Trigonometry: An Essential Tool in Engineering Mechanics

Objective

In this laboratory, students will learn how the trigonometric functions (sin(x), cos(x), and tan(x)) are applied in engineering mechanics to analyze systems in mechanical equilibrium. They will use this knowledge to design the cables in a shipping container lift system.

Introduction and Background

Engineers are problem solvers. For example, engineers developed a solution to the problem of lifting extremely heavy containers. We call the solution a crane (Fig. 1). Engineers also developed a solution for getting across bodies of water without having to swim or use a boat. We call this solution a bridge (Fig. 2).

Building cranes and bridges like the ones depicted in Figs. 1 and 2 is expensive. Consequently, engineers must ensure that, once built, these structures will not fail. To do so, engineers analyze their designs using a branch of the physical sciences called **Mechanics**. Mechanics deals with the state of rest or motion of bodies that are subjected to the action of forces. The crane can be at rest or move; the bridge remains at rest. Both bodies are subject to forces, such as the weight of the container, the weight of the cars, and the force of wind.

In this laboratory, you will learn how trigonometry plays an important role in engineering mechanics and the analysis of designs of devices such as cranes and bridges.

Figure 1 (top right). A large crane moves a container onto a ship (© Andrew Peacock/Lonely Planet Images/Getty Images).

Figure 2 (bottom right). The Golden Gate Bridge near San Francisco, CA.

Force

You can think of a force as a “push” or a “pull” exerted by one body on another. This interaction can occur through direct contact between the bodies, such as a person pulling on a rope to make it taught, or it can occur through a distance when the bodies are physically separated, as is the case with gravity and magnets. If you want more information on what a force is, the 3.5 minute video here provides an excellent summary using easy-to-understand language.

A force is an example of a **vector** because it requires two pieces of information to be completely defined, a magnitude and a direction. Imagine a soccer player kicking a soccer ball during a game. If you had to describe this action to a friend wearing a blindfold, how would you do it?

Task 1 - Write a short description of the soccer player kicking the soccer ball for someone who is not able to see this action:

____

____

____

Share your description with someone sitting next to you. How is your description similar to or different from your neighbor’s description? Perhaps you wrote something like “The player cleared the ball towards the sideline.” This description contains two pieces of information. It tells us how hard the player kicked the ball because when a player clears the ball, they kick it hard in order to move the ball as far away as possible from their team’s goal. It also tells us in what direction the player kicked the ball: “towards the sideline.” To fully understand what it means to kick a soccer ball, we must describe both how hard it is kicked (the magnitude of the force) and where it is kicked (the direction of the force).

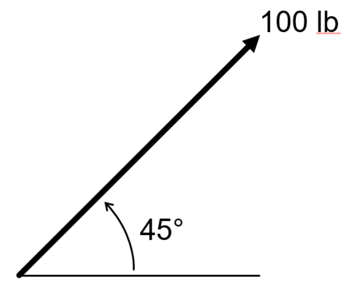
While the statement “the player cleared the ball towards the sideline” describes both the magnitude and direction of the force that a soccer player imparts on the ball, it does so subjectively. The listener must rely on their own experience to judge what the magnitude and direction of the kick are. Engineering requires objective descriptions of the magnitude and direction of the force.

Magnitude

An engineer objectively describes the magnitude of a force using a unit of measurement. Some units of measurement for force include the Newton and the pound. You are familiar with the latter if you have ever weighed yourself on a scale in the United States. Your weight is described in pounds. For example, the person standing on the scale depicted to the right weighs 115 lb, where “lb” is shorthand for “pounds”.

Direction

One of the ways that an engineer objectively describes the direction of a force is with an angle and an arrowhead. Angles are especially useful when the force is restricted to two spatial dimensions (2D). To picture a 2D system, imagine that you could only view the soccer field by looking straight down on it from a hot air balloon stationed high above the field. In the sketch at right, the direction of a force with a magnitude of 100 lb—say that of a soccer player exerted on a ball on the field— is noted as acting in a direction up and to the right via the head of the arrow at an angle of 45° above a horizontal line, where the horizontal line may represent the midfield line.



So where does trigonometry fit in?

