The right weak order on $\mathfrak{S}_n$ is a poset structure on $\mathfrak{S}_n$ whose cover relations are defined as follows: $v \lessdot w$ if $w = vs_i$ where s_i is some simple reflection and $\ell(w) = \ell(v) + 1$.

Proposition 9. *If v and w are fully commutative permutations with $v < w$ in the right weak order, then $\mathsf{Row}_2(P(v)) \subseteq \mathsf{Row}_2(P(w))$.*

In the previous example, $b = 412563 = [32145]$ is smaller than $w = 456123 = [321453423]$ in the right weak order, and we have

$$P(b) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}\,, \qquad P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}\,.$$

Due to Proposition 9, the set of fully commutative permutations whose insertion tableaux are crowded form a dual order ideal under the right weak order, and a dual order ideal is generated by its minimal elements. We characterize the minimal crowded permutations, which involves some vincular permutation patterns.

Theorem 10. *Let w be a fully commutative permutation. Then w is minimal crowded if and only if several conditions are satisfied. One of the conditions is that w contains the pattern* 415263 *such that every occurrence of the pattern is consecutive.*

Theorem 11. *Suppose that v and w are fully commutative permutations with $w = vs_i$, $\ell(w) = \ell(v) + 1$, and $s_i \in \mathsf{supp}(v)$. Suppose $P(w)$ is an uncrowded tableau, then $P(w) = P(v)$.*

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Combinatorial Exploration: An Algorithmic Framework for Enumeration

Jay Pantone Marquette University

This talk is based on joint work with Michael H. Albert, Christian Bean, Anders Claesson, Émile Nadeau, and Henning Ulfarsson.

Since 2016, the Combinatorial Exploration project has sought to develop a rigorous, algorithmic framework to discover combinatorial specifications—from which one can obtain counting formulas, generating functions, random sampling routines, and more—for sets of combinatorial objects. While this work was originally focused specifically on permutation classes, it broadened into a domain-agnostic approach that can be effectively applied to many different kinds of combinatorial objects. We have recently released the first article of a series about Combinatorial Exploration [1].

Various aspects of this project have been discussed at several Permutation Patterns conferences in recent years. This talk will give a broad summary of Combinatorial Exploration, with a particular focus on its applications to permutation patterns. We will also introduce our new website, the Permutation Pattern Avoidance Library (PermPAL) [2], available at `https://permpal.com`, which provides a reference database for our enumerative results.

In the next section of this abstract, we give an abbreviated[1)] list of permutation classes for which Combinatorial Exploration is able to rigorously compute a combinatorial specification. In the section that follows that, we show heatmaps derived by sampling uniformly at random from each of the 55 non-finite 2×4 classes.

Successes of Combinatorial Exploration in Permutation Patterns

- ◇ We can find specifications automatically for six out of the seven symmetry classes of permutations avoiding one pattern of length 4, all but $\mathrm{Av}(1324)$. These include the first *direct* enumerations of $\mathrm{Av}(1342)$ and $\mathrm{Av}(2413)$, as previous enumerations were via bijections to each other and to other objects. The final class, $\mathrm{Av}(1324)$, currently remains out of reach, but we are optimistic that several not-yet-implemented strategies may lead to progress.
- ◇ We can find specifications for all 56 symmetry classes of permutations avoiding two patterns of length 4. 53 have specifications that allow us to derive their algebraic generating functions. The remaining three are conjectured to be non-D-finite, and for these we can derive polynomial-time counting algorithms.
- ◇ Out of the 317 symmetry classes of permutations avoiding three patterns of length 4, again we can find specifications for all of them. One is conjectured to be non-D-finite; for the remaining 316 we find algebraic generating functions.

1) A more complete list, including citations to the work mentioned here, can be found in Section 2.4 of [1].

- ◇ Similarly, we can find specifications and generating functions for all symmetry classes avoiding n patterns of length 4 for $4 \leq n \leq 24$. We have not yet done a comprehensive search for specifications for classes avoiding only length 5 patterns, although we have found specifications for around 200 of these avoiding between one and forty patterns.

- ◇ Bevan, Brignall, Elvey Price, and Pantone found improved lower and upper bounds on the exponential growth rate of $\text{Av}(1324)$ by considering a set of gridded permutations that they called "domino permutations". The enumeration of these was challenging, requiring a bijection to a type of arch systems and several pages of work to enumerate these arch systems. We can find a specification and the algebraic generating function for the domino permutations.

- ◇ Defant recently studied the preimage of various permutation classes under the West-stack-sorting operation, derives that the preimage of $\text{Av}(321)$ is $\text{Av}(34251, 35241, 45231)$, and gives rough bounds on its exponential growth rate, but is unable to enumerate it. We find a specification that permits us to compute 636 terms in the counting sequence. We are unable to conjecture the generating function from these terms, and thus we predict that it is non-D-finite. We estimate that the growth rate is $6 + 2\sqrt{5}$.

- ◇ Bóna and Pantone used Combinatorial Exploration to assist with the study of five classes avoiding four patterns of length 5, and one class avoiding five patterns of length 6.

- ◇ Egge conjectured that a group of permutation classes defined by avoiding two patterns of length 4 and one of length 6 are all counted by the Schröder numbers. Burstein and Pantone proved one of these conjectures, and then Bloom and Burstein proved the remainder. We are able to find specifications and generating functions for all of these classes.

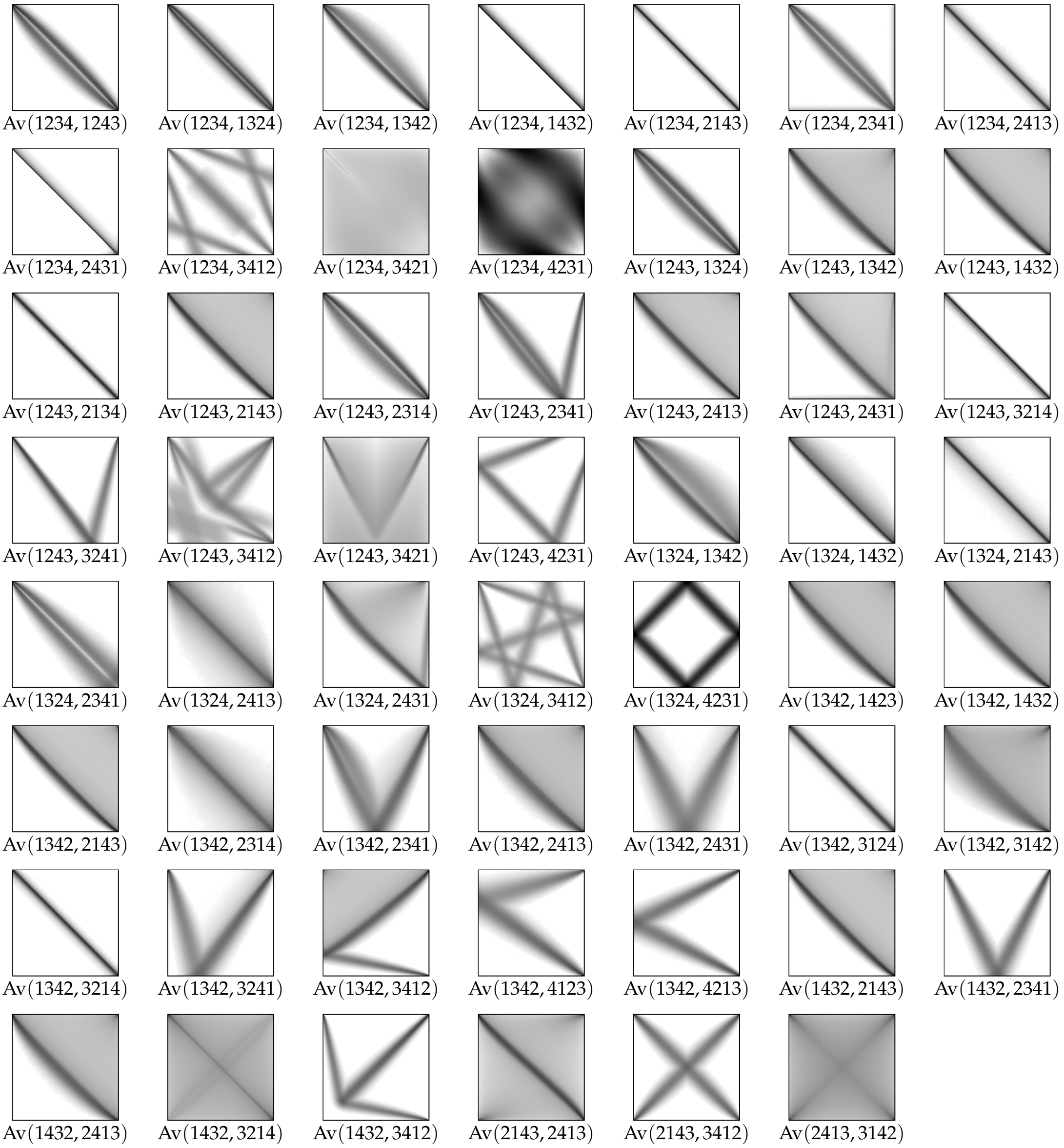
- ◇ Guo and Kitaev explore the notion of "partially ordered permutations". We are able to find specifications for many of the classes they consider.