Engineering systems usually involve more than one force. What an engineer is often interested in is the net effect of all forces acting on a system. To work out the net effect of all forces, an engineer uses—you guessed it—trigonometry!

The Resultant Force of a System of Two Forces

Imagine hanging a 10 lb weight from a ring as shown in Fig. 3. To hold the weight and the ring at rest, one must apply a 10 lb force to the rope attached to the top of the ring. The 10 lb force acts along the direction of the rope, entirely upwards. Now imagine hanging the same weight from the ring, but using two ropes instead of one with the ropes set such that they make an angle of 45 degrees relative to the vertical or horizontal (Fig. 4). Our intuition tells us that the

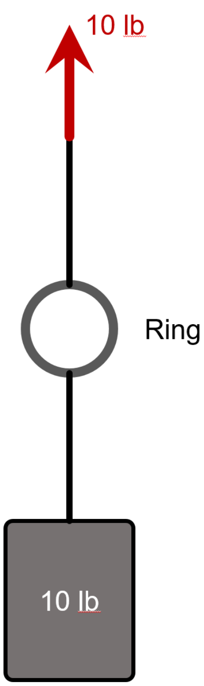


Figure 3. One rope supports the 10 lb weight.

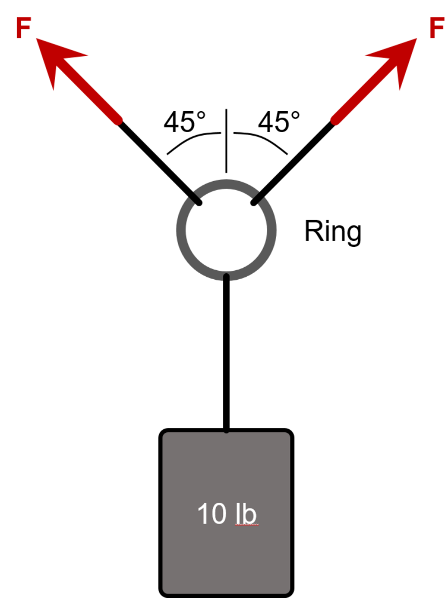


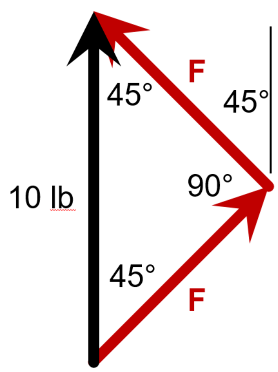
Figure 4. Two ropes support the 10 lb weight.

net effect of the two forces, **F**, must be the same as the single force because both systems are in equilibrium (not moving). That is, if we add the two forces in the two ropes together, the net effect must be to lift upwards and only upwards with a force of 10 lb. Before we apply this idea to calculate the magnitude of the **F** forces, make a guess.

Task 2 - Do you think that the magnitude of the force in the two ropes will be larger than, smaller than, or equal to 10 lb? That is, will the forces in each of the two ropes be larger than, smaller than, or equal to the force in a single rope? Circle one of the options.

LARGER (> 10 lb)   SMALLER (< 10 lb)   EQUAL (= 10 lb)

To add the forces together and calculate their net effect, we apply the triangle rule. We attach the tail of one of the forces to the tip of the other without changing the directions of the forces. The arrow connecting the tail of the first force to the tip of the second represents the net effect, or the "resultant force", of the two forces. The result is shown in this figure:



The triangle rule reveals that the addition of the two forces represents a 45-45-90 triangle. The hypotenuse of the triangle is the resultant force of 10 lb. It acts entirely upwards, as we suspected. The length of the hypotenuse must be 10 lb (in order to lift the 10 lb weight). Knowing this, we can calculate the length of the equal-length-sides of the triangle using the Pythagorean theorem from geometry:

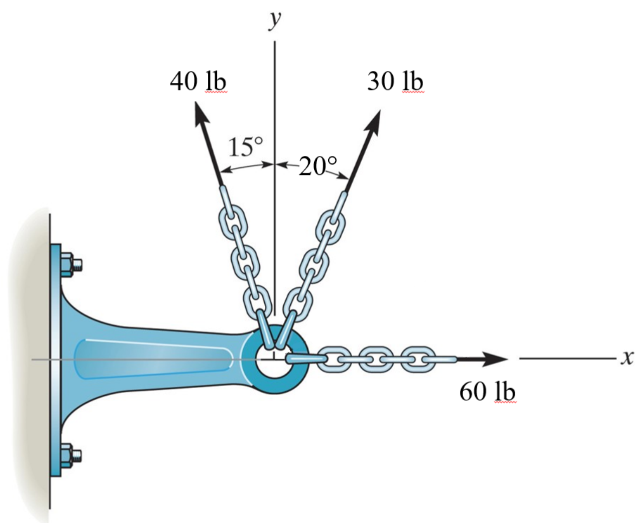
$$c = \sqrt{a^2 + b^2}$$

Task 3 - Calculate the magnitude of the force in each of the two ropes suspending the *c* = 10 lb weight, and then check your guess. Do you find the result surprising?

F = ____________ lb

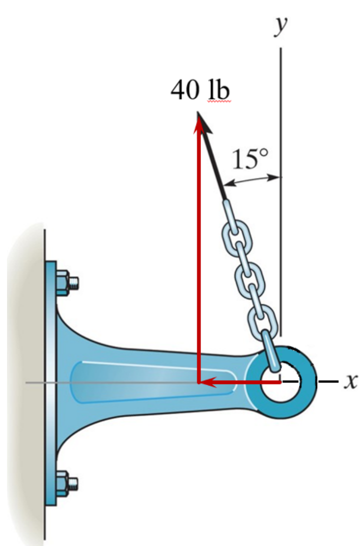
The Resultant Force of a System of More than Two Forces

The triangle rule is effective for adding two forces together. It loses its effectiveness in systems that include more than two forces. For example, consider the following system.

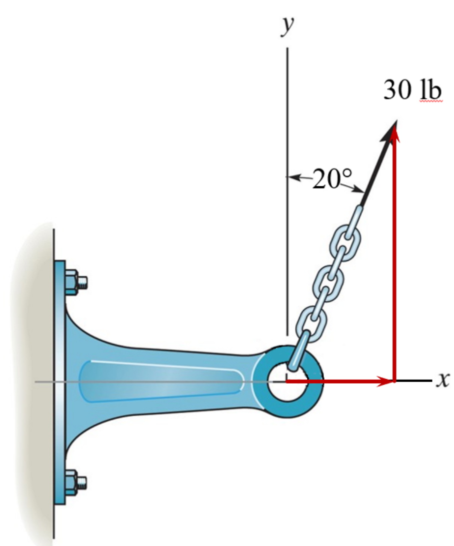


To calculate the net effect of the forces in the three chains on the eyelet, we would need to apply the triangle rule two times successively. The first time would reveal the resultant of two of the forces, e.g. the 40 lb force and the 30 lb. The second time would reveal the resultant of the resultant of the 30 lb and 40 lb forces and of the third 60 lb force.

There is a better way. We can think of each of the forces in the three chains as being the resultant of two forces where the two component forces act entirely in the horizontal direction (parallel to the line labeled *x* in the figure above) and in the vertical direction (parallel to the line labeled *y* in the figure above). Here is what this so-called “Cartesian decomposition” looks like for the 40 lb and 30 lb forces. Note that the 60 lb force is already aligned with the x-axis.



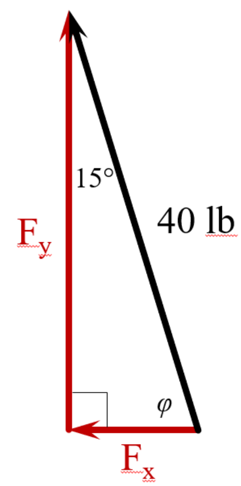
The 40 lb force as the resultant of a force aligned with the x-axis and force aligned with the y-axis.



The 30 lb force as the resultant of a force aligned with the x-axis and force aligned with the y-axis.

When we decompose forces into their Cartesian components, we form right triangles. The hypotenuse of the triangle represents the original force. The remaining two legs represent what are called the "Cartesian components" of the original force. There are always two Cartesian components, one for the x-direction and a second for the y-direction. Since the Cartesian components of the force and the original force form a right triangle, we can use trigonometry to work out the magnitude of the Cartesian component forces.