- ◇ Alland and Richmond recently showed that for a permutation π, the Schubert variety X_π has a complete parabolic bundle structure if and only if $\pi \in \text{Av}(3412, 52341, 635241)$. We are able to find a specification with a property that guarantees that this class has an algebraic generating functions, but the system is too large for us to solve. We can, however, compute the first 400 terms of the counting sequence and conjecture a value for the generating function; it appears to be algebraic with a minimal polynomial of order 6.

Heatmaps for Permutation Classes

When Combinatorial Exploration finds a specification for a permutation class, it typically allows us to sample permutations from that class uniformly at random. For each of the 55 non-finite 2×4 classes (up to symmetry), we have sampled one million permutations of length 300 and drawn their plots on top each other to form a *heatmap*.

Darker areas of the heatmap indicate that many of the sampled permutations have entries in this location, while lighter areas indicate that few do.



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Occurrences of a specific pattern in hypercube orientations, aka Statistics on reciprocal sign epistasis in fitness landscapes

Manda Riehl

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This talk is based on joint work with Lara Pudwell, Nate Chenette, and Reed Philipps

Fitness landscapes help model the theory of adaption, and can be used in applications from designing antibiotic cycling regimens to finding speciation events and hopefully in the future, to predicting evolution. In this work we will consider genetic fitness landscapes abstractly as acyclic orientations of Boolean lattices. We focus on occurrences of reciprocal sign epistasis (RSE), which appears in the hypercube orientation as a set of four edges oriented in a particular way. We computationally study which combinations of peaks and RSEs are possible, and we determine bounds and limits on occurrences of RSEs in both single-peaked and multi-peaked hypercube orientations. Our results can therefore be described as theorems on the joint distribution of two patterns (peaks and RSEs) in acyclic Boolean lattices, and likewise finding the maximum number of RSEs can be considered a form of pattern packing. Our main theorem extends a theorem of Poelwijk to show that any orientation with k peaks contains at least $k-1$ occurrences of reciprocal sign epistasis, or in other words, at least $k-1$ occurrences of a face with no path lengths greater than 1.

We will consider genetic fitness landscapes abstractly as oriented Boolean lattices, or equivalently oriented hypercubes. Each allele in a genotype is assumed to have two possible configurations: the wild type (0) and a mutation (1). A genotype can then be expressed as a bitstring describing which of the two options for each allele is present. Each genotype has a particular fitness value associated with it, where larger fitness values are better-adapted. The wild type, represented by the string $000\cdots 0$, is assumed to have the lowest fitness unless otherwise stated.

Under the strong-selection weak-mutation regime, a population is assumed to only be able to travel from one genotype to a *neighbor* genotype which differs in only one allele. Thus we will think of the Boolean lattice Q_n as the graph whose vertex set is labeled with binary words of length n and where two vertices are adjacent if and only if their labels differ in exactly one bit. In an experimental setting, not all of these allele combinations would exist as viable genomes, however we can consider those as occurring in the landscape but having low enough fitness that they would never fix in a population.

While Crona et al. [2] mostly worked with these lattices using the fitness values, we use the values to assign each edge a direction (adjacent genotypes are assumed to have different fitness values) and consider the directed Boolean lattice (hypercube). Note that these orientations must be acyclic, or a genotype would have higher fitness than itself. Accordingly we define a *peak* to be a vertex which has no edges directed away from itself, and a *valley* to be a vertex with all of its adjacent edges directed away

from itself. (In other fields those would be called sinks and sources respectively.) We can then use the directed edges to examine whether reciprocal sign epistasis exists on a face of the lattice, without considering the strength of that epistasis. Figure 18 shows the three different categories of faces, with both directed edges and vertices labelled with hypothetical fitness values. If we imagine that the lowest vertex in each diagram is wild type 00, and the topmost vertex is 11, then we see that no sign epistasis corresponds to two beneficial mutations resulting in more benefit than either single mutation. A casual way of expressing this would be "good + good = better". An example of sign epistasis, the center diagram, is when both mutations are beneficial, but exactly one of the single mutations is more beneficial than having both mutations occur. Casually we might describe this as "good + good = better than one but not the other". Reciprocal sign epistasis (commonly abbreviated as RSE), the rightmost diagram, is when both single mutants are more fit than the double mutant. Casually we might say "good + good = worse".

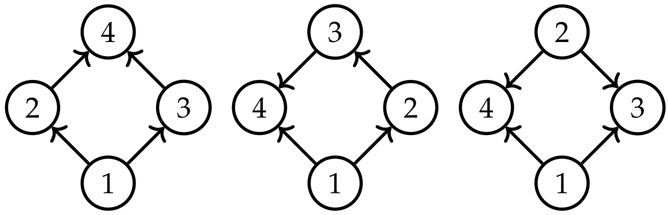

Figure 18: From left to right: no sign epistasis, sign epistasis, reciprocal sign epistasis; vertices are labeled with fitness values.

Note however, that these examples assume that the single mutants are both beneficial. We can also have no sign epistasis with deleterious (negative impact on fitness) mutations, or a combination of beneficial and deleterious mutations. In these cases, it is often easier to just consider the maximal paths in the faces in order to determine whether there is sign or reciprocal sign epistasis. No sign epistasis faces have two paths of length 2. Faces with sign epistasis have one path of length 1 and one path of length 3. Faces with reciprocal sign epistasis have four paths of length 1. Thus our biological question about RSEs can be rephrased in terms of occurrences of four maximal paths of length 1 on a face of an acyclic orientation of a hypercube.