Let's take a closer look at the 40 lb force triangle. This force has been redrawn in the figure at right and three new labels have been added—φ, F_x, and F_y—to represent properties of the triangle that can be determined from trigonometry. All of the angles in a triangle must add to 180°. We can use this property to calculate the unknown angle φ.



$$180° = 90° + 15° + \varphi$$
$$180° - 90° - 15° = \varphi$$
$$75° = \varphi$$

There are three trigonometric functions that relate the three sides of the right triangle together, the cosine function, the sine function, and the tangent function.

$$\cos(\varphi) = \cos(75°) = F_x / 40\,lb$$
$$\sin(\varphi) = \sin(75°) = F_y / 40\,lb$$
$$\tan(\varphi) = \tan(75°) = F_y / F_x$$

You can remember these relationships using the mnemonic "SOH CAH TOA" (pronounced "Sow-Cah-Toe-Ah"). The "S" stands for the sine function, the "O" stands for the leg of the triangle opposite of the angle 75°, and the "H" stands for the hypotenuse. The order of the "O" and the "H" tells us which length goes in the numerator and the denominator of the ratio that the sine function is equivalent to. Since the "O" is first, it goes in the numerator. The "H" comes second and thus goes in the denominator.

Can you work out the rest of the mnemonic knowing that "C" stands for the cosine function, "T" stands for the tangent function, and "A" stands for the leg of the triangle that is adjacent to the angle 75°?

From the definition of the sine function, we can calculate the y-component of the 40 lb force.

$$F_y = (40\,lb)\sin(75°) = (40\,lb)(0.966) = 38.6\,lb$$

From the definition of the cosine function, we can calculate the x-component of the 40 lb force.

$$F_x = (40\,lb)\cos(75°) = (40\,lb)(0.259) = 10.4\,lb$$

As a check, the ratio of the y-component of the force to the x-component should equal to the value of the tangent function.

$$F_y / F_x = 38.6 / 10.4 = 3.7$$
$$\tan(75°) = 3.7$$

The ratio of the forces and the value of the tangent function are equal! Note that we could have also used the 15° angle to calculate the x- and y-components of the 40 lb force.

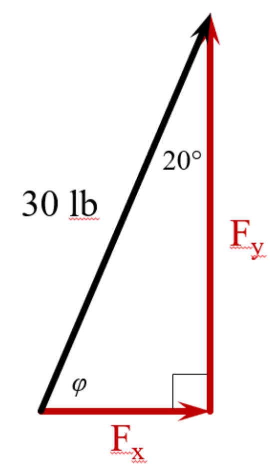
Your turn to practice! Complete the following two tasks. When you are done, have your instructor check your work.

Task 4 - Calculate the Cartesian components of the 40 lb force using the 15° angle.

F_y = ____ = __________ lb

F_x = ____ = __________ lb

Task 5 - Calculate the Cartesian components of the 30 lb force. The 30 lb force has been redrawn in the figure at right to assist with this task.



F_y = __________ lb F_x = __________ lb

Now that we have represented all of the forces in the three-force system on the eyelet as right triangles whose legs are parallel to either the x-axis or the y-axis, it is a simple matter to calculate the resultant force, $\mathbf{F}_R$. To do so, sum the horizontal components of the individual forces in the force system, i.e., "add up" all of the "F_x's", and sum the vertical components of the individual forces in the force system, i.e., "add up" all of the "F_y's". The results of the two summations are equal to the horizontal and vertical legs of the resultant force triangle.

$(\mathbf{F}_R)_x$ = (sum of all horizontal components of the forces in the force system) = $\sum F_x$

$(\mathbf{F}_R)_y$ = (sum of all vertical components of the forces in the force system) = $\sum F_y$

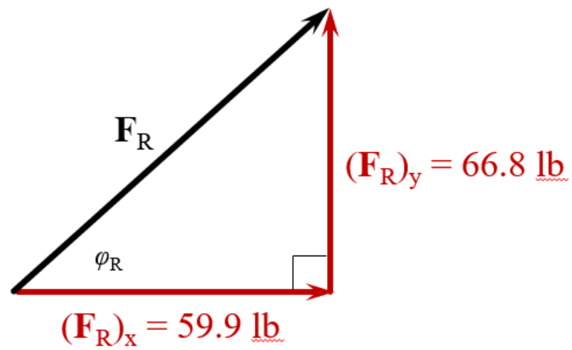
*** **Critical note** *** When doing the summation, if a horizontal force component is directed to the right, then it is added. If it is directed to the left, then it should be subtracted. Similarly, if a vertical force component is directed upwards, then it is added. If it is directed downwards, then it should be subtracted. Once the x- and y-components of the resultant force are known, then trigonometry can be applied to calculate the magnitude and direction of the resultant force.