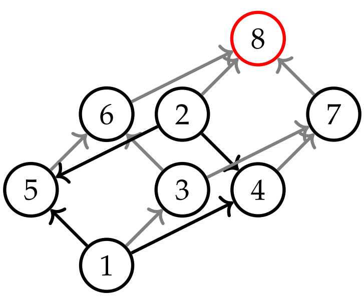

Figure 19: A 3-cube with one peak (red) and one RSE (edges highlighted in black); vertices are labeled with fitness values.

Our primary focus is on which combinations of peak counts and RSE counts are possible, in other words the nonzero entries in the joint distribution of the two patterns

peak and RSE. For example, the 3-lattice in Figure 19 has one RSE and one peak. Many of our arguments rely on building larger lattices from smaller lattices with known numbers of RSEs and peaks, and using a variety of "gluing methods" to combine lattices. We list some of our results below as a sample. In all of our reported results, we force the wild state to be a valley, fixing the orientation of its n edges outward. Our first three results below all relate to the pattern-packing question: the maximal number of RSEs in a single-peaked lattice.

Results

Theorem 1. *Single-peaked n-dimensional lattices exist with r_n RSEs, where*

$$r_n = 2^{n-3}(n^2 - 5n + 8) - 1. \tag{1}$$

Notably, the most significant term in this expression is $2^{n-3}n^2$. The number of faces in a n-dimensional lattice can be written as $2^{n-3}(n^2 - n)$, which has the same most significant term. This means that, in high enough dimensions, an arbitrarily large proportion of the faces in a lattice can be RSEs while still having only one peak.

Theorem 2. *A single-peaked n-dimensional lattice cannot have more than*

$$2^{n-3}(n^2 - n - 2\lfloor n/2 \rfloor) \tag{2}$$

RSE faces.

Conjecture 3. *The maximum number of RSEs in a single-peaked n-dimensional lattice is* $2^{n-3}(n^2 - 4n + 4)$.

n	2	3	4	5	6	7	8
Lower bound (known to be possible)	0	1	7	31	111	351	1023
Conjectured maximum	0	1	8	36	128	400	1152
Upper bound (more is impossible)	0	4	16	64	192	576	1536

Table 4: The values provided by Theorems 1 and 2 and Conjecture 3 for small n.

Theorem 4. *In any dimension, a lattice with k peaks contains at least $k - 1$ RSEs.*

Theorem 5. *For $n \geq 4$, an n-dimensional lattice with $2^{n-1} - (n-1)$ peaks can have $2^{n-2}\binom{n}{2} - \binom{n}{2}$ RSEs but not $2^{n-2}\binom{n}{2} - \binom{n}{2} - 1$ RSEs.*

Theorem 6. *For $n \geq 4$, if an n-dimensional lattice has at least $2^{n-2}\binom{n}{2} - (n-1) - (n-2)$ RSEs, then it must have exactly:*

- $2^{n-2}\binom{n}{2}$ *(every face),*
- $2^{n-2}\binom{n}{2} - (n-1)$, *or*
- $2^{n-2}\binom{n}{2} - (n-1) - (n-2)$

RSEs.

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Stirling numbers of type B

Bruce E. Sagan Michigan State University

This talk is based on joint work with Joshua Swanson

Let $[n] = \{1, 2, \ldots, n\}$ and $\mathbb{Z}$ be the integers. If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group $\mathfrak{S}_n$ then an *inversion* of π is a copy of the pattern 21. Equivalently, an inversion of π is a pair of indices (i, j) with $i < j$ and $\pi_i > \pi_j$. We let $\operatorname{inv}\pi$ be the number of inversions of π. If q is a variable, then we have the following well-known generating function

$$\sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}\pi} = [n]_q! \tag{1}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ is the standard q-analogue of n and $[n]_q! = [1]_q[2]_q \cdots [n]_q$. So equation (1) is a q-analogue of the fact that $\#\mathfrak{S}_n = n!$.

It is also possible to study q-analogues of the the Stirling numbers using inversions. Let $S([n], k)$ be the set of all partitions ρ of $[n]$ into k subsets $B_1, \ldots, B_k$, called *blocks*. We write $\rho = B_1/\ldots/B_k$ for such a partition. The *Stirling numbers of the second kind* are $S(n, k) = \#S([n], k)$ where the hash symbol denotes cardinality. These numbers can also be defined by the initial condition $S(0, 0) = 1$ and, for $n \geq 1$,

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \tag{2}$$

where $S(n, k) = 0$ if $k < 0$ or $k > n$, conventions that we will continue to use for the other objects defined below. The Stirling numbers satisfy many interesting identities.

The q-Stirling numbers $S[n, k]$ were introduced by Carlitz. They can be defined by $S[0, 0] = 1$ and, for $n \geq 1$,

$$S[n, k] = S[n-1, k-1] + [k]_q S[n-1, k].$$

Wachs and White defined four inversion-like statistics on restricted growth functions whose generating functions are the $S[n, k]$ or a closely related q-analogue. One of them, when translated into the language of set partitions, can be expressed as follows. We will write all of our partitions $\rho = B_1/\ldots/B_k$ in *standard form* meaning that $1 = \min B_1 < \ldots < \min B_k$. An *inversion* of ρ is a pair (b, B_j) such that

1. $b \in B_i$ for some $i < j$, and
2. $b > \min B_j$.

Letting $\operatorname{inv}\rho$ be the number of inversions of ρ, one can then prove

$$\sum_{\rho \in S([n], k)} q^{\operatorname{inv}\rho} = S[n, k].$$

Similarly, there are the *signless Stirling numbers of the first kind*, $c(n,k)$, which count the number of permutations in $\mathfrak{S}_n$ which have k cycles in their disjoint cycle decomposition. They satisfy $c(0,0) = 1$ and

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

for $n \geq 1$. A q-analogue can be constructed in the obvious way. And there is an inversion statistic on disjoint cycle decompositions of permutations (as opposed to the one-line notation used above) whose generating function gives these polynomials.

In order to define type B analogues of these Stirling numbers, it will be convenient to look at them from the viewpoint of posets (partially ordered sets). Let P be a finite poset with a unique minimum element $\hat{0}$. Suppose also that P is *ranked*, meaning that for any $x \in P$ the lengths of all saturated chains from $\hat{0}$ to x are the same. This common length is called the *rank of* x and denoted $\operatorname{rk} x$. We also let $\operatorname{Rk}(P,k)$ be the set of all $x \in P$ with $\operatorname{rk} x = k$. The *Whitney numbers of the second kind* for P are

$$W(P,k) = \# \operatorname{Rk}(P,k).$$

To define the Whitney numbers of the first kind, we need the (one-variable) *Möbius function of* P which is the function $\mu : P \to \mathbb{Z}$ defined recursively by $\mu(\hat{0}) = 1$ and

$$\mu(x) = -\sum_{y<x} \mu(y)$$

for $x > \hat{0}$. Then the *Whitney numbers of the first kind* for P are

$$w(P,k) = \sum_{x \in \operatorname{Rk}(P,k)} \mu(x).$$

Now consider the *partition lattice*, Π_n, consisting of all partitions ρ of $[n]$ ordered by refinement. It is immediate that $W(\Pi_n,k) = S(n,k)$ and can be proved that $w(\Pi_n,k) = s(n,k)$ where $s(n,k) = (-1)^{n-k}c(n,k)$ are the (signed) Stirling numbers of the first kind. If G is a Coxeter group then let $L(G)$ be G's intersection lattice. It is well known that Π_n is isomorphic to this lattice for the Coxeter group A_{n-1}. So define *Stirling numbers of type B* by

$$S^B(n,k) = W(L(B_n),k) \quad \text{and} \quad s^B(n,k) = w(L(B_n),k)$$

where B_n is the type B Coxeter group of rank n.

As an example, it follows from Zaslavsky's theory of signed graphs that the $S^B(n,k)$ count partitions of the following type. A *type B partition* is a partition of the set

$$\langle n \rangle := \{-n, -n+1, \ldots, n-1, n\}$$

of the form

$$\rho = B_0/B_1/B_2/\ldots/B_{2k}$$

which satisfies

1. $0 \in B_0$ and if $i \in B_0$ then $-i \in B_0$, and

2. for $i \geq 1$ we have $B_{2i} = -B_{2i-1}$,

where $-B = \{-b \mid b \in B\}$. Letting $S_B(\langle n \rangle, k)$ denote the set of all B_n partitions with $2k+1$ blocks one can show that $S^B(n,k) = \#S_B(\langle n \rangle, k)$. From this description, it follows that we have $S^B(0,0) = 1$ and

$$S^B(n,k) = S^B(n-1,k-1) + (2k+1)S^B(n-1,k).$$

One can now define a q-analogue $S^B[n,k]$ by replacing the factor $2k+1$ by $[2k+1]_q$. We have found an inversion statistic on type B partitions whose generating function is $S^B[n,k]$. Further, we have been able to prove that these objects, as well as corresponding ones for Stirling numbers of the first kind, have many interesting properties. In particular, we calculate their ordinary and exponential generating functions, show that they can be expressed in terms of elementary and complete homogeneous symmetric functions, and discuss their relationship with a recent conjecture of Zabrocki about super diagonal covariants.

On Combinatorial Models of Affine Crystals

Adam Schultze Saint Olaf College

This talk is based on joint work with Cristian Lenart and Carly Briggs

We biject two combinatorial models for tensor products of (single-column) Kirillov-Reshetikhin crystals of any classical Lie type $A-D$: the quantum alcove model, which is based on a variation of the Bruhat graph of (signed) permutation groups, and the tableau model. This allows us to translate calculations in the former model (of the energy function, the combinatorial R-matrix, keys, etc.) to the latter, which is simpler.

Introduction

Kashiwara's *crystals* are colored directed graphs encoding the structure of certain bases, called *crystal bases*, for representations of quantum groups, as the quantum parameter goes to zero. Kirillov-Reshetikhin (KR) crystals are finite dimensional crystals corresponding to certain affine Lie algebras. In classical types, there are (type-specific) models for KR crystals based on fillings of Young diagrams. While they are simpler, they have less easily accessible information; so it is generally hard to use them in specific computations: of the energy function (which induces a grading on KR crystals), the combinatorial R-matrix (the unique affine crystal isomorphism interchanging tensor factors), etc. On the other hand, these computations are more easily carried out in the quantum alcove model where the vertices of the crystal graph are given by certain chains the the corresponding quantum Bruhat graph. Thus, our goal is to translate these computations to the tableau models, via an explicit bijection between the two models.

The map from the quantum alcove model to the tableau model is a "forgetful map", while the inverse map is nontrivial. In this talk we extend the previously known bijections in types A and C to types B and D. There are significant complications in constructing the new inverse maps, which we address by considering a new concept and modifications in the algorithms, primarily through a permutation pattern avoidance.

Background

Consider a finite root system with positive roots Φ^+ and let W be the corresponding *Weyl group*. Let ρ be the half sum of positive roots. Recall that W is the symmetric group for root systems of type A and certain signed permutation groups for root systems of types B, C, D. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat order* on W is defined by its covers $w \lessdot ws_\alpha$, for $\ell(ws_\alpha) = \ell(w)+1$, where $\alpha \in \Phi^+$.

Definition 1. The *quantum Bruhat graph* $\mathrm{QBG}(W)$ on W is defined by adding downward edges, denoted $w \triangleleft ws_\alpha$, to the covers of the Bruhat order, i.e., its (labeled) edges are $w \xrightarrow{\alpha} ws_\alpha$ if $w \lessdot ws_\alpha$ or $\ell(ws_\alpha) = \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1\,,$ where $\alpha \in \Phi^+\,.$

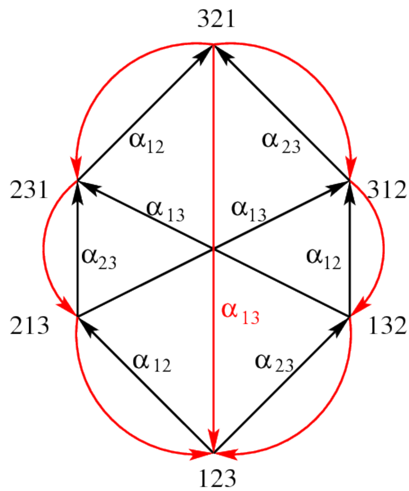

Figure 20: The quantum Bruhat graph for S_3, the Weyl group for the root system A_2. The classical Bruhat edges are in black and the quantum edges are in red.

Remark 2. The vertices for KR crystals in the *quantum alcove model* are indexed by certain paths in the quantum Bruhat graph.

Remark 3. The vertices for KR crystals in the *tableau model* are realized in terms of Kashiwara-Nakashima (KN) columns of height k. These are strictly increasing fillings of the column with entries $\{1 < 2 < \ldots < n\}$ in type A_{n-1}, and entries $\{1 < \ldots < n < 0 < \overline{n} < \ldots < \overline{1}\}$ in types B_n, C_n, and D_n with some additional conditions.

Mapping the quantum alcove model to the tableau model

To map between these two models, we use the following convention for filling a column with a given permutation as well as for assigning a permuation to a given column filling.

1. Given a permutation $w = w_1 w_2 w_3 \ldots w_n$, we fill a column of height $k < n$ with the letters $w_1 w_2 \ldots w_k$.

   Example 4. Consider the permutation 35421. We would then fill the column of height 3 as follows: $\begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline 4 \\ \hline\end{array}$.

2. Given a filling of a column C of height k, we build a permutation on n letters by listing out the values of the column and then concatenating the unused letters in increasing order.

   Example 5. Given the column $\begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline 4 \\ \hline\end{array}$, the permutation on 5 letters associated to it would be 25413.

There are certain stopping points in the quantum Bruhat graph paths that are determined by the choice of crystal. By considering the permutations at these stopping points as fillings of columns and then reordering the columns increasingly, we get a map from the vertices of the quantum alcove model to the tableau model.

Theorem 6. *This map is an affine crystal isomorphism between the quantum alcove model and the tableau model for KR crystals.*

In type A, the inverse map is given by two algorithms: *reorder*, which is the inverse of having sorted the columns increasingly, and *path*, which maps the resulting filling (now viewed once again as a permutation) back to a path in the quantum Bruhat graph. Similar algorithms work for type C. In types B and D, these algorithms both fail unless we include a certain pattern avoidence between two columns, referred to as being *blocked-off*.

Definition 7. We say that two columns $C = (l_1, l_2, ..., l_k)$ and $C' = (r_1, r_2, ..., r_k)$ are blocked off at i by $b := r_i$ if and only if the following hold:

1. $|l_i| \leq b < k$, where $|l_i| = b$ if and only if $l_i = \overline{b}$;
2. $\{1, 2, ..., b\} \subset \{|l_1|, |l_2|, ..., |l_i|\}$ and $\{1, 2, ..., b\} \subset \{|r_1|, |r_2|, ..., |r_i|\}$;
3. $|\{j : 1 \leq j \leq i, l_j < 0, r_j > 0\}|$ is odd.