Let's apply this idea to the three-force system to determine the resultant force. First, we calculate the Cartesian components of the resultant force using the summations.

$(\mathbf{F}_R)_x = \sum F_x = (60 \text{ lb}) + (30 \text{ lb})_x - (40 \text{ lb})_x = (60 \text{ lb}) + (10.3 \text{ lb}) - (10.4 \text{ lb}) = +59.9 \text{ lb}$

$(\mathbf{F}_R)_y = \sum F_y = (30 \text{ lb})_y + (40 \text{ lb})_y = (28.2 \text{ lb}) + (38.6 \text{ lb}) = +66.8 \text{ lb}$

The results show us that the resultant force can be represented by a right triangle whose legs have the length 59.9 and 66.8, as depicted at right. Note that the summations both yielded a positive result. When the result of the summation for the horizontal, x-components is positive, it means that the resultant points to the right. If the result is negative, then the resultant points to the left. When the result of the summation for the vertical, y-components is positive, it means that the resultant points upwards. If the result is negative, then the resultant points downwards.



Task 6 - Apply the Pythagorean theorem to calculate the magnitude of the resultant force, $\mathbf{F}_R$, and a trigonometric function to calculate the angle that defines the direction of the resultant force, φ_R.

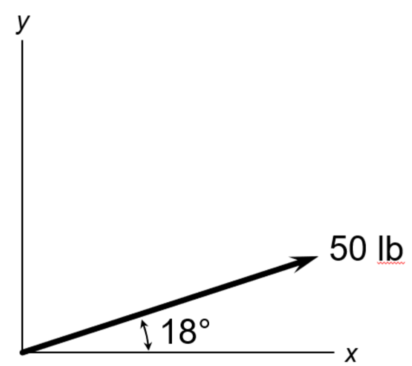
$\mathbf{F}_R$ = ____________ lb φ_R = __________ °

****** Before moving on to the next section, have your instructor check your work. ******

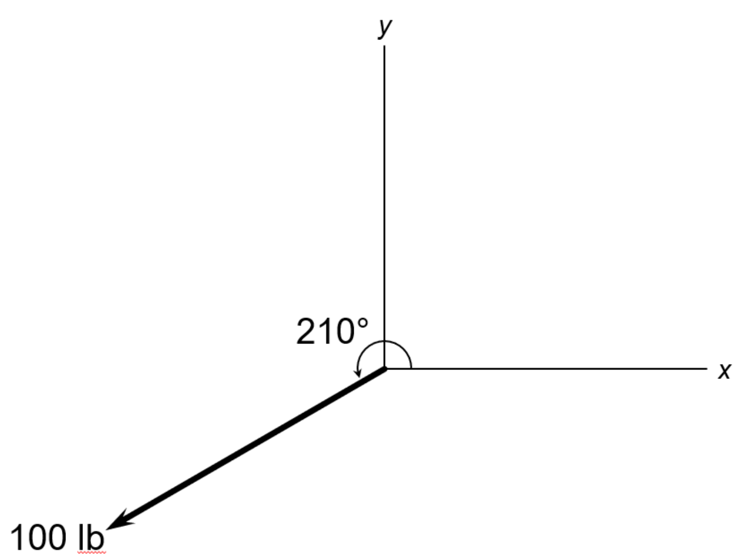
Additional Practice

You now possess all of the knowledge required to analyze two-dimensional engineering systems in mechanical equilibrium. In the next section, you will put this knowledge into practice to design the cables on a shipping container lift system. Before then, put your new knowledge to the test with the following skill development exercises.

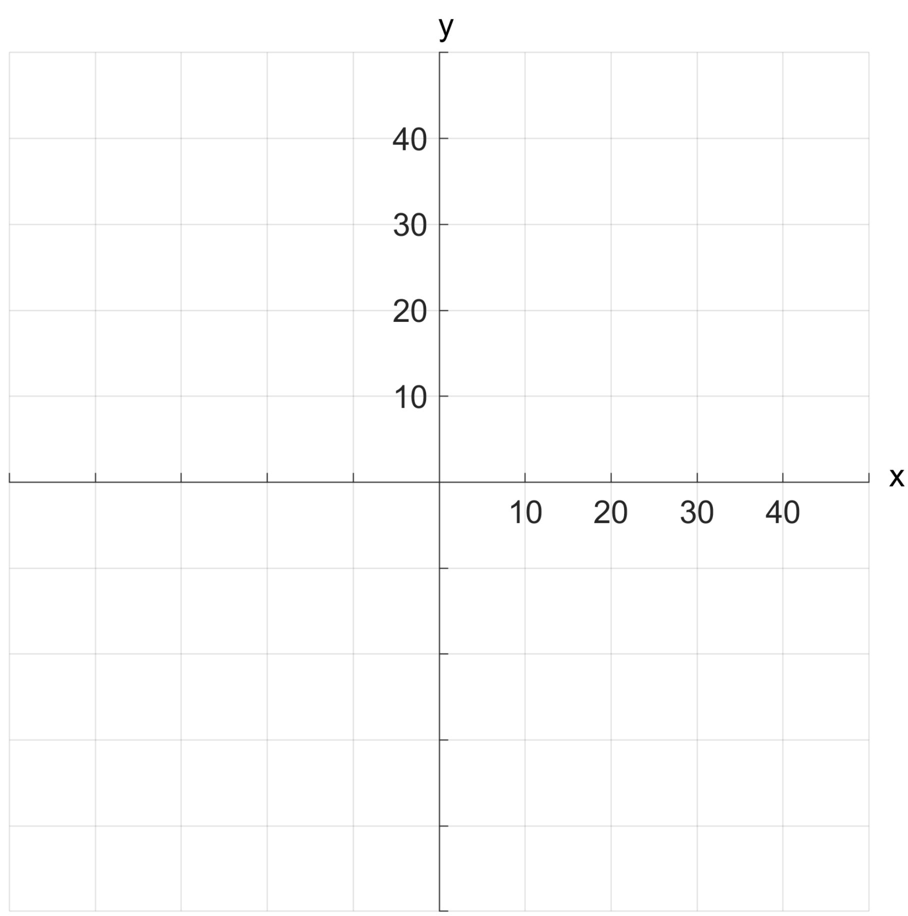
1. Draw the right-triangle representation of the 50-lb force and label the x-and y-components of the force on the axes provided. Then calculate the magnitude of the Cartesian components.



2. Draw the right-triangle representation of the 100-lb force and label the x-and y-components of the force on the axes provided. Then calculate the magnitude of the Cartesian components.



3. In the process of calculating the resultant of a system of forces, it was determined that the x-component of the resultant force is 30 lb and the y-component is -40 lb. What is the magnitude of the resultant force and in which direction does it act? Report the direction as an angle measured from the horizontal, x-axis. A positive angle indicates counter-clockwise rotation and a negative angle indicates a clockwise rotation. Draw the resultant force on the axes provided. On the axes, each tick represents a length of 10 lb.



Cable Design Problem


Figure 5. A gantry crane for moving shipping containers.

Shipping containers have become an essential part of the global economy, responsible for moving goods all over the world. If you'd like to learn more about why, check out this [Planet Money Podcast](). Shipping containers are transported on massive container ships. When a container ship arrives at a port, the containers are lifted off of the ship and onto a truck or the dock. To lift the containers off of the ship, a gantry crane like the one in Fig. 5 is often used. You can see a gantry crane in operation if you watch [this video]().

Imagine that you work for a company that is developing a new shipping container crane that will allow a container ship to be unloaded twice as fast. A sketch of the side view of the new crane is shown in Fig. 6. The new crane uses a cable-and-magnet system to attach to the shipping container rather than the mechanical clamps of the gantry crane of Fig. 5. You have been tasked with selecting the lengths of the <u>two</u> cables that extend from the central ring on the pulley to the magnet attachment points. In addition to supporting their load (the force in the cables when the crane is in operation), the cables carry the electricity that is used to activate and deactivate the magnet clamps.