Example 8. The following columns, considered as fillings from signed permutations, are blocked-off at row 4 by the value 3:

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 4 & 5 \\ \hline \overline{2} & \overline{2} \\ \hline \overline{3} & 3 \\ \hline 5 & 4 \\ \hline \end{array}\ .$$

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Pattern-Avoiding Involutions and Brownian Bridge

Erik Slivken University of North Carolina Wilmington

This talk is based on joint work with Christopher Coscia

In this talk, we show that 321-avoiding involutions convergence in some appropriate sense to Brownian Bridge.

The limiting shape of 321-avoiding involutions

Let $\mathcal{I}v_n(321)$ denote the involutions of size n that avoid the pattern 321. The asymptotics for fixed points of permutations sample uniformly from $\mathcal{I}v_n(321)$ were explored in [2]. This talk will focus on the asymptotic shape of such permutations.

Let $\mathcal{W}_n$ denote the collection of Simple Random Walk Bridges that start at $(0, -1/2)$ and end at $(n, (-1)^n 1/2)$. Finally let b be the Brownian bridge from $(0, 0)$ to $(1, 0)$.

A somewhat classic result (see, for example, [1] is that for $w \in \mathcal{W}_n$,

$$\lim_{n\to\infty} \left(\frac{1}{n^{1/2}} w(nt)\right)_{0<t\leq 1} \longrightarrow_d (B_t)_{0<t\leq 1},$$

where the convergence is in distribution with respect to the appropriate topology.

Utilizing a bijection between $\mathcal{I}v_n(321)$ and $\mathcal{W}_n$, we show that for $\pi \in \mathcal{I}v_n(321)$ chosen uniformly at random,

$$\frac{1}{n^{1/2}} \left(\pi(\lceil nt\rceil) - \lceil nt\rceil\right)_{0<t\leq 1} \longrightarrow_d (B_t)_{0<t\leq 1},$$

where again the convergence is in distribution with respect to appropriate topology.

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Shuffle Sorting Permutations

Rebecca Smith SUNY Brockport

This talk is based on joint work with Lara Pudwell

We introduce four new, but related sorting algorithms.

Introduction

There are many well-known sorting algorithms that can be applied to permutations.

Definition 1. A *sorting function* is a function $f : \mathcal{S}_n \to \mathcal{S}_n$ such that for all $\pi \in \mathcal{S}_n$, there exists $i \in \mathbb{Z}$ such that $f^i(\pi) = 123\cdots n$.

In classifying our sortable permutations, the use of ascents and/or descents to describe these permutations is often helpful.

Definition 2. An *ascent* of permutation π is an index i where $\pi_i < \pi_{i+1}$, while a *descent* is an index i where $\pi_i > \pi_{i+1}$.

The new sorting functions we developed are motivated by shuffling cards. A common way to shuffle is to cut a deck into two non-empty parts and then to *riffle* the two parts together, so that each part remains in order, but the two parts are interleaved. In practice, a deck can be cut anywhere, and a riffle may interleave the two parts of the deck in many different ways.

To create shuffling algorithms that are both well-defined and sorting functions, we

- Determine where the cut is made and
- Create rules on how to riffle the parts.

In all four algorithms, the cut will be made immediately following the longest increasing prefix of the permutation. We will use the following notation. Given a permutation π with first descent at $\pi_{i-1} > \pi_i$, let $\pi' = \pi_1 \cdots \pi_{i-1}$ and $\pi'' = \pi_i \cdots \pi_n$.

What varies is what priorities we take when riffling the two parts back together and whether we keep the second part of the permutation in the original order or if we reverse it. When we cut a deck and riffle it together, we may view this as a system of two queues. When we cut a deck, then reverse the second half before riffling, this acts a system of a queue and a stack.

Definition 3. A *stack* is a last-in, first-out data structure with push and pop operations. A *queue* is a first-in, first-out data structure.

Shuffle algorithms with the order of the second part retained

Our first two algorithms simply based on a cut-and-riffle shuffle algorithm. In both cases, the permutation is cut after the longest increasing prefix. However, we use two different conventions governing the riffle that interleaves the two parts. The Prefix-Preserving Shuffle (PRE) prioritizes keeping all of the original prefix as part of the maximum increasing prefix of the newly shuffled permutation. The Minimum Shuffle (MIN) instead prioritizes shuffling so that smaller entries appear before larger entries whenever possible.

While the intermediate outputs may be different for these two algorithms, the permutations that require exactly k iterations of algorithm PRE to be sorted are exactly the same as the permutations that require exactly k iterations of algorithm MIN to be sorted. Indeed any combination of the two algorithms implemented a total of k times will also sort these same permutations.

Proposition 4. *For any permutation π is sorted to the increasing permutation after exactly* $\mathrm{des}(\pi)$ *iterations of algorithm PRE or after exactly* $\mathrm{des}(\pi)$ *iterations of algorithm MIN.*

Corollary 5. *The number of permutations in $\mathcal{S}_n$ that are sortable after exactly k passes of algorithm PRE or the algorithm MIN is given by the Eulerian numbers (OEIS A008292).*

Shuffle algorithms with the order of the second part reversed

While the two previous algorithms were relatively straightforward to analyze, in this section, we consider two new algorithms (PRE-REV) and (MIN-REV) which act as algorithms PRE and MIN respectively, but where the second part of the original permutation is reversed before being interleaved with the longest increasing prefix.

The following proposition can be shown by verifying the increasing prefix increases in length with each iteration of the algorithm until the identity permutation is reached.

Proposition 6. *Algorithm PRE-REV is a sorting function.*

With the following definition, we can characterize the permutations requiring k iterations of PRE-REV to sort.

Definition 7. Define *prefix-suffix decomposition* of π as follows: Let $\pi^{(1)} = \pi' = \pi_1 \cdots \pi_{i-1}$ be the longest increasing prefix of π and let $\pi^{rev(1)} = (\pi'')^{rev}$ be the reversal of $\pi_i \cdots \pi_n$.

If $\pi^{rev(1)}$ is empty, then we are done.

Otherwise, given $\pi^{(1)}, \ldots, \pi^{(\ell)}$, set $\pi^{(\ell+1)}$ to be the longest increasing prefix of $\pi^{rev(\ell)}$ and recursively define $\pi^{rev(\ell+1)}$ to be the reversal of the remaining digits.

Theorem 8. *Consider $\pi \in \mathcal{S}_n$. If there are $k+1$ parts in the prefix-suffix decomposition of π, then algorithm PRE-REV requires k iterations to sort π.*

The algorithm MIN-REV is seems to be the most difficult of the four algorithms to handle in general, but we can classify sortable permutations based on placements of ascents and descents.

Theorem 9. *Consider a non-identity permutation $\pi \in \mathcal{S}_n$. Suppose there are d descents before n and a ascents after n. Then π requires exactly* $\max(2d, 2a+1)$ *applications of MIN-REV to be sorted.*

Note that both PRE-REV and MIN-REV sort exactly the unimodal permutations with one iteration, but unlike the PRE and MIN algorithms, this commonality in the case of $k = 1$ does not extend to more applications of the algorithms.

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Unimodality of q-twotorials via alternating gamma vectors

Jordan Tirrell Washington College

This talk is based on joint work with Gabriel Johnson, Chloe Sass, and Max Tucker

We study the polynomials $(1+q)(1+q^2)(1+q^3)\cdots(1+q^n)$, which we call *$q$-twotorials*. These are symmetric and unimodal polynomials which are closely related to the q-factorials $[n]_q! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$.