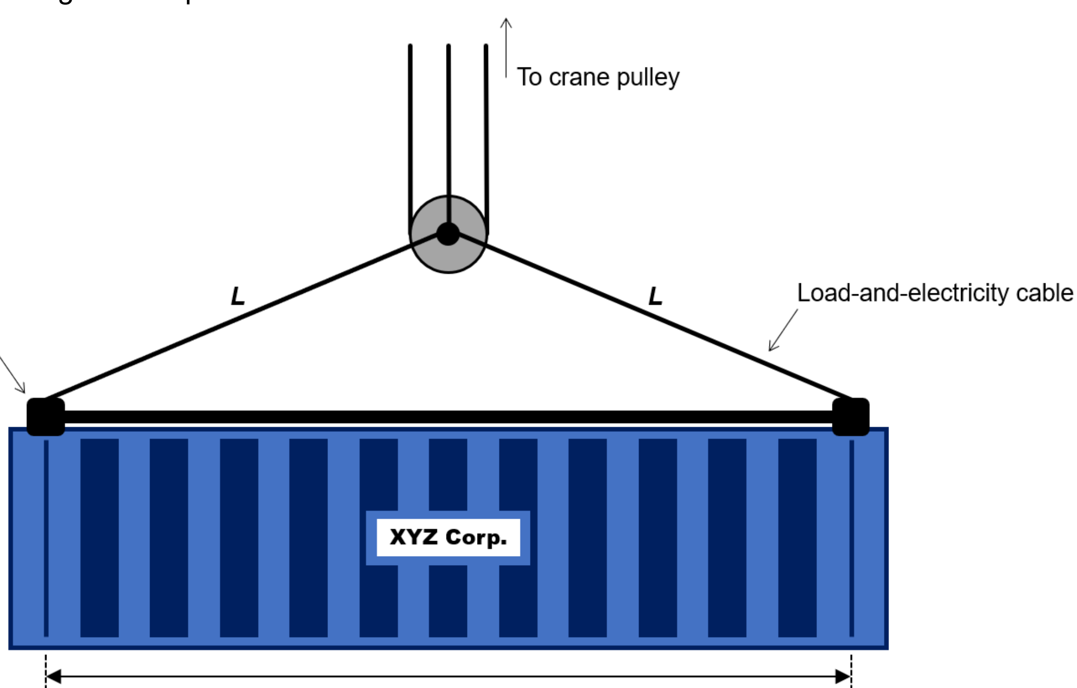


Figure 6. Sketch of the cable-and-magnet crane system for lifting shipping containers

The cable design is subject to two important constraints. The first is cost. The finance team has determined that it must cost less than $2000 for the new crane design to be financially viable for the company. The second is durability. The materials group that developed the novel load-and-electricity bearing cable have determined that the maximum force that the cable should sustain during operation is 40,000 lb. If the force becomes larger than this value, the life of the cabling system is substantially shortened.

Use what you have learned about engineering mechanics and trigonometry to select the length, ***L***, of the cables. The cable material costs $20 per foot. The gross weight (weight of the container and its contents) of the 40-ft shipping container is 67,200 lb when it carries its maximum payload.

What length would you suggest?
L = __________ ft

The following are helpful hints:

- Imagine creating a right triangle with a hypotenuse of length ***L*** that is aligned with the crane cable. This triangle is "similar" to the right triangle representation of the force in the cable. To be similar means that all of the angles in the two triangles are the same, and their side lengths differ by a constant factor. These triangles are depicted below. Use them to help you develop a link between the selection of the cable length ***L*** and the force in the cable ***F***.
- Note that the crane system is like an inverse of the weight-and-ring system depicted in Fig. 4 on page 3. Just like the net effect of the two forces in the weight-and-ring system had to be 10 lb upwards, the net effect of the forces in the cables in the cable-and-magnet crane must be to lift up upwards with a force of 67,200 lb, the weight of the container. That is, if you add the two F_y's together, the result must be equal to 67,200 lb.