Question 1. *Is there a combinatorial proof that the q-twotorials are unimodal?*

We want to make progress on this question by exploiting the fact that they have *alternating γ-vectors*. We give a technique that provides an affirmative answer to Question 1 for all $n < 25$ except 11, 13, 16, and 19, and hopefully provides a foothold for future progress. Our approach also works for some infinite families of cases of a more general question (Question 2 below). The combinatorial interpretation requires objects we call *g-trees*. These come from lists of permutations that can be arranged into a binary tree. For example, let $\bar{\pi}$ be the following list.

1234
1243
1324
1324

We can draw this as a binary tree with root 1, having two children 2 and 3, themselves each having two children, (34 and 43) and (24 and 24), respectively. We hope further study of these permutations and g-trees can provide insights towards our main questions.

Introduction

A polynomial is called *symmetric* or *palindromic* with *palindromic degree d* if it has the form $f(q) = a_0 + a_1 q + \cdots + a_d q^d$ where $a_i = a_{d-i}$ for all $0 \le i \le d$. We say $f(q)$ is *unimodal* if $a_0 \le \cdots \le a_k \ge \cdots \ge a_d$ for some $0 \le k \le d$.

Unlike the q-integers $[n]_q = 1 + q + \cdots + q^{n-1}$, the factors $1 + q^n$ of q-twotorials are not unimodal in general. So while unimodality of q-factorials follows immediately from the unimodality of the factors and the fact that unimodality is preserved by products, the unimodality of q-twotorials is harder to establish. It is proven [3], but not combinatorially. Also unlike the q-factorials, similar products like $(1+q)(1+q)(1+q^3)$ are not unimodal. This suggests we expand our search. First we need to mention that a palindromic $f(q)$ can be written in the form

$$f(q) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i q^i (1 + q + \cdots + q^{d-2i}),$$

where $g = (g_0, \ldots, g_{\lfloor d/2 \rfloor})$ is called the *g-vector* of $f(q)$. Unimodality is equivalent to non-negativity of its g-vector.

Question 2. *Given a composition $\alpha = (\alpha_1, \ldots, \alpha_n) \vdash d$, consider the palindromic polynomial*

$$(1 + q^{\alpha_1})(1 + q^{\alpha_2}) \cdots (1 + q^{\alpha_n}).$$

Is there a cancellation-free combinatorial interpretation of the g-vector?

We follow a roadmap laid out by Brittenham, Carroll, Petersen, and Thomas [1]. A palindromic $f(q)$ can also be written in the form

$$f(q) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i q^i (1+q)^{d-2i},$$

where $\gamma = (\gamma_0, \ldots, \gamma_{\lfloor d/2 \rfloor})$ is called the *γ-vector* of $f(q)$. These have some nice properties. Unlike *g-polynomials*, the *γ-polynomials* $\gamma(f; z) = \sum \gamma_i z^i$ are multiplicative, $\gamma(fg; z) = \gamma(f; z)\gamma(g; z)$. Both the q-integers and q-twos have γ-vectors that alternate in sign. This is preserved by products, so all polynomials we have mentioned have alternating γ-vectors. They also have nice combinatorial interpretations using domino tilings, as do many other alternating γ-vectors [2]. The linear transformation from γ-vectors to g-vectors is given by a matrix of ballot numbers, counted by ballot paths. The strategy suggested in [1] is therefore to combine these domino tilings and ballot paths to give a combinatorial interpretation for the g-vectors. We do exactly this, and in some cases we are able to show it is cancellation-free.

Results

Formally, a *binary tree of permutations of $n > 1$* is a list of 2^{n-2} permutations of n, say $\bar{\pi} = (\pi_1, \pi_2, \ldots, \pi_{2^{n-2}})$ such that $\pi_{i,k} = \pi_{j,k}$ when $\lceil i/2^{n-k-1} \rceil = \lceil j/2^{n-k-1} \rceil$. Here $\pi_{i,k}$ indicates the kth entry of the ith permutation. So for $n = 3$ there are the 12 pairs of permutations of 3, with each pair agreeing on their first entry (for example, $(123, 132)$). In general, the number of binary trees of permutations of n is

$$n \cdot (n-1)^2 (n-2)^{2^2} \cdots 3^{2^{n-3}} 2^{2^{n-2}}.$$

We call a binary tree of permutations *synchronous* when each permutation is the same.

Roughly, a g-tree is obtained by applying $\bar{\pi}$ to a composition α (which may have multiplicities), assigning $\pm$ signs to entries of each permutation, and removing anything with a partial sum of zero. If we chose $\bar{\pi}$ well, all partial sums may remain positive and we would have a positive g-tree and a combinatorial proof of unimodality.

Formally, to define a *g-tree* for $\alpha \vdash d$, we begin with $\bar{\pi}$. For $1 \le i \le 2^n$ we define

$$P_i = (1, \epsilon_{i,1}\alpha_{\pi_{\lceil i/4 \rceil,1}}, \epsilon_{i,2}\alpha_{\pi_{\lceil i/4 \rceil,2}}, \ldots, \epsilon_{i,n}\alpha_{\pi_{\lceil i/4 \rceil,n}}) \text{ where } \epsilon_{i,k} = (-1)^{1+\lceil i/2^{n-k} \rceil}.$$

Our g-tree is then defined as

$$T := T(\alpha; \bar{\pi}) := \{P_i : \text{no partial sum is zero}\}.$$

We say T is a *positive g-tree* if no remaining partial sums are negative.

Theorem 3. *Given any positive g-tree T for $\alpha \vdash d$, we can interpret the g-vector of $(1 + q^{\alpha_1})(1 + q^{\alpha_2}) \cdots (1 + q^{\alpha_n})$ as*

$$g_i = \#\{P \in T : \mathrm{sum}(P) = d - 2i + 1\},$$

which immediately leads to a combinatorial proof of unimodality.

In some cases it is not difficult to find positive g-trees.

Proposition 4. *Given a composition of the form $\alpha = (1, 2, 4, \ldots, 2^{k-1}, \alpha_{k+1}, \ldots, \alpha_n) \vdash d$, there is a positive g-tree of α beginning $+1{+}1{+}2{+}4{+} \cdots {+}2^{k-1}$ with size 2^{n-k} or $2^{n-k} - 1$ if and only if*

$$\sum_{i=k+1}^{n} \alpha_i \leq 2^k.$$

Moreover α corresponds to a synchronous g-tree.

For small n, we summarize our progress so far in Table 5.

len α	1	2	3	4	5	6	7	8	9
$(1 + q^{\alpha_1}) \cdots (1 + q^{\alpha_n})$ unimodal	1	2	5	13	42	149	653	3369	21304
α has a positive g-tree	1	2	5	13	41	145	626	3203	20047
α has a synchronous pos g-tree	1	2	5	13	40	141	595	3019	18831
Proposition 4 applies	1	2	5	13	37	121	477	2328	14328

Table 5: Number of partitions with given length and properties.

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Connections between permutation clusters and generalized Stirling permutations

Justin M. Troyka California State University, Los Angeles

This talk is based on joint work with Yan Zhuang

Permutation clusters are studied in the context of consecutive patterns in permutations, in the cluster method of Elizalde and Noy [2] which counts the permutations avoiding a given consecutive pattern. We count $2134\ldots(r+1)$ clusters according to inverse descent number and according to inverse peak number, and we find connections to the r-Stirling permutations and rth-order Eulerian polynomials of [4, 3] and the $(1/2)$-Eulerian polynomials of Savage and Viswanathan [5].

From these results, we can count the permutations avoiding the consecutive pattern $2134\ldots(r+1)$. This is made possible using Zhuang's [7] new analog of the cluster method that applies to the Malvenuto–Reutenauer algebra of permutations, an algebra which generalizes the symmetric functions, quasisymmetric functions, and noncommutative symmetric functions. Applying various homomorphisms to this new cluster method recovers Elizalde and Noy's cluster method and Elizalde's [1] q-analog which counts according to inversions, and it also yields several other new permutation cluster methods refined by various permutation statistics. The ones refined by inverse descent number and inverse peak number allow us to count the pattern-avoiding permutations according to these statistics, directly from our enumeration of clusters.

Definitions and notation

Given a permutation π:

- the *descent number* of π is $\operatorname{des}(\pi) = \#\{j\colon \pi(j) > \pi(j+1)\}$;
- the *peak number* is $\operatorname{pk}(\pi) = \#\{j\colon \pi(j-1) < \pi(j) > \pi(j+1)\}$;
- the *inverse descent number* of π is $\operatorname{ides}(\pi) = \operatorname{des}(\pi^{-1})$;
- the *inverse peak number* of π is $\operatorname{ipk}(\pi) = \operatorname{pk}(\pi^{-1})$.

For $r \geq 2$ and $k \geq 1$, we say $\pi \in S_{rk+1}$ is a $2134\ldots(r+1)$ *cluster* if for every i the consecutive pattern $\pi(ri+1)\,\pi(ri+2)\,\ldots\,\pi(ri+r+1)$ has the same relative order as $2134\ldots(r+1)$. We define $\mathcal{P}_k$ to be the set of permutations $\pi \in \mathcal{S}_{rk+1}$ such that π^{-1} is a $2134\ldots(r+1)$ cluster, and we define $\mathcal{P} = \bigsqcup_{k\geq 0}\mathcal{P}_k$ (with the convention that $\mathcal{P}_0 = \{1\}$). In these definitions, the r is fixed but suppressed from the notation. Our study is primarily concerned with the enumeration of $2134\ldots(r+1)$ clusters refined by ides and refined by ipk. Our primary focus is the distribution of ides and of ipk on $2134\ldots(r+1)$ clusters, or equivalently the distribution of des and of pk on $\mathcal{P}_k$.

An *r-Stirling permutation of degree k* is a permutation ρ of the multiset $\{1^r, \ldots, k^r\}$ such that the values between two t's are all at least t; that is, if $a < b < c$ and $\rho(a) = \rho(c)$, then $\rho(b) \geq \rho(a)$. We write $\mathcal{Q}_k$ to denote the set of r-Stirling permutations of degree k, and we define $\mathcal{Q} = \bigsqcup_{k \geq 0} \mathcal{Q}_k$ as the set of all r-Stirling permutations. Again, the r is fixed but suppressed from the notation. Given an r-Stirling permutation ρ:

- the *descent number* of ρ is $\mathrm{des}(\rho) = \#\{j \colon \rho(j) > \rho(j+1)\}$;
- the *plateau–descent number* of ρ is $\mathrm{plde}(\rho) = \#\{j \colon \rho(j-1) = \rho(j) > \rho(j+1)\}$.

Let $\mathrm{st} \colon \mathcal{P} \to \mathbb{N}$ be a statistic on permutations in $\mathcal{P}$. For $i \in \mathbb{N}$, we define

$$\mathbb{P}^{\mathrm{st}}(k,i) = \#\{\pi \in \mathcal{P}_k \colon \ \mathrm{st}(\pi) = i\};$$

that is, $\mathbb{P}^{\mathrm{st}}(k,i)$ is the number of permutations in $\mathcal{P}_k$ whose st value is i. Similarly, if $\mathrm{st} \colon \mathcal{Q} \to \mathbb{Z}$ is now a statistic on r-Stirling permutations, then we define

$$q^{\mathrm{st}}(k,i) = \#\{\rho \in \mathcal{Q}_k \colon \ \mathrm{st}(\rho) = i\};$$

that is, $q^{\mathrm{st}}(k,i)$ is the number of r-Stirling permutations of degree k whose st value is i. By convention, we set $\mathbb{P}^{\mathrm{st}}(k,i) = 0$ and $q^{\mathrm{st}}(k,i) = 0$ if i is a negative integer.

Main results

Recall the classical recurrence relation on the Eulerian numbers $A(n,i)$:

$$A(n,i) = i\, A(n-1,i) + (n-i+1)\, A(n-1,i-1)$$

(see e.g. [6, Sec. 1.4]). Since $A(n,i)$ is the number of length-n permutations with $i-1$ descents, the recurrence relation can be proved combinatorially by looking at where the value n is inserted into a length-$(n-1)$ permutation and whether it creates a new descent. We have established the same kind of recurrence relation for the numbers $\mathbb{P}^{\mathrm{des}}(k,i)$ and $q^{\mathrm{des}}(k,i)$, using the same method of proof as for the Eulerian numbers. In fact, we have shown that the recurrence relation for $\mathbb{P}^{\mathrm{des}}(k,i)$ is exactly the same as for $q^{\mathrm{des}}(k,i)$ (but with i shifted by 1), from which it will follow that $\mathbb{P}^{\mathrm{des}}(k,i+1) = q^{\mathrm{des}}(k,i)$.

Proposition 1. *If $r \geq 2$ and $k \geq 1$ and $i \in \mathbb{Z}$, then*

$$\mathbb{P}^{\mathrm{des}}(k,i) = i\, \mathbb{P}^{\mathrm{des}}(k-1,i) + (rk-r-i+2)\, \mathbb{P}^{\mathrm{des}}(k-1,i-1).$$

Next, we proved the result for $q^{\mathrm{des}}(k,i)$ with a straightforward generalization of the elementary counting argument used by Gessel and Stanley [4] to prove the $r = 2$ case:

Proposition 2. *If $r \geq 2$ and $k \geq 1$ and $i \in \mathbb{Z}$, then*

$$q^{\mathrm{des}}(k,i) = (i+1)\, q^{\mathrm{des}}(k-1,i) + (rk-r-i+1)\, q^{\mathrm{des}}(k-1,i-1).$$

From these (and suitable base cases) we immediately obtain:

Theorem 3. *If $r \geq 2$ and $k \geq 0$ and $i \in \mathbb{Z}$, then $\mathbb{P}^{\mathrm{des}}(k, i+1) = q^{\mathrm{des}}(k, i)$; that is, the number of permutations in $\mathcal{P}_k$ with $i+1$ descents is equal to the number of r-Stirling permutations of degree k with i descents.*

The same type of recurrence relations for $\mathbb{P}^{\mathrm{pk}}(k, i)$ and $q^{\mathrm{plde}}(k, i)$ allow us to obtain the analogous result for peaks on $\mathcal{P}$ and plateau–descents on $\mathcal{Q}$:

Theorem 4. *If $r \geq 2$ and $k \geq 0$ and $i \in \mathbb{Z}$, then $\mathbb{P}^{\mathrm{pk}}(k, i) = q^{\mathrm{plde}}(k, i)$; that is, the number of permutations in $\mathcal{P}_k$ with i peaks is equal to the number of r-Stirling permutations of degree k with i plateau–descents.*

Further considerations

The numbers $p^{\mathrm{des}}(k, i+1) = q^{\mathrm{des}}(k, i)$ are the *rth-order Eulerian numbers*, introduced by Gessel [3] in the context of r-Stirling permutations. For $r = 2$, the numbers $p^{\mathrm{pk}}(k, i) = q^{\mathrm{plde}}(k, i)$ are the $(1/2)$-Eulerian numbers, introduced by Savage and Viswanathan [5]. Unfortunately, we do not get the $(1/r)$-Eulerian numbers for $r \geq 3$.

By taking the reverse, the complement, or the reverse–complement of $2134\ldots(r+1)$ clusters, we obtain results on $(r+1)\ldots4312$ clusters, $r(r+1)(r-1)(r-2)\ldots1$ clusters, and $1\ldots(r-2)(r-1)(r+1)r$ clusters, which translate in a straightforward way from our results on $2134\ldots(r+1)$ clusters.

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The shallow permutations are the unlinked permutations

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There are many measures for how far a given permutation $w \in S_n$ is from being the identity. The most classical are length and reflection length, which are defined as follows. Let s_i denote the adjacent transposition $s_i = (i\ i+1)$ and t_{ij} the transposition $t_{ij} = (i\ j)$. The **length** of w, denoted $\ell(w)$, is the smallest integer ℓ such that there exist indices $i_1, \ldots, i_\ell$ with $w = s_{i_1} \cdots s_{i_\ell}$. It is classically known that the length of w is equal to the number of inversions of w; an **inversion** is a pair (a, b) such that $a < b$ but $w(a) > w(b)$. The **reflection length** of w, which we will denote $\ell_T(w)$, is the smallest integer r such that there exist indices $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ with $w = t_{i_1 j_1} \cdots t_{i_r j_r}$. It is classically known that $\ell_T(w)$ is equal to $n - \mathrm{cyc}(w)$, where $\mathrm{cyc}(w)$ denotes the number of cycles in the cycle decomposition of w.

Another such measure is **total displacement**, defined by Knuth [4] as $\mathrm{td}(w) = \sum_{i=1}^{n} |w(i) - i|$ and first studied by Diaconis and Graham [2] under the name Spearman's disarray. Diaconis and Graham showed that $\ell(w) + \ell_T(w) \leq \mathrm{td}(w)$ for all permutations w and asked for a characterization of those permutations for which equality holds. More recently, Petersen and Tenner [6] defined a statistic they call **depth** on arbitrary Coxeter groups and showed that, for any permutation, its total displacement is always twice its depth. Following their terminology, we call the permutations for which the Diaconis–Graham bound is an equality the **shallow** permutations.

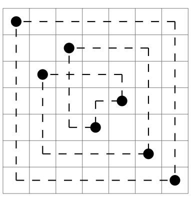

Figure 21: Knot diagram for $w = 7563421$

In a recent paper, Cornwell and McNew [1] interpreted the cycle diagram of a permutation as a knot diagram and studied the permutations whose corresponding knots are the trivial knot or the trivial link. Given a permutation w, to obtain the **cycle diagram**, draw a horizontal line between the points (i, i) and $(w^{-1}(i), i)$ for each i and a vertical line between (j, j) and $(j, w(j))$ for each j. Turn the cycle diagram into a **knot diagram** by designating every vertical line to cross over any horizontal line it meets. For example, Figure 21 shows the knot diagram for $w = 7563421$. They say that a permutation is **unlinked** if the knot diagram of the permutation is a diagram for the unlink, a collection of circles embedded trivially in $\mathbb{R}^3$. In their paper, they mainly consider derangements, but it is easy to modify their definitions to consider all permutations by treating each fixed point as a tiny unknotted loop.

Our main result is the following:

Theorem 1. *A permutation is shallow if and only if it is unlinked.*

Readers can check that Figure 21 shows that the diagram of $w = 7563421$ is a diagram of the unlink with 2 components, and $\ell(w) = 19$, $\ell_T(w) = 5$, and $\mathrm{td}(w) = 24$, so $\ell(w) + \ell_T(w) = \mathrm{td}(w)$.

Using this theorem and further results of Cornwell and McNew [1, Theorem 6.5], we obtain a generating function counting shallow permutations. Let P be the set of shallow permutations, and let

$$G(x) = \sum_{n=0}^{\infty} \sum_{P \cap S_n} x^n.$$

Then G satisfies the following recurrence.

Corollary 2. *The generating function G satisfies the following recurrence:*

$$x^2G^3 + (x^2 - 3x + 1)G^2 + (3x - 2)G + 1 = 0.$$

This is sequence A301897 (defined as the number of shallow permutations) in the OEIS [5].

Our proof relies on a recursive description of the set of unlinked permutations due to Cornwell and McNew and a different recursive description of the set of shallow permutations due to Hadjicostas and Monico [3]. We show by induction that all permutations satisfying the description of Cornwell and McNew are shallow and separately that all permutations satisfying the description of Hadjicostas and Monico are unlinked.

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A lifting of the Goulden–Jackson cluster method to the Malvenuto–Reutenauer algebra

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The study of consecutive patterns in permutations, initiated by Elizalde and Noy [2] in 2003, extends the study of classical patterns in permutations. Consecutive patterns in permutations are analogous to consecutive subwords in words, where repetition of letters is allowed. In the latter realm, the cluster method of Goulden and Jackson [5] provides a very general formula expressing the generating function for words by occurrences of prescribed subwords in terms of a "cluster generating function", which is easier to compute. In 2012, Elizalde and Noy [3] adapted the Goulden–Jackson cluster method to the setting of permutations, which allows one to count permutations by occurrences of prescribed consecutive patterns. Their adaptation of the cluster method has become a standard tool in the study of consecutive patterns, and a notable recent development is a q-analogue [1] which also keeps track of the inversion number statistic.

The main result of this talk is a lifting of the Goulden–Jackson cluster method for permutations to the Malvenuto–Reutenauer Hopf algebra. Since the basis elements of the Malvenuto–Reutenauer algebra correspond to permutations, our cluster method in Malvenuto–Reutenauer is in a sense the most general cluster method possible for permutations. By applying standard homomorphisms, we recover Elizalde and Noy's cluster method for permutations as well as its q-analogue as special cases of our generalized cluster method. We also construct other homomorphisms which lead to new specializations of our generalized cluster method that can be used to count permutations by occurrences of prescribed patterns while keeping track of other permutation statistics.

The permutation statistics that we consider are "inverses" of several classical permutation statistics related to descents and peaks, including the descent number, major index, comajor index, peak number, and left peak number. Given a statistic st, we define its *inverse statistic* ist by $\mathrm{ist}(\pi) := \mathrm{st}(\pi^{-1})$. For example, given $\pi = 72163584$, its inverse descent number is given by $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1}) = \mathrm{des}(32586417) = 4$.

We consider these inverse statistics because they are inverses of "shuffle-compatible" statistics. In [4], it is proven that if a permutation statistic st is shuffle-compatible and is a coarsening of the descent set, then st induces a quotient of the algebra QSym of quasisymmetric functions, denoted $\mathcal{A}_{\mathrm{st}}$. By composing the quotient map from QSym to $\mathcal{A}_{\mathrm{st}}$ with the canonical surjection from the Malvenuto–Reutenauer algebra to QSym, we obtain a homomorphism on the Malvenuto–Reutenauer algebra which can be used to count permutations by the corresponding inverse statistic. Applying these homomorphisms to our generalized cluster method yields specializations that refine by these inverse statistics, and these new specializations lead to new generating function formulas that count permutations by occurrences of various consecutive patterns refined by inverse statistics.

Some preliminary results of this work were presented at Permutation Patterns 2021,

and I am grateful to Sergi Elizalde for posing a question at PP 2021 which helped spawn additional results, leading to the completion of the paper [6] on which this talk is based.

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